三重大学教育学部研究紀要　第71巻　自然科学（2020）15－30頁

The values of Hilbert-Eisenstein series at cusps, III

Shigeaki TSUYUMINE

ヒルベルト・アイゼンシュタイン級数の尖点での値, III

露　峰　茂　明

Abstract. In our previous papers [2],[3], we obtain the values of some specific Hilbert-Eisenstein series at cusps. In the present paper we address generalization of characters of Hilbert-Eisenstein series, and give the values with moderating condition on characters which are not necessarily Hecke characters.

1. Introduction

In our previous papers [2],[3], we obtain the values of some specific Hilbert-Eisenstein series at cusps. The result is useful in the study of the Shimura lifting maps [4],[6], or of quadratic forms [5]. In the present paper, we consider the Hilbert-Eisenstein series of weight k associated with two characters ψ, ψ' of larger groups than the ideal class group where the product $\psi\overline{\psi}'$ is a Hecke character with same parity as k. We obtain their values at all the cusps, which may be also useful in the further study of the Shimura lifting maps or of quadratic forms. The argument is parallel to [4]. We note that we make use of different notations from the previous paper [4], and unify the notation with [6].

Let K be a totally real algebraic number field of degree g over $\mathbf{Q}$, and let $\mathcal{O}_K$ be the ring of algebraic integers. We denote by $\mathfrak{d}_K$ and D_K, the different of K and the discriminant respectively, and denote by $\mathcal{O}_K^\times$, the group of the units in K. For $\alpha \in K$, $\alpha^{(1)}, \cdots, \alpha^{(g)}$ denotes the conjugates of α in a fixed order. If $\alpha^{(i)}$ is positive for every i, then we call α *totally positive*, and denote it by $\alpha \succ 0$. We denote by N and tr, the norm map and the trace map of K over $\mathbf{Q}$ respectively, namely $\mathrm{N}(\alpha) = \prod_{i=1}^{g} \alpha^{(i)}$ and $\mathrm{tr}(\alpha) = \sum_{i=1}^{g} \alpha^{(i)}$. Let μ_K denote the Möbius function on K and let φ_K denote the Euler function on K. If $\mathfrak{P}$ is a prime ideal, then $v_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-adic valuation. If $\mathfrak{M}$ is an integral ideal, then $\{\mathfrak{M}\}_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-part of $\mathfrak{M}$, namely, $\{\mathfrak{M}\}_{\mathfrak{P}} = \mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{M})}$. For nonzero fractional ideals $\mathfrak{A}, \mathfrak{B}$, we denote by $(\mathfrak{A}, \mathfrak{B})$, their greatest common divisor, precisely, $(\mathfrak{A}, \mathfrak{B}) = \prod_{\mathfrak{P}} \mathfrak{P}^{\min\{v_{\mathfrak{P}}(\mathfrak{A}), v_{\mathfrak{P}}(\mathfrak{B})\}}$. Let $\mathfrak{N}$ be an integral ideal of K. We denote by $C_{\mathfrak{N}}$, the class group modulo $\mathfrak{N}$ in the narrow sense, and denote by $C_{\mathfrak{N}}^*$, the group of characters of $C_{\mathfrak{N}}$. An element of $C_{\mathfrak{N}}^*$ is called a (classical) Hecke character. Let $\mathcal{E}_{\mathfrak{N}}$ denotes the group of totally positive units congruent to 1 modulo $\mathfrak{N}$, while $\mathcal{E}_{\mathcal{O}_K}$ denotes the group of all totally positive units. For $\mathbf{a} = (a_1, \cdots, a_g) \in \{0,1\}^g$, we define $\mathrm{sgn}^{\mathbf{a}}(\alpha)$ by setting $\mathrm{sgn}^{\mathbf{a}}(\alpha) := \mathrm{sgn}(\alpha^{(1)})^{a_1} \cdots \mathrm{sgn}(\alpha^{(g)})^{a_g}$ for $\alpha \in K, \neq 0$ where $\mathrm{sgn}^{\mathbf{a}}(0) := 1$. Let $\mathbf{e}_\psi = (e_1, \cdots, e_n) \in \{0,1\}^g$ be so that $\psi(\mu) = \mathrm{sgn}^{\mathbf{e}_\psi}(\mu)$ for $\mu \equiv 1 \bmod \mathfrak{N}$, $\mu \neq 0$. The character $\psi \in C_{\mathfrak{N}}^*$ is called *even* (resp. *odd*) if $\mathbf{e}_\psi = (0, \cdots, 0)$ (resp. $\mathbf{e}_\psi = (1, \cdots, 1)$). We note that $\mathbf{e}_\psi = \mathbf{e}_{\overline{\psi}}$. The identity element of $C_{\mathfrak{N}}^*$ is denoted by $\mathbf{1}_{\mathfrak{N}}$, for which $\mathbf{1}_{\mathfrak{N}}(\mathfrak{A})$ is 1 or 0 according as an integral ideal $\mathfrak{A}$ is coprime to $\mathfrak{N}$ or not. Let $\mathcal{I}_K$ be the function on the set of fractional ideals defined by $\mathcal{I}_K(\mathfrak{A})$ is 1 or 0 according as $\mathfrak{A}$ is integral or not.

We denote by $\mathfrak{f}_\psi$, the conductor of a character ψ, and denote by $\mathfrak{e}_\psi$, the ideal given by

$$\mathfrak{e}_\psi := \mathfrak{f}_\psi \prod_{\psi(\mathfrak{P})=0, \mathfrak{P} \nmid \mathfrak{f}_\psi} \mathfrak{P}. \tag{1}$$

For an integral ideal $\mathfrak{M}$, we denote by $\mathcal{R}(\mathfrak{M}, \psi)$, the set of all square free products of prime divisors of $\mathfrak{M}$ coprime to $\mathfrak{f}_\psi$, where $\mathcal{O}_K$ is always in $\mathcal{R}(\mathfrak{M}, \psi)$. The primitive character associated with $\psi \in C_{\mathfrak{N}}^*$ is denoted by $\widetilde{\psi}$. For any integral ideal $\mathfrak{M}$, we define $\psi_{\mathfrak{M}} := \widetilde{\psi}\mathbf{1}_{\mathfrak{M}}$. Then $\psi_{\mathfrak{M}} = \widetilde{\psi}$ for an integral ideal $\mathfrak{M}$ with $\mathfrak{M} | \mathfrak{f}_\psi$, and $\psi_{\mathfrak{N}} = \psi$. For $\psi \in C_{\mathfrak{N}}^*$, $L_K(s, \psi)$ denotes the Hecke L-function, that is,

$$L_K(s, \psi) := \sum_{\mathfrak{A}} \frac{\psi(\mathfrak{A})}{\mathrm{N}(\mathfrak{A})^s}$$

Mie University, Faculty of Education, Department of Mathematics (*E-mail address:* tsuyu@edu.mie-u.ac.jp).

where $\mathfrak{A}$ runs over the set of all the integral ideals.

Let $\mathfrak{H}^g$ denote the product of g copies of the upper half plane $\mathfrak{H}=\{z\in\mathbf{C}\mid \Im z>0\}$, $\Im z$ being the imaginary part of z. For $\gamma,\delta\in K$ and for $\mathfrak{z}=(z_1,\cdots,z_g)\in\mathfrak{H}$, $\mathrm{N}(\gamma\mathfrak{z}+\delta)$ stands for $\prod_{i=1}^{g}(\gamma^{(i)}z_i+\delta^{(i)})$, and $\mathrm{tr}(\gamma\mathfrak{z})$ stands for $\sum_{i=1}^{g}\gamma^{(i)}z_i$. For a matrix

$$A=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\in\mathrm{SL}_2(K), \tag{2}$$

we put

$$A\mathfrak{z}=\left(\tfrac{\alpha^{(1)}z_1+\beta^{(1)}}{\gamma^{(1)}z_1+\delta^{(1)}},\cdots,\tfrac{\alpha^{(g)}z_g+\beta^{(g)}}{\gamma^{(g)}z_g+\delta^{(g)}}\right).$$

We define

$$\Gamma_0(\mathfrak{D}^{-1},\mathfrak{N}\mathfrak{D}):=\{\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\in\mathrm{SL}_2(K)\mid\alpha,\delta\in\mathcal{O}_K,\beta\in\mathfrak{D}^{-1},\gamma\in\mathfrak{N}\mathfrak{D}\}$$

for a fractional ideal $\mathfrak{D}$ and for an integral ideal $\mathfrak{N}$.

