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Rate of convergence to invariant density function for distribution of iterated beta transformation and linear mod 1 transformation

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ベータ変換 および linear mod 1 変換の繰り返しにおける分布密度関数の不変密度関数への収束の速さ

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Abstract

The β-transformation T_β and the linear mod 1 transformation $T_{\beta,\alpha}$ are transformations on [0,1) defined by $T_\beta(t) = \beta t - \lfloor \beta t \rfloor$ and $T_{\beta,\alpha}(t) = \beta t + \alpha - \lfloor \beta t + \alpha \rfloor$ $(\beta > 1, 0 < \alpha < 1)$. We consider how fast the distribution of $T_\beta^k([0,1))$ and $T_{\beta,\alpha}^k([0,1))$ approaches to its invariant distribution, and give explicit rate of convergence to invariant density function using β or β and α. The proof is proceeded by counting the number of same kind of lines which appear in the graph of $T_\beta^k([0,1))$ or $T_{\beta,\alpha}^k([0,1))$. The base of proof is to show that the ratio of two numbers counted above (or ratio of two numbers obtained from the numbers counted above) approaches to β^{-j} as $k \to \infty$. In the appendix we give numerical evaluations of approximate density functions as an application of our theorems.

Key words: beta transformation, linear mod 1 transformation, invariant density function, convergence of distribution, rate of convergence, ergodic theory.

§1. Definitions and results

The β-transformation T_β and the linear mod 1 transformation $T_{\beta,\alpha}$ are transformations on [0,1) defined by $T_\beta(t) \equiv \beta t - \lfloor \beta t \rfloor$, $\beta > 1$, and $T_{\beta,\alpha}(t) \equiv \beta t + \alpha - \lfloor \beta t + \alpha \rfloor$, $0 < \alpha < 1$, where $\lfloor x \rfloor$ is the largest integer not exceeding x. These transformations frequently appear in the ergodic theory. The T_β and $T_{\beta,\alpha}$ are also written such as $T_\beta(t) = \beta t$ mod 1 and $T_{\beta,\alpha}(t) = \beta t + \alpha$ mod 1. We assume $\beta \neq \lfloor \beta \rfloor$ and $\beta + \alpha \neq \lfloor \beta + \alpha \rfloor$ for T_β and $T_{\beta,\alpha}$ respectively. In the graph of $T_\beta([0,1))$ there appear two kinds of lines; one is the line which extends from 0 to 1 and another is the line which extends from 0 to $T_\beta(1)$. We call the former *full_Branch* (*f_B* in short) which is denoted by **F** and the latter *low_Branch* (*l_B*)

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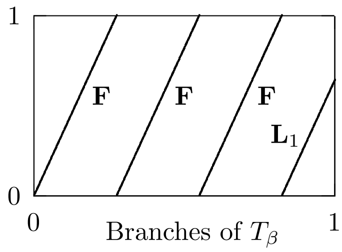
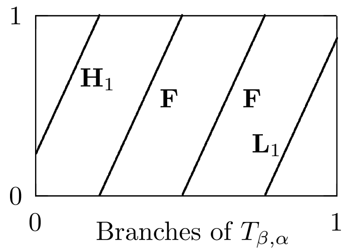
which is denoted by $\mathbf{L}_1$. The subscript 1 in $\mathbf{L}_1$ stands for 1 in $T_\beta \equiv T_\beta^1$. In the graph of $T_{\beta,\alpha}([0,1))$ there appears one more kind of line called *high_Branch* (*h_B*) which extends from $T_{\beta,\alpha}(0)$ to 1 and is denoted by $\mathbf{H}_1$.



Branches of T_β



Branches of $T_{\beta,\alpha}$

In the graph of $T_\beta^k([0,1))$ there are many $\mathbf{F}$s and $\mathbf{L}_i$'s, $i = 1, 2, \cdots, k$. Here $\mathbf{L}_i$ is the line which extends from 0 to $T_\beta^i(1)$. The number of lines $\mathbf{F}$s and $\mathbf{L}_i$'s in the graph of $T_\beta^k([0,1))$ are denoted by $\#\mathbf{F}$ and $\#\mathbf{L}_i$ respectively. An invariant density function h_β of T_β is given by Parry [2] such as

$$h_\beta(x) = \mathbf{1}_{[0,1)}(x) + \tfrac{1}{\beta}\mathbf{1}_{[0,T_\beta(1))}(x) + \tfrac{1}{\beta^2}\mathbf{1}_{[0,T_\beta^2(1))}(x) + \cdots + \tfrac{1}{\beta^k}\mathbf{1}_{[0,T_\beta^k(1))}(x) + \cdots$$

allowing multiplication by a constant. ($\mathbf{1}_{[a,b)}(x) = 1$ if $x \in [a,b)$, $= 0$ otherwise.)

