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実直線上の自己アフィンフラクタルと次元集合

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Dimension Set of Self-Affine Fractals on the Real Line

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要　旨

S がコンパクト距離空間 X 上で定義された縮小写像の族のとき，S の有限部分族 Λ に対して，極限集合 J_Λ (アトラクターあるいはIFSフラクタルともよばれる) が定義できる．$\mathrm{HD}(J_\Lambda)$ で極限集合のハウスドルフ次元を表すとき，ハウスドルフ次元の集合 $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq S\}$ を次元集合とよぶことにする．次元集合は，閉区間 $[0, \mathrm{HD}(J_S)]$ の部分集合である．極限集合の次元と測度については，Mauldin と Urbański による詳細な研究結果 [5] があるが，我々は，次元集合が閉区間 $[0, \mathrm{HD}(J_S)]$ で稠密になるのは S がどのような条件をみたす場合か，あるいはどのような場合に疎になるか，という問題に興味を持っている．このような問題は，文献 [3] において Kesseböhmer と Zhu によって初めて議論された．特に「単純正則連分数に展開したとき，指定された有限個の項だけを持つ無理数の集合のハウスドルフ次元は，単位区間に稠密に存在する」という彼らの結果は非常に興味深い．本稿では，X が区間であり更に S が X 上のアフィン変換 (１次関数) 族の場合に限って，この問題を議論する．関数をアフィン変換に限ることで，この問題に対する明確な判定条件を与えることができた．本稿の結果は，X が区間でなくとも，d 次元ユークリッド空間の連結なコンパクト集合であればそのまま成り立つと思われるが，その検証はこれからの研究課題である．

Abstract. Let X be a non-empty compact connected subset of d-dimensional euclidean space and S a conformal iterated function system on X. For $\Lambda \subseteq S$, we denote the limit set with respect to Λ by J_Λ and the Hausdorff dimension of it by $\mathrm{HD}(J_\Lambda)$. The dimensional and measure-theoretical properties for the limit set were studied in details by Mauldin and Urbański [5]. We are interested in the problem of deciding whether the dimension set $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq S \textit{ finite}\}$ is dense or nowhere dense in the interval $[0, \mathrm{HD}(J_S)]$. Such problem was studied by Kesseböhmer and Zhu [3] for the first time. In this paper we shall discuss the problem in the case where X is a closed interval and S a collection of affine transformations on X. Then the clear sufficient conditions of this problem will be obtained.

1 Introduction

Let (X, ρ) be a non-empty compact metric space and $S = \{\phi_i : i \in \mathbb{N}\}$ a collection of *injective* maps from X to X. We assume that there exists a constant $0 < r < 1$ such that we have

$$\rho(\phi_i(x), \phi_i(y)) \le r\rho(x, y) \quad \textit{for any } i \in \mathbb{N} \textit{ and for any } x, y \in X,$$

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then S is called a uniformly contractive *iterated function system* (IFS).

For each $\mathbf{i} = \{i_k\} \in \mathbb{N}^\infty$ and $n \in \mathbb{N}$, we set $\phi_{\mathbf{i}|_n} = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}$. Then the set $\cap_n \phi_{\mathbf{i}|_n}(X)$ is a singleton, because $\{\phi_{\mathbf{i}|_n}(X)\}$ is a non-increasing sequence of compact subsets of X and we have $\rho(\phi_{\mathbf{i}|_n}(x), \phi_{\mathbf{i}|_n}(y)) \leq r^n \rho(x, y)$ for any $x, y \in X$. Let $\Lambda \subseteq \mathbb{N}$ be a non-empty set, then the *limit set* J_Λ with respect to the restricted IFS $\{\phi_i \in S : i \in \Lambda\}$ is defined by

$$J_\Lambda = \bigcup_{\mathbf{i} \in \Lambda^\infty} \bigcap_{n=1}^{\infty} \phi_{\mathbf{i}|_n}(X) .$$

Obviously, we have $J_\Lambda = \cup_{i \in \Lambda} \phi_i(J_\Lambda)$. Especially if Λ is finite set then J_Λ is compact.

The dimensional and measure-theoretical properties for the limit set were studied in details by Mauldin and Urbański [5] in the case where S is a *conformal* iterated function system. We are going to introduce and use some of their results below.

We are interested in the problem of deciding whether the *restricted dimension set* :

$$\{ \mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N} \textit{ finite} \}$$

is dense or nowhere dense in the interval $[\,0,\ \mathrm{HD}(J_\mathbb{N})\,]$. Such problem was discussed for the first time by Kesseböhmer and Zhu [3]. For example, as an application of their arguments, they proved the following. Let $a_1, a_2, \ldots$ are positive integers and $J[a_1, a_2, \ldots]$ a space of irrationals in $[0,\ 1]$ with entries in the continued fraction expansion restricted to $a_1, a_2, \ldots$. Then the dimension set

$$\{ \mathrm{HD}(J[a_1, a_2, \ldots, a_n]) : a_1, a_2, \ldots, a_n \in \mathbb{N} \}$$

is dense in $[\,0,\ 1\,]$, especially we have $\{ \mathrm{HD}(J[a_1, a_2, \ldots]) : a_1, a_2, \ldots \in \mathbb{N} \} = [\,0,\ 1\,]$.