2. Gauss sums

For a prime $\mathfrak{P}$ of K, let $K_{\mathfrak{P}}$ be the $\mathfrak{P}$ adic completion of K, and let $\mathcal{O}_{\mathfrak{P}}$ be its maximal local ring. We denote by J_K, the idele group of K, and put $U_K:=\prod_{\mathfrak{P}}\mathcal{O}_{\mathfrak{P}}^{\times}\times(\mathbf{R}^+)^g$ with $\mathbf{R}^+=\{x\in\mathbf{R}\mid x>0\}$. For an integral ideal $\mathfrak{N}$ of K, let $J(\mathfrak{N})$ denote the subgroup of J_K consisting of ideles whose $\mathfrak{P}$-th components are in $\mathcal{O}_{\mathfrak{P}}$ for $\mathfrak{P}|\mathfrak{N}$. Namely for $j=(\cdots,j_{\mathfrak{P}},\cdots,j^{(1)},\cdots,j^{(g)})\in J_K$ with $j_{\mathfrak{P}}\in(K_{\mathfrak{P}})^{\times}$ and $(j^{(1)},\cdots,j^{(g)})\in(\mathbf{R}^{\times})^g$, j is in $J(\mathfrak{N})$ if and only if $j_{\mathfrak{P}}$ are in $\mathcal{O}_{\mathfrak{P}}^{\times}$ for all $\mathfrak{P}|\mathfrak{N}$. Let $K^{\times}(\mathfrak{N})$ denote the group of elements in $K^{\times}$ whose denominators and numerators are both coprime to $\mathfrak{N}$. Then the equality $K^{\times}(\mathfrak{N})=K\cap J(\mathfrak{N})$ holds. We denote by $K_{\mathfrak{N}}^{\times}$, the subgroup consisting of totally positive elements multiplicatively congruent to 1 modulo $\mathfrak{N}$. A homomorphism of the finite idele to $C_{\mathfrak{N}}$ by sending $j=(j_{\mathfrak{P}})$ to an ideal class containing fractional ideal $\prod_{\mathfrak{P}}\mathfrak{P}^{v_{\mathfrak{P}}(j_p)}$ is surjective. So the homomorphism of $J(\mathfrak{N})$ to $C_{\mathfrak{N}}$ is surjective and its kernel is $K_{\mathfrak{N}}^{\times}U_K$. Hence $C_{\mathfrak{N}}=J(\mathfrak{N})/(K_{\mathfrak{N}}^{\times}U_K)$. For a Hecke character $\phi\in G_{\mathfrak{N}}^{*}$ with $\mathbf{e}_{\phi}=(a_1,\cdots,a_g)\in\{\pm1\}^g$, it is idelically described as $\phi_J(j)=\prod_{\mathfrak{P}}\phi_{\mathfrak{P}}(j_{\mathfrak{P}})\,\mathrm{sgn}^{\mathbf{e}_{\phi}}(j)$ for $j=(\cdots,j_{\mathfrak{P}},\cdots,j^{(1)},\cdots,j^{(g)})\in J(\mathfrak{N})$ where $\phi_{\mathfrak{P}}$ is defined by $\phi_{\mathfrak{P}}(j_{\mathfrak{P}}):=\phi(\mathfrak{P}^{v_{\mathfrak{p}}(j_{\mathfrak{P}})})$ and where $\mathrm{sgn}^{\mathbf{e}_{\phi}}(j):=\prod_{k=1}^{g}\mathrm{sgn}(j^{(k)})^{a_k}$ which is called the infinity-type of ϕ_J. Let $U_{\mathfrak{N}}$ be the subgroup of U_K consisting of ideles whose $\mathfrak{P}$-th components are in $1+\mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{N})}\mathcal{O}_{\mathfrak{P}}$. There is the natural surjective homomorphism of $J(\mathfrak{N})/(K_{\mathfrak{N}}^{\times}U_{\mathfrak{N}})$ onto $C_{\mathfrak{N}}=J(\mathfrak{N})/(K_{\mathfrak{N}}^{\times}U_K)$ with the kernel $U_K/U_{\mathfrak{N}}$ isomorphic to the direct product $(\mathcal{O}_K/\mathfrak{N})^{\times}$, indeed there is an isomorphism of $(\mathcal{O}_K/\mathfrak{N})^{\times}$ onto $U(\mathfrak{N})/U_{\mathfrak{N}}$ sending $\xi+\mathfrak{N}$ to $\xi U_{\mathfrak{N}}$. Actually $J(\mathfrak{N})/(K_{\mathfrak{N}}^{\times}U_{\mathfrak{N}})$ and $(\mathcal{O}_K/\mathfrak{N})^{\times}\times C_{\mathfrak{N}}$ are isomorphic to each other as groups. We fix local parameters $\varpi_{\mathfrak{P}}$ of $\mathcal{O}_{\mathfrak{P}}$, which fix the isomorphism of $U(\mathfrak{N})/U_{\mathfrak{N}}$ onto $(\mathcal{O}_K/\mathfrak{N})^{\times}$. For a character χ of $(\mathcal{O}_K/\mathfrak{N})^{\times}$, we define a character of $J(\mathfrak{N})$ as as $j\longmapsto(j_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{N}}\longmapsto\chi((j_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{N}}\ (\mathrm{mod}\,U_{\mathfrak{N}}))$, which we denote also by χ. Then the product $\psi:=\chi\phi$ gives a character of $J(\mathfrak{N})/(K_{\mathfrak{N}}^{\times}U_{\mathfrak{N}})$. As in the case of a Hecke character, $\mathbf{e}_{\psi}$ is defined, and indeed $\mathbf{e}_{\psi}$ is equal to $\mathbf{e}_{\phi}$ because $\chi(\xi)=1$ for ξ multiplicatively congruent to 1 modulo $\mathfrak{N}$. We note that for $\varepsilon\in\mathcal{O}_K^{\times}$, $\psi(\varepsilon)$ is not necessarily equal to 1 while the value is always 1 if it is a Hecke character. For an idele $j\in J(\mathfrak{N})$, the idelic description of ψ is given by $\psi_J(j)=\chi((j_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{N}})\phi_J(j)\mathrm{sgn}^{\mathbf{e}_{\phi}}(j)=\chi((j_{\mathfrak{P}})_{\mathfrak{P}|\mathfrak{N}})\phi(\mathfrak{A})\mathrm{sgn}^{\mathbf{e}_{\phi}}(j)$ with the ideal $\mathfrak{A}$ corresponding to the finite idele of j, that is, $\mathfrak{A}=\prod_{\mathfrak{P}}\mathfrak{P}^{v_{\mathfrak{P}}(j_{\mathfrak{P}})}$. Conversely for a non-zero fractional ideal $\mathfrak{A}$, we put $j_{\mathfrak{A}}:=(\cdots,\varpi^{v_{\mathfrak{P}}(\mathfrak{A})},\cdots,1\,(g\,\text{times}))\in J_K$. For $\xi\in K^{\times}$ with $\xi j_{\mathfrak{A}}\in J(\mathfrak{N})$, we denote $\mathrm{sgn}^{\mathbf{e}_{\phi}}(\xi)\psi_J(\xi j_{\mathfrak{A}})$ by $\psi(\xi\cdot\mathfrak{A})$, that is,

$$\psi(\xi\cdot\mathfrak{A}):=\chi(\xi\prod_{\mathfrak{P}|\mathfrak{N}}\varpi_{\mathfrak{P}}^{v_{\mathfrak{P}}(\mathfrak{A})})\phi((\xi)\mathfrak{A}). \tag{3}$$

Obviously there holds an equality $\psi(\xi\cdot\mathfrak{A})\psi(\xi'\cdot\mathfrak{A}')=\psi(\xi\xi'\cdot\mathfrak{A}\mathfrak{A}')$. We note that $\psi(\xi)=\psi(\xi\cdot\mathcal{O}_K)=\chi(\xi)\phi((\xi))$ for $\xi\in K^{\times}(\mathfrak{N})$, and $\psi(\mathfrak{A})=\psi(1\cdot\mathfrak{A})$ for an ideal $\mathfrak{A}$. We define $\psi(\xi\cdot\mathfrak{A})=0$ if the numerator of $\xi\mathfrak{A}$ is not coprime to $\mathfrak{N}$, and so the values of ψ are roots of unity or 0.

We call ψ *primitive* if there is no integral ideal $\mathfrak{M}$ strictly larger than $\mathfrak{N}$ so that $\psi|_{U(\mathfrak{N})/U_{\mathfrak{N}}}$ is via group homomorphisms $U(\mathfrak{N})/U_{\mathfrak{N}}\simeq(\mathcal{O}_K/\mathfrak{N})^{\times}\longrightarrow(\mathcal{O}_K/\mathfrak{M})^{\times}\longrightarrow\mathbf{C}^{\times}$. In such a case we denote $\mathfrak{N}$ by $\mathfrak{f}_{\psi}$, and call it *the conductor* of ψ. When χ is trivial, this definition is compatible with that of a Hecke character. For a primitive ψ, the Gauss sum of ψ is defined by

$$\tau_K(\psi):=\psi(\rho\cdot\mathfrak{f}_{\psi}\mathfrak{d}_K)\sum_{\substack{\xi\succ0\\ \xi:\mathcal{O}_K/\mathfrak{f}_{\psi}}}\psi(\xi)\mathbf{e}(\mathrm{tr}(\rho\xi))$$

with $\rho \in K, \succ 0, (\rho\mathfrak{f}_\psi\mathfrak{d}_K, \mathfrak{f}_\psi) = \mathcal{O}_K$ where $\mathbf{e}(x)$ stands for $\exp(2\pi\sqrt{-1}x)$. The value $\tau_K(\psi)$ is determined up to the choices of ρ. The standard argument shows that $|\tau_K(\psi)| = \mathrm{N}(\mathfrak{f}_\psi)^{1/2}$, and that $\tau_K(\psi)\tau_K(\overline{\psi}) = \psi(\mathrm{N}(\mathfrak{f}_\psi) - 1)\mathrm{N}(\mathfrak{f}_\psi) = \mathrm{sgn}^{\mathbf{e}_\psi}(-1)\chi(-1)\mathrm{N}(\mathfrak{f}_\psi) = \mathrm{sgn}^{\mathbf{e}_\psi}(-1)\psi(-1)\mathrm{N}(\mathfrak{f}_\psi)$. We note that $\tau_K(\psi) = \psi(\mathfrak{d}_K)$ if $\mathfrak{f}_\psi = \mathcal{O}_K$.

In the present paper, for an primitive $\psi' = \chi'\phi'$ and an integral ideal $\mathfrak{N} \subset \mathfrak{f}_{\psi'}$, we consider a character ψ of $J(\mathfrak{N})/(K_{\mathfrak{N}}^\times U_{\mathfrak{N}})$ in the form $\psi = \psi'_{\mathfrak{N}}$ where $\psi'_{\mathfrak{N}} := \chi(\phi'\mathbf{1}_{\mathfrak{N}})$, χ being a character obtained by composing the natural surjective map $(\mathcal{O}_K/\mathfrak{N})^\times$ onto $(\mathcal{O}_K/\mathfrak{f}_\psi)^\times$ with χ'. We denote by $\widetilde{\psi}$, the associated primitive character of ψ, which is ψ' in the present case. The conductor $\mathfrak{f}_\psi$ of ψ defined to be that of the associated primitive character, and $\mathfrak{e}_\psi$ is defined similarly as in (1). We define $\mathcal{I}_K(\xi \cdot \mathfrak{A})$ to be 1 or 0 according as ξA is integral or not.

Lemma 1. *Let $\mathfrak{A}$ be a fractional ideal. Let ψ be a character of $J(\mathfrak{N})/(K_{\mathfrak{N}}^\times U_{\mathfrak{N}})$ as in (3), which is not necessarily primitive.*

(i) *Let $\mu \in \mathfrak{A}^{-1}$. Then*

$$\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0 \cdot \mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$$
$$= \mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\tau_K(\overline{\widetilde{\psi}}) \sum_{\mathfrak{R}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}} \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\widetilde{\psi}}(\mathfrak{R})(\psi_{\mathfrak{R}}\mathcal{I}_K)(\mu \cdot \mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R}\mathfrak{A}). \tag{4}$$

In the summation of the right hand side, at most one term survives. If there is a divisor $\mathfrak{R}$ of $\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}$ satisfying $(\mu\mathfrak{A}\mathfrak{N}^{-1}, \mathcal{O}_K) = \mathfrak{f}_\psi^{-1}\mathfrak{R}^{-1}$, then the term associated with $\mathfrak{R}$ survives.

(ii) *Let $\mu \in \mathfrak{A}^{-1}\mathfrak{N}^{-1}$. Then*

$$\sum_{\delta_0:\mathfrak{e}_\psi^{-1}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \psi(\delta_0 \cdot \mathfrak{e}_\psi\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$$
$$= \mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\tau_K(\widetilde{\psi}) \sum_{\mathfrak{R}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}} \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})}{\varphi_K(\mathfrak{R})}\widetilde{\psi}(\mathfrak{R})(\overline{\psi}_{\mathfrak{R}}\mathcal{I}_K)(\mu \cdot \mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{R}\mathfrak{A}). \tag{5}$$

In the summation of the right hand side, at most one term survives. If there is $\mathfrak{R}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}$ satisfying $(\mu\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{R}\mathfrak{A}, \mathcal{O}_K) = \mathfrak{R}^{-1}$, then the term associated with $\mathfrak{R}$ survives.