Analogously in the graph of $T_{\beta,\alpha}^k([0,1))$ we can define $\mathbf{F}$, $\mathbf{L}_i$, $\mathbf{H}_i$ and $\#\mathbf{F}$, $\#\mathbf{L}_i$, $\#\mathbf{H}_i$ respectively. Of course $\mathbf{H}_i$ is the line which extends from $T_{\beta,\alpha}^i(0)$ to 1. An invariant density function $h_{\beta,\alpha}$ of $T_{\beta,\alpha}$ is also given by Parry [3]:

$$h_{\beta,\alpha}(x) = \mathbf{1}_{[0,1)}(x) + \textstyle\sum_{i=1}^\infty \frac{1}{\beta^i}\left(\mathbf{1}_{[0,T_{\beta,\alpha}^i(1))}(x) - \mathbf{1}_{[0,T_{\beta,\alpha}^i(0))}(x)\right)$$

allowing multiplication by a constant.

Then we have the following two theorems:

Theorem A(T_β).

Assume $\lfloor\beta\rfloor \geq 3$.

(1) For $\mathbf{F}$ and $\mathbf{L}_1$, $\mathbf{L}_2$, $\cdots$, $\mathbf{L}_k$ appearing in $T_\beta^k([0,1))$, we have

$$\left|\left(\frac{\#\mathbf{L}_j}{\#\mathbf{F}}\right) - \frac{1}{\beta^j}\right| \leq C_j\left(\frac{2\lfloor\beta\rfloor - 1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor - 1)}\right)^{k-1}, \quad j = 1, 2, \cdots, k, \quad \text{where}$$

$$C_j = \frac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor - 1)}\frac{1 - A^j}{1 - A}\left(\frac{\lfloor\beta\rfloor - 1}{2\lfloor\beta\rfloor - 1}\right)^{j-1} \quad \text{and} \quad A = \frac{2\lfloor\beta\rfloor - 1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor - 1)}.$$

In particular, $\lim_{k\to\infty}\left(\frac{\#\mathbf{L}_j}{\#\mathbf{F}}\right) = \frac{1}{\beta^j}$.

(2) For a density function f_k of $T_\beta^k([0,1))$ given by

$$f_k(x) = \mathbf{1}_{[0,1)}(x) + \left(\frac{\#\mathbf{L}_1}{\#\mathbf{F}}\right)\mathbf{1}_{[0,T_\beta(1))}(x) + \left(\frac{\#\mathbf{L}_2}{\#\mathbf{F}}\right)\mathbf{1}_{[0,T_\beta^2(1))}(x) + \cdots + \left(\frac{\#\mathbf{L}_k}{\#\mathbf{F}}\right)\mathbf{1}_{[0,T_\beta^k(1))}(x),$$

we have

$$\|f_k - h_\beta\| \equiv \sup_{0\le x<1} |f_k(x) - h_\beta(x)| < C\left(\frac{2\lfloor\beta\rfloor - 1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\right)^{k-1} \quad \text{where}$$

$$C = \frac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\frac{1}{1-A}\frac{1}{1-B} + \frac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)^2} \quad \text{and} \quad B = \frac{\lfloor\beta\rfloor-1}{2\lfloor\beta\rfloor-1}.$$

In particular, $\lim_{k\to\infty}\|f_k - h_\beta\| = 0$.

Theorem B ($T_{\beta,\alpha}$).

Let $F_0 = \lfloor\beta+\alpha\rfloor - 1$, and assume $F_0 \ge 5$.

(0) $\#\mathbf{L}_i = \#\mathbf{H}_i$, $i = 1, 2, \cdots, k$, in the graph of $T^k_{\beta,\alpha}([0,1))$.