In this paper, we assume that X is a closed interval and that the IFS S satisfies the following :

(1.1) S satisfies *open set condition* with $U = \mathrm{Int}(X)$, that is $U \supset \cup_{i \in \mathbb{N}} \phi_i(U)$ with the union disjoint,

(1.2) ϕ_i are *affinities* and $|\phi_i'| > 0$ for every $i \in \mathbb{N}$, where $|\phi_i'|$ means the norm of the derivative.

In this case, the limit set will be called an *self-affine fractal.* Mauldin and Urbański [5] proved that the Hausdorff dimension of the self-affine fractal is given by

$$\mathrm{HD}(J_\Lambda) = \inf\{ s > 0 \,:\, \sum_{k \in \Lambda} |\phi_k'|^s \leq 1 \} .$$

In particular, it is well known [2, 6] that if Λ is finite then

$$\mathrm{HD}(J_\Lambda) \textit{ is the unique number } s \textit{ satisfying } \sum_{k \in \Lambda} |\phi_k'|^s = 1 .$$

Using these two characterizations of the dimensions, we are going to prove the following.

Theorem A. *Suppose there exists an* $0 < s_0 < \mathrm{HD}(J_\mathbb{N})$ *such that we have*

$$\sum_{k > m} |\phi_k'|^{s_0} \geq |\phi_m'|^{s_0} \quad \textit{for any} \quad m \in \mathbb{N} .$$

Then the restricted dimension set $\{ \mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N} \textit{ finite} \}$ *is dense in the interval* $[0,\ s_0]$. *In particular, we have* $\{ \mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N} \} = [0,\ s_0\,]$.

An IFS S is said to be *absolutely regular* system if $\sum_{k \in \Lambda} |\phi_k'|^s = 1$ has a unique solution for *any* $\Lambda \subseteq \mathbb{N}$.

Theorem B. *Let S be an absolutely regular system and assume $|\phi'_1| \geq |\phi'_2| \geq |\phi'_3| \geq \cdots$.*
(i) *Suppose there exists an $m \in \mathbb{N}$ such that*

$$\sum_{k>m} |\phi'_k|^{\mathrm{HD}(J_{\{1,2,\ldots,m\}})} < |\phi'_m|^{\mathrm{HD}(J_{\{1,2,\ldots,m\}})}.$$

Then there exists an interval $(\alpha, \beta) \subseteq [0, \mathrm{HD}(J_{\mathbb{N}})]$ with $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}\} \cap (\alpha, \beta) = \emptyset$.
(ii) *Suppose there exists an $m > 2$ such that*

$$\sum_{k>m} |\phi'_k|^{\mathrm{HD}(J_{\{1,2,\ldots,m-1\}})} < |\phi'_{m-1}|^{\mathrm{HD}(J_{\{1,2,\ldots,m-1\}})}.$$

Then there exists an interval $(\alpha, \beta) \subseteq [0, \mathrm{HD}(J_{\mathbb{N}\setminus\{m\}})]$ with $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N},\ m \notin \Lambda\} \cap (\alpha, \beta) = \emptyset$

Theorem C. *Let S be an absolutely regular system. Suppose there exists an $0 < s_0 < \mathrm{HD}(J_{\mathbb{N}})$ such that*

$$\sum_{k>m} |\phi'_k|^{s_0} < |\phi'_m|^{s_0} \quad \textit{for any} \quad m \in \mathbb{N}.$$

Then the dimension set $\{\mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N}\}$ is nowhere dense in $[s_0, \mathrm{HD}(J_{\mathbb{N}})]$.

In the next section we discuss some elementary properties of the Hausdorff dimensions of the self-affine fractal and prove Theorem A. In Section 3 we consider a regular system and prove Theorem B and Theorem C. As an application, we prove in Section 4 that in the case where $|\phi'| = \mathit{const.}\, r^{-k}$, $r > 1$, a dense interval narrows as r grows and a nowhere dense interval extends. A part of the discussions in this article are completely elementary and may be already known. We give a full discussion, however, in order to make the paper readable and self-contained.