Proof. (i) For ψ primitive, the equality (4) turns out to be

$$\sum_{\substack{\xi\succ 0 \\ \xi:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{f}_\psi\mathfrak{A}\mathfrak{d}_K^{-1}}} \overline{\psi}(\xi \cdot \mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\xi\mu)) = \mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\psi(\mu \cdot \mathfrak{A}\mathfrak{f}_\psi)\tau_K(\overline{\psi}) \tag{6}$$

for $\mu \in \mathfrak{A}^{-1}\mathfrak{f}_\psi^{-1}, \neq 0$. At first we prove (6). Let $\alpha \in \mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0$ with $(\alpha\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{f}_\psi\mathfrak{d}_K) = \mathcal{O}_K$. Then the left hand side is equal to $\sum_{\substack{\xi\succ 0 \\ \xi:\mathcal{O}/\mathfrak{f}_\psi}} \overline{\psi}(\alpha\xi \cdot \mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\alpha\xi\mu)) = \overline{\psi}(\alpha \cdot \mathfrak{A}^{-1}\mathfrak{d}_K)\sum_{\substack{\xi\succ 0 \\ \xi:\mathcal{O}/\mathfrak{f}_\psi}} \overline{\psi}(\xi)\mathbf{e}(\mathrm{tr}(\xi(\alpha\mu)))$. The sum $\sum_{\substack{\xi\succ 0 \\ \xi:\mathcal{O}/\mathfrak{f}_\psi}} \overline{\psi}(\xi)\mathbf{e}(\mathrm{tr}(\xi(\alpha\mu)))$ is equal to $\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\psi(\alpha\mu \cdot \mathfrak{f}_\psi\mathfrak{d}_K)\tau_K(\overline{\psi})$ from definition of the Gauss sum. Hence the left hand side of (6) is $\overline{\psi}(\alpha \cdot \mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\psi(\alpha\mu \cdot \mathfrak{f}_\psi\mathfrak{d}_K)\tau_K(\overline{\psi})$, which is equal to the right hand side.

Now we prove (4). Let us take $\alpha \in \mathfrak{N}^{-1}, \succ 0$ such that $\alpha\mathfrak{N} \subset \mathcal{O}_K$ and $(\alpha\mathfrak{N}, \mathfrak{N}) = \mathcal{O}_K$. Let $\alpha\mu = \mu_1 + \mu_2$ where all the prime divisors of the denominator of μ_1 are divisors of $\mathfrak{f}_\psi$, and all the prime divisors of the denominator of μ_2 are divisors of $\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}$. Then

$$\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0 \cdot \mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu)) = \sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\alpha\delta_0 \cdot \mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\alpha\mu))$$
$$= \overline{\psi}(\alpha \cdot \mathfrak{N}) \sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0 \cdot \mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu_1))\mathbf{e}(\mathrm{tr}(\delta_0\mu_2)),$$

which is 0 unless $\mu_1 \in \mathfrak{A}^{-1}\mathfrak{f}_\psi^{-1}$ and $\mu_2 \in \mathfrak{A}^{-1}\mathfrak{R}^{-1}$ for some $\mathfrak{R}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}$. In such a case the above is equal to

$$\begin{aligned}&\mu_K(\mathfrak{R})\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\psi}(\alpha\cdot\mathfrak{N})\sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{f}_\psi\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\overline{\overline{\psi}}(\delta_0\cdot\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu_1))\\ &=\mu_K(\mathfrak{R})\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\psi}(\alpha\cdot\mathfrak{N})\mathrm{sgn}^{\mathbf{e}_\psi}(\mu_1)\widetilde{\psi}(\mu_1\cdot\mathfrak{A}\mathfrak{f}_\psi)\tau_K(\overline{\overline{\psi}}) \qquad \text{(by (6))}\\ &=\mu_K(\mathfrak{R})\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha^{-1}\mu_1)\overline{\overline{\psi}}(\mathfrak{R})\widetilde{\psi}(\alpha^{-1}\mu_1\cdot\mathfrak{A}\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R})\tau_K(\overline{\overline{\psi}}). \qquad (7)\end{aligned}$$

Here in the notation of (3), there holds an equality $\psi(\xi\cdot\mathfrak{A}) = \mathrm{sgn}^{\mathbf{e}_\psi}(\xi\xi')\psi(\xi'\cdot\mathfrak{A})$ if $\xi\prod_{\mathfrak{P}|\mathfrak{N}}\varpi_\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{A})}$ and $\xi'\prod_{\mathfrak{P}|\mathfrak{N}}\varpi_\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{A})}$ are multiplicatively congruent to 1 modulo $\mathfrak{f}_\psi$. Hence we may replace $\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha^{-1}\mu_1)\widetilde{\psi}(\alpha^{-1}\mu_1\cdot\mathfrak{A}\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R})$ by $\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\widetilde{\psi}(\mu\cdot\mathfrak{A}\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R})$ in (7). The factor $(\psi_\mathfrak{R}\mathcal{I}_K)(\mu\cdot\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\mathfrak{R})$ appearing in the right hand side of (4) is equal to $\widetilde{\psi}(\mu\cdot\mathfrak{A}\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R})$ if $(\mu\mathfrak{A}\mathfrak{N}^{-1},\mathcal{O}_K) = \mathfrak{f}_\psi^{-1}\mathfrak{R}^{-1}$ with $\mathfrak{R}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}$, and it is 0 if otherwise. This shows (4).

(ii) In the summation of the left hand side of (5), we may assume that δ_0 satisfies $(\delta_0\mathfrak{e}_\psi\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K,\mathfrak{N}) = \mathcal{O}_K$. Such δ_0 are written as products $\delta_0 = \delta_1\delta_2$ where δ_1 are the representatives of $\mathfrak{f}_\psi^{-1}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}$ modulo $\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}$ with $(\delta_0\mathfrak{f}_\psi\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K,\mathfrak{N}) = \mathcal{O}_K$, and δ_2 are the representatives of $\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi$ modulo $\mathcal{O}_K$ with $(\rho_2\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathcal{O}_K) = \mathcal{O}_K$. We can take δ_1 (resp. δ_2) so that they are totally positive and that the differences of δ_1's (resp. δ_2's) are in $\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}$ (resp. $\mathfrak{f}_\psi$). We write μ in the form $\mu = \mu_1+\mu_2$ with $\mu_1 \in \mathfrak{A}^{-1}\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}\mathfrak{N}^{-1}$ and $\mu_2 \in \mathfrak{A}^{-1}\mathfrak{f}_\psi\mathfrak{N}^{-1}$. Then the left hand side of (5) is equal to

$$\sum_{\delta_1,\delta_2}\psi(\delta_1\delta_2\cdot\mathfrak{f}_\psi\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_1))\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2)).$$

Since $\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2))$ is independent of δ_1, this is equal to

$$\begin{aligned}&\sum_{\delta_2}\left\{\sum_{\delta_1}\psi(\delta_1\delta_2\cdot\mathfrak{e}_\psi\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_1))\right\}\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2))\\ &=\sum_{\delta_2}\mathrm{sgn}^{\mathbf{e}_\psi}(\mu_1)\overline{\overline{\psi}}(\mu_1\cdot\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{N}\mathfrak{A})\tau_K(\widetilde{\psi})\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2)) \qquad \text{(by (6))}\\ &=\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\overline{\overline{\psi}}(\mu\cdot\mathfrak{N}\mathfrak{A})\tau_K(\widetilde{\psi})\sum_{\delta_2}\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2)).\end{aligned}$$

The last summation is equal to $\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})}{\varphi_K(\mathfrak{R})}$ if $(\mu_2\mathfrak{A}\mathfrak{e}_\psi^{-1}\mathfrak{N},\mathcal{O}_K) = \mathfrak{R}^{-1}$ $(\mathfrak{R}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})$ or equivalently if $(\mu\mathfrak{A}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{N},\mathcal{O}_K) = \mathfrak{R}^{-1}$. Thus for this $\mathfrak{R}$, the left hand side of (5) is equal to

$$\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})}{\varphi_K(\mathfrak{R})}\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\tau_K(\widetilde{\psi})\widetilde{\psi}(\mathfrak{R})\overline{\overline{\psi}}(\mu\cdot\mathfrak{N}\mathfrak{A}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{R}).$$

The similar argument of the last part of the proof of (i), shows our assertion. □

Let X be a function on the set $\{\mathfrak{N}\mathfrak{M}^{-1}\mid\mathfrak{M}|\mathfrak{N},(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K\}$ of ideals. Then we define $\Lambda_k(\mathfrak{N},\psi)$ by

$$\begin{aligned}\Lambda_k(\mathfrak{N},\psi)X :=\, &\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}\\ &\times\sum_{\mathfrak{M}|\mathfrak{N},(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K}\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\overline{\psi}}(\mathfrak{M})X(\mathfrak{N}\mathfrak{M}^{-1}). \qquad (8)\end{aligned}$$

Proposition 1. *Let $\mathfrak{N}$ be an integral ideal and let ψ be a character of $J(\mathfrak{N})/(K_\mathfrak{N}^\times U_\mathfrak{N})$ as in (3). Let $\mathfrak{A}$ be a fractional ideal. Let $X_\mu(\mathfrak{M}) = \sum_{\delta_0:\mathfrak{M}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\overline{\psi}_\mathfrak{M}(\delta_0\cdot\mathfrak{M}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$ for $\mu\in\mathfrak{A}^{-1}$ and for an integral ideal $\mathfrak{M}$ contained in $\mathfrak{f}_\psi$. Then*

$$\Lambda_k(\mathfrak{N},\psi)X_\mu = \mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}\tau_K(\overline{\overline{\psi}})\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)(\psi\mathcal{I}_K)(\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A}). \qquad (9)$$

Proof. Unless $(\mu\mathfrak{f}_\psi\mathfrak{N}^{-1},\mathfrak{f}_\psi) = \mathcal{O}_K$, the both sides of (9) are 0 and the equality holds. We assume that $(\mu\mathfrak{f}_\psi\mathfrak{N}^{-1},\mathfrak{f}_\psi) = \mathcal{O}_K$. Put $\mathfrak{R}_{\mathfrak{N},\psi} := \prod_{\mathfrak{P}|\mathfrak{N}}\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{N})}$. Let $\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}$ and put $\mathfrak{R} = \mathfrak{f}_\psi^{-1}(\mu\mathfrak{N}^{-1}\mathfrak{M},\mathcal{O}_K)^{-1}$.