(1) For $\mathbf{F}$ and $\mathbf{L}_1$, $\mathbf{L}_2$, $\cdots$, $\mathbf{L}_k$ appearing in $T^k_{\beta,\alpha}([0,1))$, we have

$$\left|\left(\frac{\#\mathbf{L}_j}{\#\mathbf{F} + \sum_{i=1}^k \#\mathbf{L}_i}\right) - \frac{1}{\beta^j}\right| \le \tilde{C}_j\left(\frac{2F_0-1}{(F_0-1)(F_0-2)}\right)^{k-1} \quad \text{where}$$

$$\tilde{C}_j = \frac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\left(\frac{F_0-2}{2F_0-1}\right)^{j-1}\frac{1-\tilde{A}^j}{1-\tilde{A}} \quad \text{and} \quad \tilde{A} = \frac{2F_0-1}{(F_0-1)(F_0-2)}.$$

In particular, $\lim_{k\to\infty}\left(\frac{\#\mathbf{L}_j}{\#\mathbf{F}+\sum_{i=1}^k \#\mathbf{L}_i}\right) = \frac{1}{\beta^j}$.

(2) For a density function $\tilde{f}_k$ of $T^k_{\beta,\alpha}([0,1))$ given by

$$\tilde{f}_k(x) = \mathbf{1}_{[0,1)}(x) + \sum_{i=1}^k\left(\frac{\#\mathbf{L}_i}{\#\mathbf{F}+\sum_{j=1}^k \#\mathbf{L}_j}\right)\left(\mathbf{1}_{[0,T^i_{\beta,\alpha}(1))}(x) - \mathbf{1}_{[0,T^i_{\beta,\alpha}(0))}(x)\right)$$

we have

$$\|\tilde{f}_k - h_{\beta,\alpha}\| < \tilde{C}\left(\frac{2F_0-1}{(F_0-1)(F_0-2)}\right)^{k-1} \quad \text{where}$$

$$\tilde{C} = \left(\frac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\right)\frac{1}{1-\tilde{A}}\frac{1}{1-\tilde{B}} + \frac{1}{(F_0-1)(F_0-2)} \quad \text{and} \quad \tilde{B} = \frac{F_0-2}{2F_0-1}.$$

In particular, $\lim_{k\to\infty}\|\tilde{f}_k - h_{\beta,\alpha}\| = 0$.

Numerical evaluations of $\|f_k$ - $h_\beta\|$ and $\|\tilde{f}_k$ - $h_{\beta,\alpha}\|$ will be given in the appendix.

§2. Convergence of the distribution of $T^i_\beta([0,1))$

In the following we will omit proofs which are elementary and/or straightforward in order to make this paper short.

2.1 The rule of generating f_B $\mathbf{F}$ and l_B $\mathbf{L}_i$ in the β-transformation is as follows:

(i) If we apply the next T_β to $\mathbf{F}$ (i.e., in $T_\beta(\mathbf{F})$, intuitively), we have $\lfloor\beta\rfloor$ $\mathbf{F}$s and one $\mathbf{L}_1$.

(ii) If we apply the next T_β to $\mathbf{L}_i$ (i.e., in $T_\beta(\mathbf{L}_i)$, intuitively), we have $\lfloor\beta T^i(1)\rfloor$ $\mathbf{F}$s and one $\mathbf{L}_{i+1}$.

Let us put

$$F_0 = \lfloor\beta\rfloor \quad \text{and} \quad L_i = \lfloor\beta T^i(1)\rfloor, \quad i = 1, 2, 3, \cdots.$$

Remember that L_i is the number of **F**s contained in $T_\beta(\mathbf{L}_i)$. It is clear that

$$0 \le L_i \le F_0 \quad \text{for} \quad i = 1, 2, 3, \cdots.$$

The next proposition follows immediately from the rule.

Proposition 1A(T_β).