2 Properties of the Dimensions of Self-Affine Fractals

From now on, we set $a_k = |\phi'_k|$. Then we obtain $\sum_k a_k \leq 1$ and $a_k > 0$ for all k from (1.1) and (1.2). Let $\Lambda \subset \mathbb{N}$ be a non-empty set and define, for each $s \geq 0$,

$$\mu_\Lambda(s) = \sum_{k \in \Lambda} a_k^s .$$

Evidently $\mu_\Lambda(s)$ is *non-increasing* in s. It also has the following properties[5]. Denote

$$\theta_\Lambda = \inf\{s > 0 : \mu_\Lambda(s) < \infty\}$$

and

$$F(\Lambda) = \begin{cases} (\theta_\Lambda\,,\ \infty), & \textit{if}\ \ \mu_\Lambda(\theta_\Lambda) = \infty\,, \\ [\theta_\Lambda\,,\ \infty), & \textit{if}\ \ \mu_\Lambda(\theta_\Lambda) < \infty\,. \end{cases}$$

Then μ_Λ is strictly decreasing, convex and continuous on $F(\Lambda)$. In particular, if Λ is finite then $\mu_\Lambda(s)$ is convex, strictly decreasing and continuous in s.

Recall that the Hausdorff dimension of the limit set is given by

$$\mathrm{HD}(J_\Lambda) = \inf\{s > 0 \ :\ \mu_\Lambda(s) \leq 1\,\}. \tag{2.1}$$

Since $\sum_k a_k \leq 1$ we have $\mu_\Lambda(1) \leq 1$, hence $\mathrm{HD}(J_\Lambda) \leq 1$ for any $\Lambda \subseteq \mathbb{N}$.

Proposition 2.1 *Let* Λ, $\Lambda' \subseteq \mathbb{N}$.

(i) *Suppose* $\mathrm{HD}(J_\Lambda) > 0$ *and there exists an* $s_0 > 0$ *such that* $\mu_\Lambda(s) \le \mu_{\Lambda'}(s)$ *for all* $s_0 < s < \mathrm{HD}(J_\Lambda)$. *Then* $\mathrm{HD}(J_\Lambda) \le \mathrm{HD}(J_{\Lambda'})$.

(ii) *Suppose there exists an* $s_0 > \mathrm{HD}(J_{\Lambda'})$ *such that* $\mu_\Lambda(s) \le \mu_{\Lambda'}(s)$ *for all* $\mathrm{HD}(J_{\Lambda'}) < s < s_0$. *Then* $\mathrm{HD}(J_\Lambda) \le \mathrm{HD}(J_{\Lambda'})$.

PROOF. (i) Choose an s such that $s_0 < s < \mathrm{HD}(J_\Lambda)$, then $\mu_\Lambda(s) > 1$ by (2.1). Thus the assumption implies $\mu_{\Lambda'}(s) > 1$, which proves $\mathrm{HD}(J_{\Lambda'}) \ge s$ by (2.1) again. So we have $\mathrm{HD}(J_\Lambda) \le \mathrm{HD}(J_{\Lambda'})$. Similarly we may prove (ii). □

Proposition 2.2 *If* $\Lambda \subseteq \Lambda'$ *then* $\mathrm{HD}(J_\Lambda) \le \mathrm{HD}(J_{\Lambda'})$. *Moreover if* Λ *is finite,* $\Lambda \ne \Lambda'$ *then* $\mathrm{HD}(J_\Lambda) < \mathrm{HD}(J_{\Lambda'})$.

PROOF. Let $\Lambda \subseteq \Lambda' \subseteq \mathbb{N}$ then $\mu_\Lambda \le \mu_{\Lambda'}$, thus the assertion is proved by Proposition 2.1. Next we suppose Λ is finite. Since $\mathrm{HD}(J_\Lambda)$ is the unique number s satisfying $\mu_\Lambda(s) = 1$ and since $\Lambda \ne \Lambda'$, we have $\mathrm{HD}(J_\Lambda) < \mathrm{HD}(J_{\Lambda'})$. □

Proposition 2.3 *Suppose that* $\{\Lambda_n\}$ *is a non-decreasing sequence of sets of positive integers. Then we have* $\mathrm{HD}(J_{\cup_n \Lambda_n}) = \sup_n \mathrm{HD}(J_{\Lambda_n})$. *As a result,* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N} \text{ finite}\}$ *is dense in* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N}\}$.

PROOF. Since it holds $\mu_{\cup_n \Lambda_n} = \sup_n \mu_{\Lambda_n}$, the assertion is concluded by (2.1). □

Proposition 2.4 $\{\mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N} \text{ finite}\}$ *has no isolated point.*

PROOF. Let $\Lambda \subset \mathbb{N}$ be a finite set and set $\Gamma_n = \Lambda \cup \{\max \Lambda + n\}$. Then we have $\mu_\Lambda(s) = \inf_n \mu_{\Gamma_n}(s)$, hence the assertion is observed by (2.1). □

Denote for each $m \in \mathbb{N}$

$$\Lambda_m = \{1, 2, \ldots, m\} \quad \textit{and} \quad \Omega_m = \mathbb{N} \backslash \{m\}.$$

For convenience sake we put $\Lambda_0 = \emptyset$, $\Omega_0 = \mathbb{N}$ and $\mu_\emptyset = 0$. Then the following follows from Proposition 2.2.