Then by (4), $X(\mathfrak{N}\mathfrak{M}^{-1}) = \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N}\mathfrak{M}^{-1})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\overline{\widetilde{\psi}}(\mathfrak{R})\widetilde{\psi}(\mu\cdot\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{f}_\psi\mathfrak{R})\tau_K(\overline{\widetilde{\psi}})$. Let $Y(\mathfrak{N}\mathfrak{M}^{-1}) = \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})}{\varphi_K(\mathfrak{R})}\overline{\widetilde{\psi}}(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})$. Then $X(\mathfrak{N}\mathfrak{M}^{-1})$ is equal to the product of $\frac{\varphi_K(\mathfrak{N}\mathfrak{R}_{\mathfrak{N},\psi}^{-1})}{\varphi_K(\mathfrak{f}_\psi)}\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\widetilde{\psi}(\mu\cdot\mathfrak{N}^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{A}\mathfrak{f}_\psi)\tau_K(\overline{\widetilde{\psi}})$ and $Y(\mathfrak{N}\mathfrak{M}^{-1})$, where the former is the constant on $\mathfrak{M}$. We must compute $\Lambda_k(\mathfrak{N},\psi)Y$. We note that $\mathfrak{R} = \mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K$. Then

$$\begin{aligned}\Lambda_k(\mathfrak{N},\psi)Y &= \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi)\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}\sum_{\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}}\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\\ &\quad\times\overline{\widetilde{\psi}}(\mathfrak{M})\mu_K(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K)\frac{\varphi_K(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})}{\varphi_K(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K)}\overline{\widetilde{\psi}}(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})\\ &= \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi)\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}\prod_{\mathfrak{P}|\mathfrak{R}_{\mathfrak{N},\psi}}Z(\mathfrak{P}),\end{aligned}$$

where

$$Z(\mathfrak{P}) = \sum_{i=\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-1,0\}}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\prod_{\mathfrak{P}|\mathfrak{P}^i}(1-\mathrm{N}(\mathfrak{P}))\ \ \mu_K(\mathfrak{P}^{\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-i,0\}})\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})}{\varphi_K(\mathfrak{P}^{\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-i,0\}})}.$$

If $v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})\le 0$, then

$$Z(\mathfrak{P}) = \varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}) + (1-\mathrm{N}(\mathfrak{P}))\sum_{i=1}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i}) = 0.$$

If $v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}) = 1$, then

$$Z(\mathfrak{P}) = \sum_{i=0}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\prod_{\mathfrak{P}|\mathfrak{P}^i}(1-\mathrm{N}(\mathfrak{P}))\ \ \mu_K(\mathfrak{P}^{\max\{1-i,0\}})\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})}{\varphi_K(\mathfrak{P}^{\max\{1-i,0\}})} = -\mathrm{N}(\mathfrak{P})^{v_P(\mathfrak{R}_{\mathfrak{N},\psi})}.$$

If $v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}) > 1$, then

$$Z(\mathfrak{P}) = (1-\mathrm{N}(\mathfrak{P}))\{-\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})+1-v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})})}{\varphi_K(\mathfrak{P})} + \sum_{i=v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})\} = 0.$$

Thus

$$\Lambda_k(\mathfrak{N},\psi)Y = \begin{cases}\mathrm{N}(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi)\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k} & (v_\mathfrak{P}(\mu) = v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-1 \text{ for } \mathfrak{P}|\mathfrak{R}_{\mathfrak{N},\psi}),\\ 0 & (\text{otherwise}).\end{cases}$$

Then

$$\begin{aligned}\Lambda_k(\mathfrak{N},\psi)X_\mu &= \frac{\varphi_K(\mathfrak{N}\mathfrak{R}_{\mathfrak{N},\psi}^{-1})}{\varphi_K(\mathfrak{f}_\psi)}\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\widetilde{\psi}(\mu\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{A})\tau_K(\overline{\widetilde{\psi}})\\ &\quad\times\begin{cases}\mathrm{N}(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi)\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k} & (v_\mathfrak{P}(\mu) = v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-1 \text{ for } \mathfrak{P}|\mathfrak{R}_{\mathfrak{N},\psi})\\ 0 & (\text{otherwise})\end{cases}\\ &= \mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}\tau_K(\overline{\widetilde{\psi}})\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\psi(\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A}).\end{aligned}$$

□

3. Eisenstein series

Let $k\in\mathbf{N}$. Let $\mathfrak{N},\mathfrak{N}'$ be fixed integral ideals of K and let $\mathfrak{D}$ be a fixed fractional ideal. Let $\mathfrak{A},\mathfrak{B}$ be fractional ideals of K. Let $\gamma_0\in\mathfrak{A}\mathfrak{B}\mathfrak{D}, \delta_0\in\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}$. We define

$$E_{k,\mathfrak{A},\mathfrak{D},\mathfrak{B}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}') := \mathrm{N}(\mathfrak{A})^k\sum_{\substack{\gamma\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\\ \delta\equiv\delta_0(\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1})\\ (\gamma,\delta)/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}{}^{'}\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}|\mathrm{N}(\gamma\mathfrak{z}+\delta)|^{-s}|_{s=0}$$

where $\gamma \equiv \gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})$ implies that $\gamma \equiv \gamma_0$ modulo $\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}$ and where $\sum'$ implies that the term corresponding to $(\gamma,\delta)=(0,0)$ is omitted in the summation. For a set S, $\Delta(x,S)$ is define to be 1 or 0 according as $x\in S$ or not. Then we have the Fourier expansion

$$\begin{aligned}
&E_{k,\mathfrak{A},\mathfrak{D},\mathfrak{B}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')\\
=&\Delta(\gamma_0,\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\mathrm{N}(\mathfrak{A})^k \sum_{\substack{\mu\equiv\delta_0(\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}\\
&+\left(\tfrac{(-2\pi\sqrt{-1})^k}{(k-1)!}\right)^g D_K^{1/2}\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}(\mathfrak{B})\sum_{0\prec\nu\in\mathfrak{B}^2\mathfrak{D}}\ \sum_{\substack{\nu/\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\\ \mu:\mathfrak{A}^{-1}\mathfrak{B}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))
\end{aligned}$$

where there is the additional term

$$(-\pi\sqrt{-1})^g D_K^{1/2}\mathrm{N}(\mathfrak{B})\sum_{\substack{\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

when $k=1$, and where there is the additional term $-\pi/(\mathrm{N}(\mathfrak{N}'\mathfrak{D}\mathfrak{d}_K)\Im z)$ when $g=1$ and $k=2$.

Let ψ,ψ' be characters of $J(\mathfrak{N})/(K_{\mathfrak{N}}^{\times}U_{\mathfrak{N}}), J(\mathfrak{N}')/(K_{\mathfrak{N}'}^{\times}U_{\mathfrak{N}'})$ such as (3) respectively so that $\psi\overline{\psi'}$ is a Hecke character in $C^*_{\mathfrak{N}\mathfrak{N}'}$ with same parity as k. So if we put $\psi:=\chi\phi$ and $\psi':=\chi'\phi'$, then

$$\chi(\xi)=\chi'(\xi)\qquad (\xi\in K^{\times}((\mathfrak{N},\mathfrak{N}'))). \tag{10}$$

We assume that either $\psi\neq\mathbf{1}_{\mathfrak{N}}$ or $\psi'\neq\mathbf{1}_{\mathfrak{N}'}$ when $g=1$ and $k=2$. We assume that

$$(\mathfrak{N},\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1})=\mathcal{O}_K. \tag{11}$$

Then we put

$$\begin{aligned}
\widetilde{g}_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D})=\widetilde{g}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}^{\psi'}(\mathfrak{z};\mathfrak{D}):=&\left(\tfrac{(k-1)!}{(-2\sqrt{-1}\pi)^k}\right)^g D_K^{-1/2}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\sum_{\gamma_0,\delta_0}\overline{\psi}(\delta_0\cdot\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\\
&\times\psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})E_{k,\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')
\end{aligned}\tag{12}$$

where in the second summation, γ_0 runs over the set of totally positive representatives of $\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}$ modulo $\mathfrak{N}'\mathfrak{A}\mathfrak{D}$ with $(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1},\mathfrak{N}')=\mathcal{O}_K$, and δ_0 runs over the set of totally positive representatives of $\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}$ modulo $\mathfrak{A}\mathfrak{d}_K^{-1}$ with $(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K,\mathfrak{N})=\mathcal{O}_K$. Further let

$$\begin{aligned}
G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D})=G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{N}\mathfrak{e}_{\psi}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1};\mathfrak{D}):=&\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}\sum_{\mathfrak{M}|\mathfrak{N},(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K}\\
&\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\widetilde{\overline{\psi}}(\mathfrak{M})\widetilde{g}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}^{\psi'}(\mathfrak{z};\mathfrak{D}).
\end{aligned}\tag{13}$$

4. Constant terms of Hilbert Eisenstein series

In this section we use the following result due to Hecke [1].

Lemma 2. *Let $\mathfrak{M}$ be a fractional ideal, and let $\mathcal{E}$ be a subgroup of finite index in the group of all units. Let $\mu_0\in K$ and $k\in\mathbf{Z}$. Then there holds the functional equations*

$$\begin{aligned}
\sideset{}{'}\sum_{\substack{\mu\equiv\mu_0(\mathfrak{M})\\ \mu/\mathcal{E}}}\mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}=&\left(\tfrac{(-2\sqrt{-1}\pi)^k}{2\cdot(k-1)!}\right)^g D_K^{-1/2}\mathrm{N}(\mathfrak{M})^{-1}\sideset{}{'}\sum_{\mu:\mathfrak{M}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}}\mathbf{e}(\mathrm{tr}(\mu_0\mu))\\
&\times\mathrm{sgn}(\mathrm{Nm}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0},\ \sideset{}{'}\sum_{\substack{\mu\equiv\mu_0(\mathfrak{M})\\ \mu/\mathcal{E}}}\mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}\\
=&(-\sqrt{-1}\pi^{-1})^g D_K^{-1/2}\mathrm{N}(\mathfrak{M})^{-1}\sideset{}{'}\sum_{\mu:\mathfrak{M}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}}\mathbf{e}(\mathrm{tr}(\mu_0\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}
\end{aligned}$$

where $\sum'$ implies that the term corresponding to $\mu=0$ is omitted in the summation.

Let A be as in (2) with $\alpha \in \mathcal{O}_K, \gamma \in \mathfrak{D}\mathfrak{d}_K$. If $f(\mathfrak{z})$ is a Hilbert modular form of weight k for $\Gamma_0(\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{N}'\mathfrak{D}\mathfrak{d}_K)$, then the value $\kappa(\alpha/\gamma, f) = \kappa(\alpha,\gamma,f)$ of $f(\mathfrak{z})$ at the cusp α/γ is defined by

$$\kappa(\alpha/\gamma, f) = \kappa(\alpha,\gamma,f) := \lim_{\mathfrak{z}\to\sqrt{-1}\infty} \mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k} f(A\mathfrak{z}). \tag{14}$$

We determine the value at each cusp, of the Hilbert-Eisenstein series (13) as well as the Fourier expansion at the cusp $\sqrt{-1}\infty$.