Suppose that in $T_\beta^{k-1}([0,1))$ the number of f_Bs and l_Bs are

$$\#\mathbf{F} = q_0,\ \#\mathbf{L}_1 = q_1,\ \#\mathbf{L}_2 = q_2,\ \cdots,\ \#\mathbf{L}_{k-1} = q_{k-1}.$$

Then in the next $T_\beta^k([0,1))$ we have

$$\#\mathbf{F} = q_0F_0 + \textstyle\sum_{i=1}^{k-1} q_iL_i \ \text{ and } \ \#\mathbf{L}_1 = q_0,\ \#\mathbf{L}_2 = q_1,\ \#\mathbf{L}_3 = q_2,\ \cdots,\ \#\mathbf{L}_k = q_{k-1}.$$

For the numbers $\#\mathbf{F}$ and $\#\mathbf{L}_1$ of $T_\beta^k([0,1))$, $k = 1, 2, 3, \cdots$, let us define

$$r_k = \frac{\#\mathbf{L}_1}{\#\mathbf{F}},$$

and put

$$r_{i \rhd j} = r_ir_{i-1}\cdots r_j.$$

Applying the proposition above repeatedly to $k = 1, 2, 3, \cdots$, we obtain

Corollary of Proposition 1A(T_β).

If

$$\#\mathbf{F} : \#\mathbf{L}_1 : \#\mathbf{L}_2 : \#\mathbf{L}_3 : \cdots : \#\mathbf{L}_{k-1} = 1 : r_{k-1} : r_{k-1}r_{k-2} : r_{k-1}r_{k-2}r_{k-3} : \cdots : r_{k-1\rhd 1}$$

for $T_\beta^{k-1}([0,1))$, then for $T_\beta^k([0,1))$ it holds that

$$\begin{aligned} &\#\mathbf{F} : \#\mathbf{L}_1 : \#\mathbf{L}_2 : \#\mathbf{L}_3 : \cdots : \#\mathbf{L}_{k-1} : \#\mathbf{L}_k \\ &= \left(F_0 + \textstyle\sum_{i=1}^{k-1} r_{k-1\rhd k-i}L_i\right) : 1 : r_{k-1} : r_{k-1}r_{k-2} : \cdots : r_{k-1\rhd 2} : r_{k-1\rhd 1} \\ &= 1 : r_k : r_kr_{k-1} : r_kr_{k-1}r_{k-2} : \cdots : r_{k\rhd 2} : r_{k\rhd 1}. \end{aligned}$$

In particular for $T_\beta^k([0,1))$

$$r_k = \left(F_0 + \sum_{i=1}^{k-1} r_{k-1\rhd k-i}L_i\right)^{-1}, \quad \text{and}$$

$$\frac{\#\mathbf{L}_j}{\#\mathbf{F}} = r_{k\rhd k-(j-1)}, \quad j = 1, 2, \cdots k.$$

2.2 We prepare here several lemmas for the proof of Theorem A (T_β). In order to evaluate r_k we define new variables $\bar{r}_k$ and $\underline{r}_k$, $k = 0, 1, 2, \cdots$, recursively. Indeed let

$$\bar{r}_0 = F_0^{-1} \quad \text{and} \quad \underline{r}_0 = (F_0 + \textstyle\sum_{i=1}^{\infty} \bar{r}_0^i F_0)^{-1},$$

and for $k = 1, 2, 3, \cdots$, define

$$\bar{r}_k = \left(F_0 + \textstyle\sum_{i=1}^{k} \underline{r}_{k-1 \rhd k-i} L_i\right)^{-1} \quad \text{and} \quad \underline{r}_k = \left(F_0 + \textstyle\sum_{i=1}^{k} \bar{r}_{k-1 \rhd k-i} L_i + \sum_{i=k+1}^{\infty} \bar{r}_0^i F_0\right)^{-1}.$$

(Notations $\underline{r}_{i \rhd j}$ and $\bar{r}_{i \rhd j}$ are defined analogously to $r_{i \rhd j}$.)

Lemma 1A (T_β).
(a) $\underline{r}_k < r_{k+1} \le \bar{r}_k$, $k = 0, 1, 2, \cdots$.
(b) $\bar{r}_0 \ge \bar{r}_1 \ge \bar{r}_2 \ge \cdots \ge \bar{r}_k \ge \bar{r}_{k+1} \ge \cdots$ and $\underline{r}_0 \le \underline{r}_1 \le \underline{r}_2 \le \cdots \le \underline{r}_k \le \underline{r}_{k+1} \le \cdots$.