Lemma 2.5 *We have* $\mathrm{HD}(J_{\Lambda_0}) < \mathrm{HD}(J_{\Lambda_1}) < \mathrm{HD}(J_{\Lambda_2}) < \mathrm{HD}(J_{\Lambda_3}) < \cdots$.

Lemma 2.6 *Let* $\Gamma \subseteq \mathbb{N}$ *and* $k \in \Gamma$, $l \in \mathbb{N}$. *If* $a_k \ge a_l$ *then we have* $\mathrm{HD}(J_{\Gamma \cap \Omega_k}) \le \mathrm{HD}(J_{\Gamma \cap \Omega_l})$.

PROOF. If $a_k \ge a_l$, $k \in \Gamma$, $l \in \mathbb{N}$ then we have $\mu_{\Gamma \cap \Omega_k}(s) \le \mu_{\Gamma \cap \Omega_l}(s)$ for any $0 \le s \le 1$. Hence the assertion follows from Proposition 2.1. □

Proposition 2.7 *Suppose there exists an* $0 < s < \mathrm{HD}(J_\mathbb{N})$ *such that*

$$\sum_{k > m} a_k^s \ge a_m^s \quad \textit{for any} \quad m \in \mathbb{N}.$$

Then there exists a non-decreasing sequence of finite sets of positive integers $\{\Gamma_n\}$ *such that*

$$\sup_n \mathrm{HD}(J_{\Gamma_n}) = s.$$

PROOF. First we shall construct a non-decreasing sequence of finite sets of positive integers $\{\Gamma_n\}$ and a strictly increasing sequence $\{m_n\}$ such that

($\star$) $\quad \mu_{\Gamma_n}(s) \le 1 < \mu_{\Gamma_n \cup \{m_n\}}(s)$ *for all* n.

Since $s < \mathrm{HD}(J_\mathbb{N})$ we have $\mu_\mathbb{N}(s) > 1$ by (2.1), hence

$$m_1 = \min\{\, m \ge 2 \,:\, \mu_{\{1,2,\ldots,m\}}(s) > 1\}$$

is well defined. Set $\Gamma_1 = \{1, 2, \ldots, m_1 - 1\}$. Suppose Γ_n and m_n are given, then by hypothesis

$$\mu_{\Gamma_n \cup \{m_n+1, m_n+2, \ldots\}}(s) = \mu_{\Gamma_n}(s) + \sum_{k>m_n} a_k^s \geq \mu_{\Gamma_n}(s) + a_{m_n}^s = \mu_{\Gamma_n \cup \{m_n\}}(s) > 1.$$

Thus the following is well defined:

$$m_{n+1} = \min\{\, m \geq m_n + 1 \;:\; \mu_{\Gamma_n \cup \{m_n+1, m_n+2, \ldots, m\}}(s) > 1 \,\}\,.$$

We denote

$$\Gamma_{n+1} = \begin{cases} \Gamma_n\,, & \textit{if } m_{n+1} = m_n + 1\,, \\ \Gamma_n \cup \{m_n + 1, m_n + 2, \ldots, m_{n+1} - 1\}\,, & \textit{if } m_{n+1} > m_n + 1\,. \end{cases}$$

Then $\{\Gamma_n\}$ and $\{m_n\}$ satisfy $(\star)$. Observing $0 \leq 1 - \mu_{\Gamma_n}(s) \leq a_{m_n}^s \to 0$, we have $\sup_n \mu_{\Gamma_n}(s) = 1$.

Next we shall prove $\sup_n \mathrm{HD}(J_{\Gamma_n}) = s$. Since $\mu_{\Gamma_n}(s) \leq 1$ for any n, $\sup_n \mathrm{HD}(J_{\Gamma_n}) \leq s$ is trivial by (2.1). Assume $\sup_n \mathrm{HD}(J_{\Gamma_n}) < s$ and set $\Gamma = \cup_n \Gamma_n$, then $\mathrm{HD}(J_\Gamma) = \sup_n \mathrm{HD}(J_{\Gamma_n})$ by Proposition 2.3. Since μ_Γ is strictly decreasing on $F(\Gamma)$ and since $\mathrm{HD}(J_\Gamma) < s$, we have $\mu_\Gamma(s) < 1$. On the other hand, we have $\mu_{\Gamma_n}(s) < \mu_\Gamma(s) < 1$ for all n, which contradicts to the fact $\sup_n \mu_{\Gamma_n}(s) = 1$. Thus $\sup_n \mathrm{HD}(J_{\Gamma_n}) = s$. □

Proof of Theorem A. Suppose $\sum_{k>m} |\phi_k'|^{s_0} \geq |\phi_m'|^{s_0}$ for any $m \in \mathbb{N}$. Then we have $\sum_{k>m} a_k^s \geq a_m^s$ for any $0 < s \leq s_0$ and for any $m \in \mathbb{N}$, thus the restricted dimension set is dense in $[0, s_0]$ by Proposition 2.7. The second assertion follows from Proposition 2.3.