For a cusp $\alpha/\gamma \in K \cup \{\infty\}$, we can take $\alpha \in \mathcal{O}_K, \gamma \in \mathfrak{D}\mathfrak{d}_K$ so that

$$\mathfrak{B} := (\alpha, \gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$$

satisfies

$$(\mathfrak{B}, \mathfrak{N}\mathfrak{N}') = \mathcal{O}_K. \tag{15}$$

Since $A \in \mathrm{SL}_2(K)$, we can take $\beta \in \mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \delta \in \mathcal{O}_K$ so that $\mathfrak{B}^{-1} = (\beta\mathfrak{D}\mathfrak{d}_K, \delta)$ and that $(\beta, \mathfrak{N}\mathfrak{N}') = (\delta, \mathfrak{N}\mathfrak{N}') = \mathcal{O}_K$. Then the equality $\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k} E_{k,\mathfrak{N}'\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(A\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}') = E_{k,\mathfrak{N}'\mathfrak{A},\mathfrak{D},\mathfrak{B}}(\mathfrak{z}, \alpha\gamma_0 + \gamma\delta_0, \beta\gamma_0 + \delta\delta_0; \mathfrak{N}, \mathfrak{N}')$ holds, and the constant term of the Fourier expansion of $E_{k,\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')$ at α/γ, is equal to

$$\mathrm{N}(\mathfrak{A})^k \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \Delta(\alpha\gamma_0 + \gamma(\delta_0+\delta'), \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}) \sideset{}{'}\sum_{\substack{\mu\equiv\beta\gamma_0+\delta(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

where there is the additional term

$$(-\pi\sqrt{-1})^g D_K^{1/2}\mathrm{N}(\mathfrak{N}'^{-1}\mathfrak{B}) \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \sideset{}{'}\sum_{\substack{\mu\equiv\alpha\gamma_0+\gamma(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

when $k=1$. The modular form $\widetilde{g}^{\psi'}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}(\mathfrak{z};\mathfrak{D})$ is a linear combination of $E_{k,\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')$'s by (12), and we obtain the following;

Lemma 3. *Let $A, \alpha, \beta, \gamma, \delta$ be as above. Assume the condition (15) holds. Let $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ denote the constant term of $\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}\widetilde{g}^{\psi'}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}(A\mathfrak{z};\mathfrak{D})$. Then $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ is given by*

$$\left(\frac{(k-1)!}{(-2\sqrt{-1}\pi)^k}\right)^g D_K^{-1/2}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1} \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}} \overline{\psi}(\delta_0\cdot\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)$$
$$\times\, \psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathrm{N}(\mathfrak{A})^k \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'), \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})$$
$$\times \sideset{}{'}\sum_{\substack{\mu\equiv\beta\gamma_0+\delta(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

where in the second summation, γ_0 and δ_0 satisfy $(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}, \mathfrak{N}') = \mathcal{O}_K$, $(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$ respectively. When $k=1$, there is the additional term $C^1_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ with

$$C^1_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$$
$$:= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{N}(\mathfrak{N}'^{-1}\mathfrak{B}) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}} \overline{\psi}(\delta_0\cdot\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)$$
$$\times\, \psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}) \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \sideset{}{'}\sum_{\substack{\mu\equiv\alpha\gamma_0+\gamma(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

For $\gamma \in \mathcal{O}_K$, we put

$$\mathfrak{M}'_\gamma := \mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}')^{-1}.$$

By the assumption (11), $\mathfrak{M}'_\gamma$ is coprime to $\mathfrak{N}$ if it is integral. The purpose of this section is to prove the following;

Theorem 1. *Let* $\alpha \in \mathcal{O}_K, \gamma \in \mathfrak{D}\mathfrak{d}_K, \mathfrak{B} = (\alpha, \gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$ *with* $(\mathfrak{B}, \mathfrak{N}\mathfrak{N}') = \mathcal{O}_K$. *Put* $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}) := C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}, k, \psi_{\mathfrak{N}\mathfrak{M}^{-1}}, \psi')$ *for a divisor* $\mathfrak{M}$ *of* $\mathfrak{N}$ *with* $(\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K$. *Let* $\Lambda_k(\mathfrak{N}, \psi)$ *be as in (8). If there is no divisor* $\mathfrak{M}_\gamma$ *of* $\mathfrak{N}$ *with* $(\mathfrak{M}_\gamma, \mathfrak{f}_\psi) = \mathcal{O}_K$ *and* $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ *for* $\mathfrak{M}'_\gamma$ *integral, then* $\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma} = 0$. *Suppose otherwise. Let* $\mathfrak{M}_\gamma$ *be the largest ideal satisfying* $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$. *Then*

$$
\begin{aligned}
&\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma} \\
&= \mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}'))\overline{\widetilde{\psi}}(\alpha \cdot \mathfrak{B}^{-1}\mathfrak{M}_\gamma\mathfrak{M}'_\gamma(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')^{-1}) \\
&\quad \times \psi'(-\gamma \cdot \mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}(\mathfrak{M}_\gamma^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1} \\
&\quad \times \mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau_K(\overline{\widetilde{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M}_\gamma^{-1}) \prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1 - \mathrm{N}(\mathfrak{P}))\ L_K(1-k, \widetilde{\psi\overline{\psi}'}) \\
&\quad \times \prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}, \mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1 - \widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-k}) \prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{P}\nmid\mathfrak{M}_\gamma\mathfrak{N}'}(1 - \widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{k-1}).
\end{aligned} \tag{16}
$$

If $\gamma = 0$, *then* $\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma}$ *is non-zero only when* $\mathfrak{N}' = \mathcal{O}_K$, *and the value is obtained by replacing* γ *in (16) by* $\mathrm{N}(\mathfrak{N})$. *If* $\alpha = 0$, *then then* $\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma}$ *is non-zero only when* $\mathfrak{f}_\psi = \mathcal{O}_K$, *and the value is obtained by replacing* α *in (16) by* 1.

Several preparations are necessary to give the proof of Theorem 1.

Lemma 4. *Unless* $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{N}') = \mathfrak{N}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ *for any* $\mathfrak{M}'_\gamma$ *integral, then* $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$ *vanishes. Suppose the equality holds for some* $\mathfrak{M}'_\gamma$ *integral. Then* $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$ *equals*

$$
\begin{aligned}
&2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\overline{\psi}(\alpha \cdot \mathfrak{B}^{-1}\mathfrak{M}'_\gamma) \\
&\times \psi'(-\gamma \cdot \mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma) \sum_{\mathfrak{A} \in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \\
&\times \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} (\overline{\psi}\psi')(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.
\end{aligned}
$$

Proof. Since $(\alpha\mathfrak{B}^{-1}, \gamma\mathfrak{B}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}) = \mathcal{O}_K$ and since $(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}, \mathfrak{N}') = (\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$ in the equation for $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$ in Lemma 3, it is possible that $\Delta(\alpha\gamma_0 + \gamma(\delta_0 + \delta'), \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}) \neq 0$ only when $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{N}') = \mathfrak{N}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ for $\mathfrak{M}'_\gamma$ integral. This shows the first assertion of Lemma 4. In particular if $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi') \neq 0$, then $\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1} \subset \mathfrak{N}$ and $(\alpha, \mathfrak{N}) = \mathcal{O}_K$. When $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{N}') = \mathfrak{N}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ for $\mathfrak{M}'_\gamma$ integral, $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$ is equal to

$$
\begin{aligned}
&2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)) \sum_{\mathfrak{A} \in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}_\gamma'^{-1}\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}, \succ 0 \\ \delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0}} \\
&\Delta(\alpha\gamma_0 + \gamma\delta_0, \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\overline{\psi}(\delta_0 \cdot \mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)(\psi'\mathcal{I}_K)(\gamma_0 \cdot \mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}(\mathfrak{B}) \\
&\times \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}\mathfrak{B}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \mathbf{e}(\mathrm{tr}((\beta\gamma_0 + \delta\delta_0)\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0},
\end{aligned}
$$

which is obtained by Lemma 2 and by Lemma 1 (ii). The map

$$
\begin{pmatrix} \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}_\gamma'^{-1}\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D} \\ \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}_\gamma'^{-1}\mathfrak{A}\mathfrak{B}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D} \\ \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1} \end{pmatrix}
$$

obtained by multiplying by $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, is bijective. Using this bijection, we have

$$C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$$

$$
\begin{aligned}
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}_\gamma'^{-1}\mathfrak{A}\mathfrak{B}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}, \succ 0 \\ \delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}, \succ 0}} \\
&\quad \Delta(\gamma_0, \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D})\mathrm{sgn}^{\mathbf{e}_\psi}(-\beta\gamma_0 + \alpha\delta_0)\overline{\psi}((-\beta\gamma_0+\alpha\delta_0)\cdot\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\delta\gamma_0-\gamma\delta_0) \\
&\quad \times (\psi'\mathcal{I}_K)((\delta\gamma_0-\gamma\delta_0)\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathrm{N}(\mathfrak{B})\mathrm{N}(\mathfrak{A})^{k-1} \\
&\quad \times \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}\mathfrak{B}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}
\end{aligned}
$$

replacing $\mathfrak{A}$ by $\mathfrak{A}\mathfrak{B}$,

$$
\begin{aligned}
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}_\gamma')^{-1}\overline{\psi}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{M}_\gamma') \\
&\quad \times \psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}_\gamma') \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{M}_\gamma'(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \\
&\quad (\overline{\psi}\psi'\mathcal{I}_K)(\delta_0\mathfrak{N}\mathfrak{M}_\gamma'^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K) \sideset{}{'}\sum_{\mu : \mathfrak{M}_\gamma'^{-1}(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0} \\
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}_\gamma')^{-1}\overline{\psi}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{M}_\gamma') \\
&\quad \times \psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}_\gamma') \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sideset{}{'}\sum_{\mu : \mathfrak{M}_\gamma'^{-1}(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \\
&\quad \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{M}_\gamma'\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{M}_\gamma'(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} (\overline{\psi}\psi')(\delta_0\mathfrak{N}\mathfrak{M}_\gamma'^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.
\end{aligned}
$$

Replacing $\mathfrak{A}$ by $\mathfrak{M}_\gamma'^{-1}\mathfrak{A}$, we obtain the result of the lemma. □

Just replacing $\mathfrak{N}$ by $\mathfrak{N}\mathfrak{M}^{-1}$ in the lemma, we obtain the following;

Corollary. *Let $\mathfrak{M}$ be a divisor of $\mathfrak{N}$ with $(\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K$. Unless $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ for $\mathfrak{M}_\gamma'$ integral, then $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}, k, \psi_{\mathfrak{N}\mathfrak{M}^{-1}}, \psi')$ vanishes. Suppose the equality. Then it equals*

$$
\begin{aligned}
&2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}_\gamma')^{-k}\overline{\psi}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{M}_\gamma') \\
&\times \psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}_\gamma') \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \\
&\sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} (\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}\psi')(\delta_0\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.
\end{aligned}
$$