Proof. (a) Since $r_1^{-1} - \bar{r}_0^{-1} = 0$ and $\underline{r}_0^{-1} - r_1^{-1} = \sum_{i=1}^{\infty} \bar{r}_0^i F_0 > 0$, we have $\underline{r}_0 < r_1 \le \bar{r}_0$. Then it is easy to show $r_{k+1}^{-1} - \bar{r}_k^{-1} > 0$ and $\underline{r}_k^{-1} - r_{k+1}^{-1} > 0$ under the assumption that $\underline{r}_i < r_{i+1} \le \bar{r}_i$, $i = 0, 1, 2, \cdots, k-1$.

(b) It is obvious that $\bar{r}_0 \ge \bar{r}_1$. Since $\underline{r}_0^{-1} - \underline{r}_1^{-1} = \bar{r}_0(F_0 - L_1) \ge 0$, we have $\underline{r}_0 \le \underline{r}_1$. Then it is easy to show $\bar{r}_k^{-1} - \bar{r}_{k-1}^{-1} \ge 0$ and $\underline{r}_{k-1}^{-1} - \underline{r}_k^{-1} \ge 0$ under the assumption that $\bar{r}_0 \ge \bar{r}_1 \ge \bar{r}_2 \ge \cdots \ge \bar{r}_{k-1}$ and $\underline{r}_0 \le \underline{r}_1 \le \underline{r}_2 \le \cdots \le \underline{r}_{k-1}$. □

Lemma 2A (T_β).

(a) $\bar{r}_{k-1 \rhd k-i} - \underline{r}_{k-1 \rhd k-i} \le \bar{r}_0^{i-1} \sum_{j=1}^{i} (\bar{r}_{k-j} - \underline{r}_{k-j}), \quad i = 1, 2, \cdots, k, \quad k = 1, 2, \cdots$.

(b) $\underline{r}_k^{-1} - \bar{r}_k^{-1} \le \frac{\bar{r}_0}{1-\bar{r}_0} \left\{ \left(\sum_{i=1}^{k} \bar{r}_0^{i-1} (\underline{r}_{k-i}^{-1} - \bar{r}_{k-i}^{-1}) \right) + \bar{r}_0^{k-1} \right\} \equiv S_k, \quad k = 1, 2, \cdots$.

(c) $\bar{r}_k - \underline{r}_k \le \frac{\bar{r}_0^2}{1-\bar{r}_0} \left(\bar{r}_0 \frac{2-\bar{r}_0}{1-\bar{r}_0} \right)^k, \quad k = 0, 1, 2, \cdots$.

Proof. (a) As an example we show the case $k = 5$, $i = 3$:

$$\bar{r}_4 \bar{r}_3 \bar{r}_2 - \underline{r}_4 \underline{r}_3 \underline{r}_2 = (\bar{r}_4 - \underline{r}_4) \bar{r}_3 \bar{r}_2 + \underline{r}_4 (\bar{r}_3 - \underline{r}_3) \bar{r}_2 + \underline{r}_4 \underline{r}_3 (\bar{r}_2 - \underline{r}_2)$$
$$\le \bar{r}_0^2 \{(\bar{r}_4 - \underline{r}_4) + (\bar{r}_3 - \underline{r}_3) + (\bar{r}_2 - \underline{r}_2)\} \quad \text{(by Lemma 1A)}.$$

(b) From the definition of $\underline{r}_k$ and $\bar{r}_k$,

$$\underline{r}_k^{-1} - \bar{r}_k^{-1} \le \left(\textstyle\sum_{i=1}^{k} (\bar{r}_{k-1 \rhd k-i} - \underline{r}_{k-1 \rhd k-i}) F_0 \right) + \bar{r}_0^{k+1} \frac{1}{1-\bar{r}_0} F_0$$
$$\le (\bar{r}_{k-1} - \underline{r}_{k-1}) F_0 + \bar{r}_0 \{(\bar{r}_{k-1} - \underline{r}_{k-1}) + (\bar{r}_{k-2} - \underline{r}_{k-2})\} F_0 + \cdots$$
$$+ \bar{r}_0^{k-1} \{(\bar{r}_{k-1} - \underline{r}_{k-1}) + (\bar{r}_{k-2} - \underline{r}_{k-2}) + \cdots + (\bar{r}_0 - \underline{r}_0)\} F_0 + \bar{r}_0^{k+1} \frac{1}{1-\bar{r}_0} F_0 \quad \text{(by (a))}$$
$$= \left(\textstyle\sum_{i=1}^{k} F_0 \bar{r}_0^{i-1} (\bar{r}_{k-i} - \underline{r}_{k-i})(1 + \bar{r}_0 + \bar{r}_0^2 + \cdots + \bar{r}_0^{k-i}) \right) + F_0 \bar{r}_0^{k+1} \frac{1}{1-\bar{r}_0}$$
$$\le F_0 \frac{1}{1-\bar{r}_0} \left(\textstyle\sum_{i=1}^{k} \bar{r}_0^{i-1} (\bar{r}_{k-i} - \underline{r}_{k-i}) \right) + F_0 \bar{r}_0^{k+1} \frac{1}{1-\bar{r}_0}.$$