3 Dimension Set of Absolutely Regular System

In this section we assume $S = \{\phi_k : k \in \mathbb{N}\}$ is absolutely regular. Thus for any $\Lambda \subseteq \mathbb{N}$, we have

$$\mathrm{HD}(J_\Lambda) = s \quad \textit{if and only if} \quad \mu_\Lambda(s) = 1$$

Then, for example, it has the following property.

Lemma 3.1 *Let $\Lambda \subseteq \mathbb{N}$, $\alpha > 0$ and assume $\mu_\Lambda(\alpha) < 1$ then $\mathrm{HD}(J_\Lambda) < \alpha$.*

Proposition 3.2 *Let $\Gamma \subseteq \mathbb{N}$. Suppose there exists an $l \in \Gamma$ such that $a_l = \min\{a_k : k \in \Gamma \cap \Lambda_l\}$ and*

$$\sum_{k>l,\, k\in\Gamma} a_k^{\mathrm{HD}(J_{\Gamma\cap\Lambda_l})} < a_l^{\mathrm{HD}(J_{\Gamma\cap\Lambda_l})}\,.$$

Then $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \Gamma\} \cap (\,\mathrm{HD}(J_{\Gamma\cap\Omega_l}),\, \mathrm{HD}(J_{\Gamma\cap\Lambda_l})\,) = \emptyset$.

Proof. First we shall prove $\mathrm{HD}(J_{\Gamma\cap\Omega_l}) < \mathrm{HD}(J_{\Gamma\cap\Lambda_l})$. By the assumption we have

$$\begin{aligned} \mu_{\Gamma\cap\Omega_l}(\mathrm{HD}(J_{\Gamma\cap\Lambda_l})) &= \mu_{\Lambda_{l-1}\cap\Gamma}(\mathrm{HD}(J_{\Gamma\cap\Lambda_l})) + \sum_{k>l, k\in\Gamma} a_k^{\mathrm{HD}(J_{\Gamma\cap\Lambda_l})} \\ &< \mu_{\Lambda_{l-1}\cap\Gamma}(\mathrm{HD}(J_{\Gamma\cap\Lambda_l})) + a_l^{\mathrm{HD}(J_{\Gamma\cap\Lambda_l})} = \mu_{\Gamma\cap\Lambda_l}(\mathrm{HD}(J_{\Gamma\cap\Lambda_l})) = 1. \end{aligned}$$

So Lemma 3.1 implies $\mathrm{HD}(J_{\Gamma\cap\Omega_l}) < \mathrm{HD}(J_{\Gamma\cap\Lambda_l})$.

Next we shall prove $\mathrm{HD}(J_\Lambda) \notin (\mathrm{HD}(J_{\Gamma\cap\Omega_l}),\, \mathrm{HD}(J_{\Gamma\cap\Lambda_l}))$ for any $\Lambda \subseteq \Gamma$.

Let $\Lambda \subseteq \Gamma$. If $[\Gamma \cap \Lambda_l] \subseteq \Lambda$ then $\mathrm{HD}(J_\Lambda) \geq \mathrm{HD}(J_{\Gamma\cap\Lambda_l})$, hence $\mathrm{HD}(J_\Lambda) \notin (\mathrm{HD}(J_{\Gamma\cap\Omega_l}),\, \mathrm{HD}(J_{\Gamma\cap\Lambda_l}))$. Assume $[\Gamma \cap \Lambda_l] \not\subseteq \Lambda$, then there exists $j \in \Gamma \cap \Lambda_l$ such that $\Lambda \subseteq [\Gamma \cap \Omega_j]$. Then we have $\mathrm{HD}(J_\Lambda) \leq \mathrm{HD}(J_{\Gamma\cap\Omega_j})$. On the other hand the hypothesis and Lemma 2.6 implies $\mathrm{HD}(J_{\Gamma\cap\Omega_j}) \leq \mathrm{HD}(J_{\Gamma\cap\Omega_l})$, so

$\mathrm{HD}(J_\Lambda) \notin (\,\mathrm{HD}(J_{\Gamma\cap\Omega_l}),\ \mathrm{HD}(J_{\Gamma\cap\Lambda_l})\,)$, which complete the proof. □

Proof of Theorem B.

(i) Choose $\Gamma = \mathbb{N}$ and $l = m$ in Proposition 3.2 then we have the conclusion.

(ii) Similarly choose $\Gamma = \mathbb{N}\backslash\{m\}$ and $l = m-1$ in Proposition 3.2 then we obtain the assertion.