Let $\mathfrak{M}_\gamma$ be the largest ideal with $\mathfrak{M}_\gamma|\mathfrak{N}, (\mathfrak{M}_\gamma, \mathfrak{f}_\psi) = \mathcal{O}_K$ satisfying $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}_\gamma'^{-1}$. Then $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}, k, \psi_{\mathfrak{N}\mathfrak{M}^{-1}}, \psi') = 0$ for $\mathfrak{M}$ with $\mathfrak{M}_\gamma \nmid \mathfrak{M}$. Suppose that $\mathfrak{M}$ is a divisor of $\mathfrak{N}\mathfrak{M}_\gamma^{-1}$ coprime to $\mathfrak{f}_\psi$ satisfying $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}_\gamma'^{-1}$. Then $(\mathfrak{M}, \mathfrak{N}') = \mathcal{O}_K$, from which there holds $(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K) = (\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)$. For such $\mathfrak{M}$, we have

$$
\begin{aligned}
&C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}, k, \psi_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}, \psi') \\
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}_\gamma')^{-k}\overline{\psi}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{M}_\gamma') \\
&\quad \times \psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}_\gamma') \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \\
&\quad \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \overline{\psi}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K) \\
&\quad \times \mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.
\end{aligned}
$$

Then

$$\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma} = \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}\overline{\widetilde{\psi}}(\mathfrak{M}_\gamma)\Big(\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)$$

$$\sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}|\mathfrak{M}\\ \mathfrak{P}\nmid\mathfrak{M}_\gamma}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M})C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}},\psi')$$

$$= 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}$$

$$\times\,\mathrm{N}((\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K))\mathrm{N}(\mathfrak{B})^k\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{M}'_\gamma)^{-k}\Big(\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\overline{\widetilde{\psi}}(\mathfrak{M}_\gamma)\overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\Big)$$

$$\times\,\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\times\sideset{}{'}\sum_{\mu:(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\mathrm{sgn}(\mathrm{N}(\mu))^kD(\mu)|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0} \tag{17}$$

with

$$D(\mu):=\sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}|\mathfrak{M}\\ \mathfrak{P}\nmid\mathfrak{M}_\gamma}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M})\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$\overline{\psi}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu)).$$

Lemma 5. *Let* $\mu\in(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}$. *Then* $D(\mu)$ *is equal to*

$$\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\cap\mathfrak{e}_{\psi'})\mathrm{N}(\mathfrak{f}_{\overline{\psi}\psi'})^{-1}\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\mathrm{N}(\mathfrak{P})^{-1})$$

$$\times\,\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R})$$

$$\times\,((\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'}), \tag{18}$$

where $\mathcal{R}(\ ,\ )$ *is defined just after (1).*

Proof. There holds

$$D(\mu)=\sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\mathfrak{P}|\mathfrak{M},\mathfrak{P}\nmid\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)(\overline{\widetilde{\psi}}\psi')(\mathfrak{M})\times$$

$$\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}(\overline{\psi}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}\psi')(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$$

$$=\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})(\widetilde{\psi\overline{\psi}'}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\mathfrak{P}|\mathfrak{M},\mathfrak{P}\nmid\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)\times$$

$$\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\cap\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\cap\mathfrak{e}_{\psi'})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}\times\begin{cases}1 & (\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A},\mathcal{O}_K)=\mathfrak{f}_{\overline{\psi}\psi'}^{-1}\mathfrak{R}^{-1}) \\ 0 & (\text{otherwise})\end{cases}$$

by Lemma 1. Then

$$D(\mu)=\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})(\widetilde{\psi\overline{\psi}'}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},\mathfrak{P}\nmid\mathfrak{f}_\psi\mathfrak{N}'}Z(\mathfrak{P})$$

$$\times\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{e}_{\psi'}\cap\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'}}\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}$$

$$\times \begin{cases} 1 & (\mu\mathfrak{A}\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'}}\mathfrak{P}^{-v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})}, \mathcal{O}_K) = \mathfrak{f}_{\overline{\psi}\psi'}^{-1}\mathfrak{R}^{-1}) \\ 0 & (\text{otherwise}) \end{cases}$$

where

$$Z(\mathfrak{P}) = \begin{cases} \sum_{i=v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})-1}^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})} (1-\mathrm{N}(\mathfrak{P}))^{\min\{1,i\}}\mu_K(\{\mathfrak{R}\}_\mathfrak{P})\frac{\varphi_K(\{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{P}^{-i}\}_\mathfrak{P})}{\varphi_K(\{\mathfrak{R}\}_\mathfrak{P})} & (\mathfrak{P}\nmid\mathfrak{M}_\gamma), \\ \sum_{i=v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})-1}^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})} \mu_K(\{\mathfrak{R}\}_\mathfrak{P})\frac{\varphi_K(\{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{P}^{-i}\}_\mathfrak{P})}{\varphi_K(\{\mathfrak{R}\}_\mathfrak{P})} & (\mathfrak{P}|\mathfrak{M}_\gamma). \end{cases}$$

A simple calculation leads to the following;
(i) The case that $\mathfrak{P}\nmid\mathfrak{M}_\gamma$ and $v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})>1:\ Z(\mathfrak{P})=0$.
(ii) The case that $\mathfrak{P}\nmid\mathfrak{M}_\gamma$ and $v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})=1:\ Z(\mathfrak{P})=-\mathrm{N}(\mathfrak{P})^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})}$.
(iii) The case that $\mathfrak{P}\nmid\mathfrak{M}_\gamma$ and $v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})\le 0:\ Z(\mathfrak{P})=0$.
(iv) The case that $\mathfrak{P}|\mathfrak{M}_\gamma$ and $v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})>1:\ Z(\mathfrak{P})=0$.
(v) The case that $\mathfrak{P}|\mathfrak{M}_\gamma$ and $v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})=1:\ Z(\mathfrak{P})=0$.
(vi) The case that $\mathfrak{P}|\mathfrak{M}_\gamma$ and $v_\mathfrak{P}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})\le 0:\ Z(\mathfrak{P})=\mathrm{N}(\mathfrak{P})^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})}$.

Then, putting $\mathfrak{R}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi,\mathfrak{N}'} := \prod_{\mathfrak{P}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},\mathfrak{P}\nmid\mathfrak{f}_\psi\mathfrak{N}'}\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi})}$ with $\mathfrak{R}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi} = \prod_{\mathfrak{P}|\mathfrak{N}\mathfrak{M}_\gamma^{-1}}\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})}$,

$$\begin{aligned} D(\mu) =& \operatorname{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi,\mathfrak{N}'}^{-1}\cap\mathfrak{e}_{\psi'})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})} \\ &\times\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi,\mathfrak{N}'})(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}) \\ &\times((\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'}), \end{aligned}$$

which is equal to (18). □

Proof of Theorem 1. By Lemma 5 and by (17), we have

$$\begin{aligned} &\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma} \\ =& 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\operatorname{sgn}^{\mathbf{e}_\psi}(\alpha)\operatorname{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')) \\ &\times\overline{\overline{\psi}}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{M}'_\gamma\mathfrak{M}_\gamma)\psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_{\overline{\psi}\psi'})^{-k} \\ &\times\mathrm{N}(\mathfrak{f}_\psi\mathfrak{M}_\gamma^{-1})\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\mathrm{N}(\mathfrak{P})^{-1})\mathrm{N}(\mathfrak{M}_\gamma^{-1}\mathfrak{f}_\psi(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))^{k-1} \\ &\times(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\mathrm{N}(\mathfrak{R})^{-k+1}\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{R}) \\ &\times\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\sum_{\mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}^{'}((\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}})(\mu\mathfrak{A})\mathrm{N}(\mu\mathfrak{A})|^{k-1-s}|_{s=0} \\ =& \tau_K(\overline{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\operatorname{sgn}^{\mathbf{e}_\psi}(\alpha)\operatorname{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))\overline{\overline{\psi}}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{M}'_\gamma\mathfrak{M}_\gamma) \\ &\times\psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{M}_\gamma^{-1})^k\mathrm{N}(\mathfrak{f}_{\overline{\psi}\psi'})^{-k} \\ &\times\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\mathrm{N}(\mathfrak{P})^{-1})\mathrm{N}((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))^{k-1} \\ &\times(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\mathrm{N}(\mathfrak{R})^{-k+1}\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{R}) \\ &\times L_K(1-k,(\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}}). \end{aligned}$$

Here

$$\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\mathrm{N}(\mathfrak{R})^{-k+1}\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{R})L_K(1-k,(\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}})$$

$$=L_K(1-k,\widetilde{\psi\overline{\psi}'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{P}\nmid\mathfrak{M}_\gamma\mathfrak{N}'}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{k-1})\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}\mathrm{N}(\mathfrak{P})(\mathrm{N}(\mathfrak{P})-1)^{-1}$$

$$\times\prod_{\mathfrak{P}|\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-k}),$$

from which, the theorem follows. □

5. The case of weight 1

We compute the additional term which appears when $k=1$. As in the preceding section, we put $\mathfrak{B}:=(\alpha,\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$ for $\alpha\in\mathcal{O}_K$, $\gamma\in\mathfrak{D}\mathfrak{d}_K$, and assume the condition (15). From Lemma 3 and Lemma 2, we have for $\mathfrak{M}|\mathfrak{N}$ with $(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K$,

$$\begin{aligned}&C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')\\&=(-\sqrt{-1}\pi^{-1})^g2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{-1/2}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{-1}\\&\times\sideset{}{'}\sum_{\mu:(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1},\mathfrak{N}')^{-1}\mathfrak{A}^{-1}\mathfrak{B}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ0}\psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu))\\&\times\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ0}\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}(\delta_0\cdot\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\gamma\delta_0\mu)).\end{aligned}\tag{19}$$

The purpose of this section is to prove the following;

Theorem 2. *Let $\alpha\in\mathcal{O}_K,\gamma\in\mathfrak{D}\mathfrak{d}_K,\mathfrak{B}=(\alpha,\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$ with $(\mathfrak{B},\mathfrak{N}\mathfrak{N}')=\mathcal{O}_K$. Let $C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1})$ denote $C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$. Put*

$$\mathfrak{L}_\gamma:=\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}.$$

If there is no divisor $\mathfrak{R}$ of $\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ so that the numerator of $\mathfrak{L}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$, then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ vanishes. Suppose that such $\mathfrak{R}$ exists. Let $\widetilde{\mathfrak{R}}_\gamma$ be the divisor of $(\mathfrak{N},\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$ satisfying $v_\mathfrak{P}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})=0$ for any prime divisor $\mathfrak{P}$ of $(\mathfrak{N},\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$. Put $\mathfrak{L}'_\gamma:=(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{N}'^{-1},\mathfrak{N}^{-1})\cap\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$. Then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is equal to

$$\begin{aligned}&\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mu_K(\widetilde{\mathfrak{R}}_\gamma)\mathrm{N}(\mathfrak{B})\overline{\psi}(\mathfrak{B})\psi(-\gamma\cdot\gamma^{-1}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})\cap\mathcal{O}_K))\overline{\widetilde{\psi}'}(\alpha\cdot\mathfrak{B}^{-1})\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\\&\times\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},\mathcal{O}_K)^{-1})\varphi_K(\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}'^{-1}_\gamma)\mathrm{N}((\mathfrak{L}_\gamma,\widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}'_\gamma)\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\psi\overline{\psi}'})\tau_K(\overline{\widetilde{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})\\&\times L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi,\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})).\end{aligned}\tag{20}$$

If $\gamma=0$, then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is non-zero only when $\mathfrak{N}=\mathcal{O}_K$, and the value is obtained by replacing γ in (20) by $\mathrm{N}(\mathfrak{N}')$. If $\alpha=0$, then then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is non-zero only when $\mathfrak{f}_{\psi'}=\mathcal{O}_K$, and the value is obtained by replacing α in (20) by 1.