Since $\bar{r}_{k-i} - \underline{r}_{k-i} = \underline{r}_{k-i}\bar{r}_{k-i}(\underline{r}_{k-i}^{-1} - \bar{r}_{k-i}^{-1}) \le \bar{r}_0^2(\underline{r}_{k-i}^{-1} - \bar{r}_{k-i}^{-1})$ and $F_0\bar{r}_0 = 1$, we finally have

$$\underline{r}_k^{-1} - \bar{r}_k^{-1} \le \tfrac{\bar{r}_0}{1-\bar{r}_0}\left\{\left(\sum_{i=1}^{k} \bar{r}_0^{i-1}(\underline{r}_{k-i}^{-1} - \bar{r}_{k-i}^{-1})\right) + \bar{r}_0^{k-1}\right\}.$$

(c) Let us write the r.h.s.of (b) by S_k. Since S_k can be rewritten such as

$$S_k = \tfrac{\bar{r}_0}{1-\bar{r}_0}(\underline{r}_{k-1}^{-1} - \bar{r}_{k-1}^{-1}) + \bar{r}_0\tfrac{\bar{r}_0}{1-\bar{r}_0}\left\{(\underline{r}_{k-2}^{-1} - \bar{r}_{k-2}^{-1}) + \cdots + (\underline{r}_0^{-1} - \bar{r}_0^{-1})\bar{r}_0^{k-2} + \bar{r}_0^{k-2}\right\},$$

the S_k satisfies the relation

$$S_k = \tfrac{\bar{r}_0}{1-\bar{r}_0}(\underline{r}_{k-1}^{-1} - \bar{r}_{k-1}^{-1}) + \bar{r}_0 S_{k-1} \quad \text{where} \quad S_0 = \tfrac{1}{1-\bar{r}_0}.$$

Then applying (b) to $\underline{r}_{k-1}^{-1} - \bar{r}_{k-1}^{-1}$ we have

$$S_k \le \tfrac{\bar{r}_0}{1-\bar{r}_0}S_{k-1} + \bar{r}_0 S_{k-1} = \left(\bar{r}_0\tfrac{2-\bar{r}_0}{1-\bar{r}_0}\right)S_{k-1},$$

which yields the inequality

$$S_k \le \tfrac{1}{1-\bar{r}_0}\left(\bar{r}_0\tfrac{2-\bar{r}_0}{1-\bar{r}_0}\right)^k, k = 0, 1, 2, \cdots.$$

Now we obtain

$$\bar{r}_k - \underline{r}_k = \underline{r}_k\bar{r}_k(\underline{r}_k^{-1} - \bar{r}_k^{-1}) \le \bar{r}_0^2 S_k \le \tfrac{\bar{r}_0^2}{1-\bar{r}_0}\left(\bar{r}_0\tfrac{2-\bar{r}_0}{1-\bar{r}_0}\right)^k. \ \square$$

It is known (Renyi [4]) that β satisfies

Proposition 2A(T_β).

The β is expressed using F_0 and L_i such as

$$\beta = F_0 + \tfrac{1}{\beta}L_1 + \tfrac{1}{\beta^2}L_2 + \tfrac{1}{\beta^3}L_3 + \cdots + \tfrac{1}{\beta^k}L_k + \cdots.$$

This proposition is easily verified if we apply substitution

$$F_0 = \lfloor \beta T_\beta^0(1) \rfloor = \beta - T_\beta^1(1) \quad \text{and} \quad L_i = \lfloor \beta T_\beta^i(1) \rfloor = \beta T_\beta^i(1) - T_\beta^{i+1}(1)$$

to the r.h.s. ($\to$ then all terms except β are canceled).