Proposition 3.3 *Suppose there exists a finite set $\Lambda_0 \subseteq \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that $m_0 \le \max\Lambda_0$ and*

$$\sum_{k>m} a_k^{\mathrm{HD}(J_{\Lambda_0})} < a_m^{\mathrm{HD}(J_{\Lambda_0})} \quad \text{for any } m \in \Lambda_0\,, \text{ and for any } m \ge m_0.$$

Then there exits two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\beta_{n+1} < \alpha_n < \beta_n$, $\inf_n \alpha_n = \mathrm{HD}(J_{\Lambda_0})$ and

$$\{\,\mathrm{HD}(J_\Lambda) :\ \Lambda \subseteq \Lambda_0 \ \ or\ \ \min[\Lambda\backslash\Lambda_0] \ge m_0\,\} \cap (\alpha_n,\, \beta_n) = \emptyset.$$

for any n.

Proof. Set $n_0 = \max\Lambda_0$ and for each $n = 1, 2, \ldots$

$$\Gamma_n = \Lambda_0 \cup \{n_0+n\},\ \Gamma'_n = \Lambda_0 \cup \{n_0+n+1,\, n_0+n+2,\, \ldots\},\ \beta_n = \mathrm{HD}(J_{\Gamma_n}),\ \alpha_n = \mathrm{HD}(J_{\Gamma'_n}).$$

First we shall prove $\beta_{n+1} < \alpha_n < \beta_n$ and $\inf_n \alpha_n = \mathrm{HD}(J_{\Lambda_0})$. Since $\Gamma_{n+1} \subset \Gamma'_n$, $\beta_{n+1} < \alpha_n$ is trivial. Since $\mathrm{HD}(J_{\Gamma_n}) > \mathrm{HD}(J_{\Gamma_0})$, the hypothesis implies

$$\begin{aligned}\mu_{\Gamma'_n}(\mathrm{HD}(J_{\Gamma_n})) &= \mu_{\Lambda_0}(\mathrm{HD}(J_{\Gamma_n})) + \sum_{k>n_0+n} a_k^{\mathrm{HD}(J_{\Gamma_n})}\\ &< \mu_{\Lambda_0}(\mathrm{HD}(J_{\Gamma_n})) + a_{n_0+n}^{\mathrm{HD}(J_{\Gamma_n})} = \mu_{\Gamma_n}(\mathrm{HD}(J_{\Gamma_n})) = 1,\end{aligned}$$

thus $\alpha_n = \mathrm{HD}(J_{\Gamma'_n}) < \mathrm{HD}(J_{\Gamma_n}) = \beta_n$ by Lemma 3.1. We also have $\inf_n \alpha_n = \mathrm{HD}(J_{\Lambda_0})$ since $\inf_n \mu_{\Gamma_n} = \mu_{\Gamma_0}$.

Now we shall prove the last assertion. If $\Lambda \subseteq \Lambda_0$ then we have $\mathrm{HD}(J_\Lambda) \notin (\alpha_n,\, \beta_n)$ for any n, since $\alpha_n > \mathrm{HD}(J_{\Lambda_0})$. So we need to prove $\mathrm{HD}(J_\Lambda) \notin (\alpha_n,\, \beta_n)$ for any n and for any $\Lambda \subseteq \mathbb{N}$ with $\min[\Lambda\backslash\Lambda_0] \ge m_0$. Fix any n and fix such Λ.

If $\Gamma_n \subseteq \Lambda$ then we observe $\mathrm{HD}(J_\Lambda) \notin (\alpha_n,\, \beta_n)$. Otherwise put $j = \min\{k \in \Gamma_n \cup \Lambda : k \notin \Gamma_n \ \mathit{or}\ k \notin \Lambda\}$, then we have $\Gamma_n \cap \Lambda_{j-1} = \Lambda \cap \Lambda_{j-1}$. Since $\max\Gamma_n = n_0+n$ and $\Gamma_n \not\subseteq \Lambda$, we have $j \le n_0+n$. If $j = n_0+n$ then since $n_0+n \notin \Lambda$, $\mathrm{HD}(J_{\Gamma'_n}) > \mathrm{HD}(J_\Lambda)$ and by the hypothesis we have

$$\begin{aligned}\mu_\Lambda(\mathrm{HD}(J_{\Gamma'_n})) &= \mu_{\Lambda\cap\Lambda_{n_0+n-1}}(\mathrm{HD}(J_{\Gamma'_n})) + \mu_{\Lambda\cap\{n_0+n+1,n_0+n+2,\ldots\}}(\mathrm{HD}(J_{\Gamma'_n}))\\ &< \mu_{\Gamma_n\cap\Lambda_{n_0+n-1}}(\mathrm{HD}(J_{\Gamma'_n})) + \sum_{k>n_0+n} a_k^{\mathrm{HD}(J_{\Gamma'_n})}\\ &= \mu_{\Lambda_0}(\mathrm{HD}(J_{\Gamma'_n})) + \sum_{k>n_0+n} a_k^{\mathrm{HD}(J_{\Gamma'_n})} = \mu_{\Gamma'_n}(\mathrm{HD}(J_{\Gamma'_n})) = 1,\end{aligned}$$

hence $\mathrm{HD}(J_\Lambda) < \mathrm{HD}(J_{\Gamma'_n}) = \alpha_n$. If $j < n_0+n$ and $j \notin \Lambda$ then $j \in \Lambda_0$ and similarly