Proof. By (19) and by Proposition 1, we have

$$\begin{aligned}\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}=&(-\sqrt{-1}\pi^{-1})^g2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{-1/2}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{-1}\\&\sideset{}{'}\sum_{\mu:(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1},\mathfrak{N}')^{-1}\mathfrak{A}^{-1}\mathfrak{B}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ0}\psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\\&\times\mathrm{sgn}^{\mathbf{e}_\psi}(\gamma\mu)(\psi\mathcal{I}_K)(\gamma\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}\end{aligned}$$

$$= (-\sqrt{-1}\pi^{-1})^g 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{-1/2}\mathrm{sgn}^{\mathbf{e}_\psi}(\gamma)\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{-1}$$
$$\sideset{}{'}\sum_{\mu:(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1},\mathcal{O}_K)^{-1}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{B}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \psi(\gamma\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\mathrm{N}(\mu)^{-1}$$
$$\times|\mathrm{N}(\mu)|^{-s}|_{s=0}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ 0}\psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu)).$$

Since $\tau_K(\widetilde{\psi'}) = \mathrm{sgn}^{\mathbf{e}_{\psi'}}(-1)\psi'(-1)\mathrm{N}(\mathfrak{f}_{\psi'})\tau_K(\overline{\widetilde{\psi'}})^{-1}$ and since $\mathrm{sgn}^{\mathbf{e}_\psi}(-1)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-1) = (-1)^g$ and $\chi(\xi) = \chi'(\xi)$ by (10), Lemma 1 leads to

$$\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$$
$$= (\sqrt{-1}\pi^{-1})^g 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{-1/2}\mathrm{N}(\mathfrak{f}_{\psi'})\tau_K(\overline{\widetilde{\psi'}})^{-1}\psi(-1)\mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$$
$$\times\sum_{\mathfrak{R}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\sideset{}{'}\sum_{\mu:(\alpha\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{R},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi)^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\psi(\gamma\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})$$
$$\times\widetilde{\psi}'(\mathfrak{R})\overline{\psi}'_{\mathfrak{R}}(\alpha\mu\cdot\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{A}\mathfrak{D}\mathfrak{d}_K\mathfrak{R})\mathrm{N}(\mu\mathfrak{A})^{-1-s}|_{s=0}$$
$$= (\sqrt{-1}\pi^{-1})^g\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{-1/2}\mathrm{N}(\mathfrak{f}_{\psi'})\tau_K(\overline{\widetilde{\psi'}})^{-1}\mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$$
$$\times\sum_{\mathfrak{R}|(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\psi(-\gamma\cdot\gamma^{-1}(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K))\widetilde{\psi}'(\mathfrak{R})$$
$$\times\overline{\psi}'_{\mathfrak{R}}(\alpha\cdot(\mathfrak{L}_\gamma\mathfrak{R}^{-1},(\alpha))^{-1})\mathrm{N}((\alpha\mathfrak{R},\mathfrak{L}_\gamma)\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{D}\mathfrak{d}_K)L_K(1,\widetilde{\psi\overline{\psi'}})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})$$
$$= \mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\tau_K(\overline{\widetilde{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}\mathfrak{d}_K^{-1}\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\sum_{\mathfrak{R}|(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})}\mu_K(\mathfrak{R})$$
$$\times\varphi_K(\mathfrak{R})^{-1}\psi(-\gamma\cdot\gamma^{-1}(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K))\widetilde{\psi}'(\mathfrak{R})\overline{\psi}'_{\mathfrak{R}}(\alpha\cdot\alpha^{-1}(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})$$
$$\times\mathrm{N}((\alpha\mathfrak{R},\mathfrak{L}_\gamma)\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{D}\mathfrak{d}_K)L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}),$$

where we use the functional equation of the L-function at the last equality.

Since $\mathfrak{B} = (\alpha,\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$ is coprime to $\mathfrak{N}\mathfrak{N}'$, we have $(\alpha\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{R},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}) = (\alpha,\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$ $\times(\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{R},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}) = \mathfrak{B}(\mathfrak{R},\mathfrak{L}_\gamma)\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$ for $\mathfrak{R}$ dividing $(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$. Then $(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K) = \alpha^{-1}\mathfrak{B}(\mathfrak{L}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)$ follows. Then

$$\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma} = \mathrm{N}(\mathfrak{B})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\tau_K(\overline{\widetilde{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\overline{\psi}(\mathfrak{B})\overline{\widetilde{\psi'}}(\alpha\cdot\mathfrak{B}^{-1})$$
$$\times\sum_{\mathfrak{R}|(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1},\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\mathrm{N}((\mathfrak{L}_\gamma,\mathfrak{R})\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(-\gamma\cdot\gamma^{-1}(\mathfrak{L}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K))\widetilde{\psi}'(\mathfrak{R})$$
$$\times\overline{\psi}'_{\mathfrak{R}}((\mathfrak{L}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}).$$

In the summation, the term corresponding to $\mathfrak{R}$ survives if the numerator of $\mathfrak{L}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$. Suppose that such $\mathfrak{R}$ exists. Then

$$\widetilde{\mathfrak{R}}_\gamma := \prod_{\mathfrak{P}|(\mathfrak{N},\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}),\ v_{\mathfrak{P}}(\mathfrak{L}_\gamma)=1}\mathfrak{P}$$

is the largest such ideal, and $\mathfrak{R}$ is written as the product of $\widetilde{\mathfrak{R}}_\gamma$ and a divisor of $\mathfrak{L}'_\gamma\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ where $\mathfrak{L}'_\gamma := (\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{N}'^{-1},\mathfrak{N}^{-1})\cap\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$. Then

$$\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$$

$$
\begin{aligned}
&= \mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\mathrm{N}(\mathfrak{B})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\tau_K(\overline{\widetilde{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\overline{\psi}(\mathfrak{B})\overline{\widetilde{\psi}}'(\alpha\cdot\mathfrak{B}^{-1}) \\
&\quad\times \sum_{\mathfrak{R}|\mathfrak{L}'_\gamma\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}} \mu_K(\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})\varphi_K(\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})^{-1}\mathrm{N}((\mathfrak{L}_\gamma,\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(-\gamma\cdot\gamma^{-1}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{R}^{-1}\cap\mathcal{O}_K)) \\
&\quad\times \widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma\mathfrak{R}}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}) \\
&= \mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\mathrm{N}(\mathfrak{B})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\tau_K(\overline{\widetilde{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\overline{\psi}(\mathfrak{B})\overline{\widetilde{\psi}}'(\alpha\cdot\mathfrak{B}^{-1})\mu_K(\widetilde{\mathfrak{R}}_\gamma)\varphi_K(\widetilde{\mathfrak{R}}_\gamma)^{-1}\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}) \\
&\quad\times \mathrm{N}((\mathfrak{L}_\gamma,\widetilde{\mathfrak{R}}_\gamma)\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(-\gamma\cdot\gamma^{-1}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K))\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},\mathcal{O}_K)^{-1})L_K(0,\widetilde{\overline{\psi}\psi'}) \\
&\quad\times \prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\sum_{\mathfrak{R}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\mathrm{N}(\mathfrak{R})\overline{\psi}(\mathfrak{R})\widetilde{\psi}'(\mathfrak{R})\prod_{\mathfrak{P}|\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}) \\
&= \mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\mathrm{N}(\mathfrak{B})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\tau_K(\overline{\widetilde{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\overline{\psi}(\mathfrak{B})\overline{\widetilde{\psi}}'(\alpha\cdot\mathfrak{B}^{-1})\mu_K(\widetilde{\mathfrak{R}}_\gamma)\varphi_K(\widetilde{\mathfrak{R}}_\gamma)^{-1} \\
&\quad\times \varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}((\mathfrak{L}_\gamma,\widetilde{\mathfrak{R}}_\gamma)\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(-\gamma\cdot\gamma^{-1}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K))\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},\mathcal{O}_K)^{-1}) \\
&\quad\times L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\mathrm{N}(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma)\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma)^{-1}\prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})),
\end{aligned}
$$

which is equal to (20). □

6. Main Theorem

Let ψ, ψ' be as in Section 3. We define

$$\sigma_{k-1,\psi}^{\psi'}(\nu;\mathfrak{M}) := \sum_{\nu\mathfrak{M}\subset\mathfrak{A}\subset\mathcal{O}_K}\psi(\mathfrak{A})\psi'(\nu\cdot\mathfrak{M}\mathfrak{A}^{-1})\mathrm{N}(\mathfrak{A})^{k-1} \tag{21}$$

for a totally positive $\nu \in K$ and for a fractional ideal $\mathfrak{M}$. We note that it is 0 if $\nu\mathfrak{M}$ is not integral. If $\nu\mathfrak{M}$ is integral and if $\psi = \chi\phi, \psi' = \chi'\phi'$ are as in Section 3, then (21) is equal to $\chi'(\nu\prod_{\mathfrak{P}|\mathfrak{N}'}\varpi_{\mathfrak{P}}^{-v_{\mathfrak{P}}(\nu)})\sum_{\nu\mathfrak{M}\subset\mathfrak{A}\subset\mathcal{O}_K}$ $\phi(\mathfrak{A})\phi'(\nu\mathfrak{M}\mathfrak{A}^{-1})\mathrm{N}(\mathfrak{A})^{k-1}$.