$$\begin{aligned}\mu_\Lambda(\mathrm{HD}(J_{\Lambda_0})) &= \mu_{\Lambda\cap\Lambda_{j-1}}(\mathrm{HD}(J_{\Lambda_0})) + \mu_{\Lambda\cap\{j+1,j+2,\ldots\}}(\mathrm{HD}(J_{\Lambda_0}))\\ &< \mu_{\Lambda_0\cap\Lambda_{j-1}}(\mathrm{HD}(J_{\Lambda_0})) + a_j^{\mathrm{HD}(J_{\Lambda_0})} \le \mu_{\Lambda_0}(\mathrm{HD}(J_{\Lambda_0})) = 1.\end{aligned}$$

hence $\mathrm{HD}(J_\Lambda) < \mathrm{HD}(J_{\Lambda_0}) < \alpha_n$. Finally assume $j < n_0+n$ and $j \in \Lambda$ then $j \notin \Lambda_0$, $j \ge m_0$ and

$$\begin{aligned}\mu_{\Gamma_n}(\mathrm{HD}(J_\Lambda)) &= \mu_{\Gamma_n\cap\Lambda_{j-1}}(\mathrm{HD}(J_\Lambda)) + \mu_{\Gamma_n\cap\{j+1,j+2,\ldots\}}(\mathrm{HD}(J_\Lambda))\\ &< \mu_{\Lambda\cap\Lambda_{j-1}}(\mathrm{HD}(J_\Lambda)) + a_j^{\mathrm{HD}(J_\Lambda)} \le \mu_\Lambda(\mathrm{HD}(J_\Lambda)) = 1,\end{aligned}$$

hence $\mathrm{HD}(J_\Lambda) > \mathrm{HD}(J_{\Gamma_n}) = \beta_n$, which complete the proof. $\square$

Proof of Theorem C. Assume there exists a $\Lambda \subseteq \mathbb{N}$ such that $\mathrm{HD}(J_\Lambda) \in (s_0, \mathrm{HD}(J_\mathbb{N}))$. By Proposition 2.3 we observe $\mathrm{HD}(J_{\Lambda_0}) \in (s_0, \mathrm{HD}(J_\mathbb{N}))$ for some finite set $\Lambda_0 \subseteq \Lambda$. Thus the assumption implies

$$\sum_{k>m} a_k^{\mathrm{HD}(J_{\Lambda_0})} < a_m^{\mathrm{HD}(J_{\Lambda_0})} \quad \text{for any } m \in \mathbb{N}.$$

According to Proposition 3.3 there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\beta_{n+1} < \alpha_n < \beta_n$, $\inf_n \alpha_n = \mathrm{HD}(J_{\Lambda_0})$ and

$$\{\,\mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N}\,\} \cap (\alpha_n, \beta_n) = \emptyset.$$

for any n, hence the dimension set $\{\,\mathrm{HD}(J_\Lambda) : \Lambda \subset \mathbb{N}\,\}$ is nowhere dense in $[s_0, \mathrm{HD}(J_\mathbb{N})]$. $\square$

4 Example

Consider a collection of affine transformations $S = \{\phi_k(x) = a_k x + b_k : k \in \mathbb{N}\}$ on the unit interval as follows : let $r > 1$ and set

$$\begin{cases} a_k = (r-1)\, r^{-k} & \textit{for} \quad k = 1, 2, \ldots \quad \textit{and} \\ b_1 = 0, \quad b_k = a_1 + a_2 + \cdots + a_{k-1} & \textit{for} \quad k = 2, 3, \ldots. \end{cases}$$

Then for any $0 < s < 1$ we have

$$\sum_{k \geq 1} a_k^s = (r-1)^s \sum_{k \geq 1} r^{-sk} = (r-1)^s \frac{1}{r^s - 1},$$

hence S is *absolutely regular* and $\mathrm{HD}(J_\mathbb{N}) = 1$ since $\sum_k a_k = 1$. Similarly we have for any $0 < s < 1$

$$\sum_{k>m} a_k^s = (r-1)^s \frac{r^{-sm}}{r^s - 1} = \frac{1}{r^s - 1} a_m^s.$$

<u>**Claim 1.**</u> *If* $1 < r \leq 2$ *then* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*$\}$ *is dense in* $[0, 1]$. .

proof. If $1 < r \leq 2$ then we have $\sum_{k>m} a_k = (r-1)\frac{r^{-m}}{r-1} \geq a_m$ for any m. Thus the assumptions of Theorem A are satisfied for $s_0 = 1$, hence $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*$\}$ is dense in $[0, 1]$.

<u>**Claim 2.**</u> *Suppose* $r > 2$ *then* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*$\}$ *is dense in* $[0, \log_r 2]$, *and nowhere dense in* $[\log_r 2, 1]$.

proof. Assume $r > 2$. Since $\sum_{k>m} a_k^{\log_r 2} = a_m^{\log_r 2}$ for any $m \in \mathbb{N}$, applying Theorem A we conclude that $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*$\}$ is dense in $[0, \log_r 2]$.

On the other hand, we have $\sum_{k>m} a_k^s < a_m^s$ for any $m \in \mathbb{N}$ and any $s \in (\log_r 2, 1)$, hence $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*$\}$ is nowhere dense in $[\log_r 2, 1]$ by Theorem C.

<u>**Claim 3.**</u> *If* $1 < r \leq \frac{1+\sqrt{5}}{2}$ *then* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*, $m \notin \Lambda\}$ *is still dense in* $[0, \mathrm{HD}(J_{\mathbb{N}\setminus\{m\}})]$ *for any* $m \in \mathbb{N}$.

proof. Let $1 < r \leq \frac{1+\sqrt{5}}{2}$ then it satisfies $1 + r - r^2 \geq 0$. Fix any $m \in \mathbb{N}$ and we shall prove $\sum_{k>l, k \neq m} a_k \geq a_l$ for any $l \neq m$, then the claim follows from Theorem A. If $l > m$ then the inequality follows from the similar arguments to claim 1. If $l = m-1$ then we need to show $\sum_{k>m} a_k \geq a_{m-1}$. In fact, we have

$$\sum_{k>m} a_k = (r-1)\frac{r^{-m}}{r-1} = (r-1) r^{-(m-1)} \frac{r^{-1}}{r-1} \geq (r-1) r^{-(m-1)} = a_{m-1}.$$

Now we suppose $l < m-1$. Then we need to show $\sum_{k>m} a_k + \sum_{k=l+1}^{m-1} a_k \geq a_l$. Actually we have

$$\sum_{k>m} a_k + \sum_{k=l+1}^{m-1} a_k = (r-1)\frac{r^{-m}}{r-1} + (r-1)\sum_{k=l+1}^{m-1} r^{-k} \geq (r-1)r^{-(l+2)} + (r-1)r^{-(l+1)} \geq a_l,$$

hence $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite*, $m \notin \Lambda\}$ is dense in $[0, 1]$.

__Claim 4.__ *Suppose* $\frac{1+\sqrt{5}}{2} < r \leq 2$. *Then, for any* $m \in \mathbb{N}$, $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite,* $m \notin \Lambda\}$ *is dense in* $[0, \log_r \frac{1+\sqrt{5}}{2}]$. *On the other hand, for large enough* $m \in \mathbb{N}$, *there exists an interval* $I \subseteq [\log_r \frac{1+\sqrt{5}}{2}, \mathrm{HD}(J_{\mathbb{N}\setminus\{m\}})]$ *such that* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}, m \notin \Lambda\} \cap I = \emptyset$

proof. Suppose $\frac{1+\sqrt{5}}{2} < r \leq 2$ and put $s_0 = \log_r \frac{1+\sqrt{5}}{2}$. Then we have $1+r^{s_0}-r^{2s_0} = 0$ and $1+r^s-r^{2s} > 0$ if and only if $0 < s < s_0$.

By the similar arguments to claim 3, we may prove that $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite,* $m \notin \Lambda\}$ is dense in $[0, s_0]$ for any $m \in \mathbb{N}$.

Next, choose m_0 so that $\mathrm{HD}(J_{\Lambda_m}) > s_0$ for any $m \geq m_0$ and fix such m. Then we have

$$\sum_{k>m+1} a_k^{\mathrm{HD}(J_{\Lambda_m})} < a_m^{\mathrm{HD}(J_{\Lambda_m})},$$

hence the second assertion follows from Theorem B(ii).

By a similar discussion, we conclude the following :

__Claim 5.__ *Let* n *be a positive integer and* r_n *a real number such that* $r_n^n(r_n-1) = 1$, $1 < r_n < 2$.

If $1 < r \leq r_n$ *then* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite,* $m_1, \ldots, m_n \notin \Lambda\}$ *is dense in* $[0, \mathrm{HD}(J_{\mathbb{N}\setminus\{m_1,\ldots,m_n\}})]$ *for any* $m_1, \ldots, m_n \in \mathbb{N}$.

Suppose $r_n < r \leq 2$. *Then, for any* $m_1 < \cdots < m_n \in \mathbb{N}$, $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}$ *finite,* $m_1, \ldots, m_n \notin \Lambda\}$ *is dense in* $[0, \log_r r_n]$. *On the other hand, for large enough* $m \in \mathbb{N}$, *there exists an interval* $I \subseteq [\log_r r_n, \mathrm{HD}(J_{\mathbb{N}\setminus\{m,\ldots,m+n-1\}})]$ *such that* $\{\mathrm{HD}(J_\Lambda) : \Lambda \subseteq \mathbb{N}, m, \ldots, m+n-1 \notin \Lambda\} \cap I = \emptyset$.

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