Main Theorem. *Let $k \in \mathbf{N}$ and let $\mathfrak{N}, \mathfrak{N}'$ be integral ideals of K. Let ψ, ψ' be characters of $J(\mathfrak{N})/(K_{\mathfrak{N}}^\times U_{\mathfrak{N}})$, $J(\mathfrak{N}')/(K_{\mathfrak{N}'}^\times U_{\mathfrak{N}'})$ as in (3) with the conductors $\mathfrak{f}_\psi, \mathfrak{f}_{\psi'}$ respectively so that $\psi\overline{\psi}'$ is a Hecke character in $C^*_{\mathfrak{N}\mathfrak{N}'}$ with same parity as k. Let $\widetilde{\psi}$ denote the primitive character associated with ψ. Let $\mathfrak{e}_\psi, \mathfrak{e}_{\psi'}$ be as in (1). We assume $(\mathfrak{N}, \mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}) = \mathcal{O}_K$. For a fixed fractional ideal $\mathfrak{D}$, let*

$$
\begin{aligned}
G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}) &= G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1};\mathfrak{D}) \\
&:= \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k} \\
&\quad\times \sum_{\mathfrak{M}|\mathfrak{N},(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K}\left(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\right)\overline{\widetilde{\psi}}(\mathfrak{M})\widetilde{g}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}^{\psi'}(\mathfrak{z};\mathfrak{D}),
\end{aligned}
$$

where we assume that $\psi \neq \mathbf{1}_{\mathfrak{N}}$ or $\psi' \neq \mathbf{1}_{\mathfrak{N}'}$ when $g = 1$ and $k = 2$. Then $G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D})$ is a Hilbert modular form for $\Gamma_0(\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{N}'\mathfrak{D}\mathfrak{d}_K)$ of weight k with character $\psi\psi'$, whose Fourier expansion is given by

$$
\begin{cases}
\overline{\psi}'(\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{D})L_K(1-k,\psi\overline{\psi}') & (k>1 \text{ or } \mathfrak{N}\subsetneq\mathcal{O}_K, \text{ and } \mathfrak{N}'=\mathcal{O}_K) \\
\overline{\psi}(\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{D})L_K(0,\overline{\psi}\psi') & (k=1,\mathfrak{N}=\mathcal{O}_K,\mathfrak{N}'\subsetneq\mathcal{O}_K) \\
\overline{\psi}'(\mathfrak{D})L_K(0,\psi\overline{\psi}')+\overline{\psi}(\mathfrak{D})L_K(0,\overline{\psi}\psi') & (k=1,\mathfrak{N}=\mathfrak{N}'=\mathcal{O}_K) \\
0 & (\text{otherwise})
\end{cases}
$$

$$+\,2^g\sum_{0\prec\nu\in\mathfrak{D}}\sigma_{k-1,\psi}^{\psi'}(\nu;\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{D}^{-1})\ \mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})).$$

Let α/γ be a cusp with $\alpha \in \mathcal{O}_K$, $\gamma \in \mathfrak{D}\mathfrak{d}_K$. We can take α, γ so that $\mathfrak{B} := (\alpha, \gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1})$ is coprime to $\mathfrak{N}\mathfrak{N}'$. The value $\kappa(\alpha/\gamma, G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}))$ of $G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D})$ at the cusp α/γ defined in (14) is 0 if there are not integral ideals $\mathfrak{M}_\gamma, \mathfrak{M}'_\gamma$ with $\mathfrak{M}_\gamma|\mathfrak{N}$, $(\mathfrak{M}_\gamma, \mathfrak{f}_\psi) = \mathcal{O}_K$, $\mathfrak{M}'_\gamma|\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}$ and with $(\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$. Suppose otherwise, and let $\mathfrak{M}_\gamma$ be the largest such ideal. Then the value $\kappa(\alpha/\gamma, G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}))$ is given by

$$\begin{aligned}
&\mathrm{sgn}^{\mathbf{e}_\psi}(\alpha)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(-\gamma)\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}'))\overline{\overline{\psi}}(\alpha\cdot\mathfrak{B}^{-1}\mathfrak{M}_\gamma\mathfrak{M}'_\gamma(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')^{-1}) \\
&\times \psi'(-\gamma\cdot\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k \\
&\times \mathrm{N}(\mathfrak{M}_\gamma^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1}\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau_K(\overline{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M}_\gamma^{-1})L_K(1-k, \widetilde{\psi\overline{\psi}'}) \\
&\times \prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P})) \prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}, \mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-k}) \prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{P}\nmid\mathfrak{M}_\gamma\mathfrak{N}'}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{k-1})
\end{aligned} \tag{22}$$

where if $\gamma = 0$, then the value is non-zero only when $\mathfrak{N}' = \mathcal{O}_K$ and it is given by replacing γ in (22) by $\mathrm{N}(\mathfrak{N})$, *and where if $\alpha = 0$, the value is non-zero only when $\mathfrak{f}_\psi = \mathcal{O}_K$ and it is given by replacing α in (22) by* 1.

When $k = 1$, the Eisenstein series may have an additional term. Let $\mathfrak{L}_\gamma := \gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ and $\mathfrak{L}'_\gamma := (\gamma\mathfrak{D}^{-1}\mathfrak{d}_K^{-1}\mathfrak{N}'^{-1}, \mathfrak{N}^{-1}) \cap \mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$. If there is an integral divisor $\mathfrak{R}$ of $\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ so that the numerator of $\mathfrak{L}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$, then there is the additional term. Let $\widetilde{\mathfrak{R}}_\gamma$ be the divisor of $(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$ satisfying $v_\mathfrak{P}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}) = 0$ for any prime divisor $\mathfrak{P}$ of $(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$. Then $\kappa(\alpha/\gamma, G_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}))$ has the additional term

$$\begin{aligned}
&\mathrm{sgn}^{\mathbf{e}_\psi}(-\gamma)\mathrm{sgn}^{\mathbf{e}_{\psi'}}(\alpha)\mu_K(\widetilde{\mathfrak{R}}_\gamma)\psi(-\gamma\cdot\gamma^{-1}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})\cap\mathcal{O}_K))\overline{\overline{\psi'}}(\alpha\cdot\mathfrak{B}^{-1})\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}, \mathcal{O}_K)^{-1}) \\
&\times \overline{\psi}(\mathfrak{B})\mathrm{N}(\mathfrak{B})\varphi_K(\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}_\gamma'^{-1})\mathrm{N}((\mathfrak{L}_\gamma, \widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}'_\gamma)\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\tau_K(\overline{\overline{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})L_K(0, \widetilde{\overline{\psi}\psi'}) \\
&\times \prod_{\mathfrak{P}|\mathfrak{e}_\psi, \mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}) \prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P}))
\end{aligned} \tag{23}$$

where if $\gamma = 0$, then the value is non-zero only when $\mathfrak{N} = \mathcal{O}_K$ and it is given by replacing γ in (23) by $\mathrm{N}(\mathfrak{N}')$, *and where if $\alpha = 0$, the value is non-zero only when $\mathfrak{f}_{\psi'} = \mathcal{O}_K$ and it is given by replacing α in (23) by* 1.

Remark: Both ψ, ψ' can be Hecke characters, and in such a case $\psi(\xi\cdot\mathfrak{A}) = \psi(\xi\mathfrak{A})$ and $\psi(\xi) = \psi((\xi))$ for $\xi \in K$.

Proof. The values at cusps are investigated in Section 4 and Section 5. We compute the higher terms. Then

$$\begin{aligned}
&\widetilde{g}_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}) = \widetilde{g}_{k,\psi_\mathfrak{N},\mathfrak{N}}^{\psi'}(\mathfrak{z};\mathfrak{D}) \\
&= C + [\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau(\overline{\overline{\psi}})^{-1}\sum_{0\prec\nu\in\mathfrak{D}}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ 0}\psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}) \\
&\qquad\times \sum_{\substack{\nu/\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{D}) \\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\overline{\psi}(\delta_0\cdot\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu)),
\end{aligned}$$

where C is the constant term. Let $X(\mathfrak{N}\mathfrak{M}^{-1}) := \widetilde{g}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}^{\psi'}(\mathfrak{z};\mathfrak{D})$ for $\mathfrak{M}|\mathfrak{N}$ with $(\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K$, and let $X_\mu(\mathfrak{N}\mathfrak{M}^{-1}) := \sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}(\delta_0\cdot\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$. Then

$$\begin{aligned}
&\Lambda_k(\mathfrak{N}, \psi)X \\
&= C' + [\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau(\overline{\overline{\psi}})^{-1}\sum_{0\prec\nu\in\mathfrak{D}}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ 0}
\end{aligned}$$

$$\sum_{\substack{\nu/\mu\equiv\gamma_0\,(\mathfrak{N}'\mathfrak{A}\mathfrak{D})\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))\quad \Lambda_k(\mathfrak{N},\psi)X_\mu(\mathfrak{N}\mathfrak{M}^{-1})$$

$$=C'+\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\sum_{0\prec\nu\in\mathfrak{D}}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}/\mathfrak{N}'\mathfrak{A}\mathfrak{D},\succ 0}\psi'(\gamma_0\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})$$

$$\sum_{\substack{\nu/\mu\equiv\gamma_0\,(\mathfrak{N}'\mathfrak{A}\mathfrak{D})\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}(\psi\mathcal{I}_K)(\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\mathrm{sgn}^{\mathbf{e}_\psi}(\mu)\mathrm{sgn}(\mathrm{N}(\mu))^{-1}\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

$$=C'+\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\sum_{0\prec\nu\in\mathfrak{D}}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}$$

$$\sum_{\substack{\nu/\mu\in\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}(\psi\mathcal{I}_K)(\mu\cdot\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\psi'(\nu/\mu\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathrm{N}(\mu\mathfrak{A})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

$$=C'+2^g\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}\sum_{0\prec\nu\in\mathfrak{D}}$$

$$\sum_{\nu\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{D}^{-1}\subset\mathfrak{A}\subset\mathfrak{N}\mathfrak{e}_\psi}\psi(\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\psi'(\nu\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1})\mathrm{N}(\mathfrak{A})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))\qquad\text{(by (10))}$$

$$=C'+2^g\sum_{0\prec\nu\in\mathfrak{D}}\sigma_{k-1,\psi}^{\psi'}(\nu;\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{D}^{-1})\ \mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})).$$

□

References

[1] E. Hecke, *Über die L-Funktionen und den Dirichletschen Primzahlsatz für einen beliebigen Zahlkörper* , Nachr. Gesell. Wiss. Göttingen, (1917), 299–318.

[2] S. Tsuyumine, *The Values of Hilbert-Eisenstein Series at Cusps*, Bull. Fac. Educ. Mie Univ. **64**, (2013), 1–20.

[3] S. Tsuyumine, *The Values of Hilbert-Eisenstein Series at Cusps, II*, Bull. Fac. Educ. Mie Univ. **67**, (2016), 7–22.

[4] S. Tsuyumine, *Shimura lifting of modular forms of weight* 3/2, Ramanujan J. **39**, (2016), 363–449.

[5] S. Tsuyumine, *Sums of three squares under congruence condition modulo a prime*, J. Number Theory, **159**, (2016), 123–159.

[6] S. Tsuyumine, *On Shimura lifting of Hilbert modular forms*, Research in Number Theory, **4**:40, (2018), 1–45.