Lemma 3A(T_β).

(a) $\bar{r}_k^{-1} \le \beta \le \underline{r}_k^{-1}$, that is, $\underline{r}_k \le \frac{1}{\beta} \le \bar{r}_k$, $k = 0, 1, 2, \cdots$.

(b) $r_{k\triangleright k-(j-1)} - \frac{1}{\beta^j} = \left(\sum_{i=0}^{j-2}(r_{k-i} - \frac{1}{\beta})\frac{1}{\beta^i}(r_{k-i-1\triangleright k-(j-1)})\right) + (r_{k-(j-1)} - \frac{1}{\beta})\frac{1}{\beta^{j-1}}$.

Proof. (a) Recalling $0 \le L_i \le F_0$ and noticing $\bar{r}_0^{-1} = F_0 \le \beta$, we see $\beta \le \underline{r}_0^{-1}$. Then it is easy to show $\bar{r}_k^{-1} \le \beta$ and $\underline{r}_k^{-1} \ge \beta$ under the assumption that $\underline{r}_i \le \frac{1}{\beta} \le \bar{r}_i$ for $i = 0, 1, 2, \cdots, k-1$.

(b) All terms cancel out in the r.h.s. except those of l.h.s. $\square$

2.3 Now we can complete the proof of Theorem A (T_β).

Proof of Theorem A (T_β). (1) From Lemma 1A (a), Lemma 3A (a) and Lemma 2A (c) it holds that

$$|r_k - \tfrac{1}{\beta}| \le \bar{r}_{k-1} - \underline{r}_{k-1} \le \tfrac{\bar{r}_0^2}{1-\bar{r}_0}\left(\bar{r}_0 \tfrac{2-\bar{r}_0}{1-\bar{r}_0}\right)^{k-1} = \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\left(\tfrac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\right)^{k-1}.$$

Hence from Corollary of Proposition 1A, Lemma 3A (b) and $r_i \le \bar{r}_0$ (by Lemma 1A), $\frac{1}{\beta} \le \bar{r}_0$ (by Lemma 3A (a)) we have

$$\begin{aligned}
\left|\tfrac{\#\mathbf{L}_j}{\#\mathbf{F}} - \tfrac{1}{\beta^j}\right| &= |r_{k\triangleright k-(j-1)} - \tfrac{1}{\beta^j}| \le \bar{r}_0^{j-1}\textstyle\sum_{i=0}^{j-1} |r_{k-i} - \tfrac{1}{\beta}| \\
&\le \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\bar{r}_0^{j-1}\textstyle\sum_{i=0}^{j-1}\left(\tfrac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\right)^{k-1-i} \equiv \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\bar{r}_0^{j-1}\textstyle\sum_{i=0}^{j-1} A^{k-1-i} \\
&= \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\bar{r}_0^{j-1}A^{k-j}\tfrac{1-A^j}{1-A} = \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\tfrac{1-A^j}{1-A}(\bar{r}_0A^{-1})^{j-1}A^{k-1} \\
&= \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\tfrac{1-A^j}{1-A}\left(\tfrac{\lfloor\beta\rfloor-1}{2\lfloor\beta\rfloor-1}\right)^{j-1}A^{k-1} \qquad (A = \tfrac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}).
\end{aligned}$$

Moreover the condition

$$\tfrac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)} < 1 \quad\Leftrightarrow\quad 2\lfloor\beta\rfloor - 1 < \lfloor\beta\rfloor(\lfloor\beta\rfloor-1) \quad\Leftrightarrow\quad \lfloor\beta\rfloor^2 - 3\lfloor\beta\rfloor + 1 > 0$$

implies $\lfloor\beta\rfloor > \frac{3+\sqrt{5}}{2} = 2.6180\cdots$.

(2) It is clear that f_k is a density function of $T_\beta^k([0,1))$. Using Proposition 2A, we have

$$f_k(x) - h_\beta(x) = \textstyle\sum_{i=1}^k\left(\left(\tfrac{\#\mathbf{L}_i}{\#\mathbf{F}}\right) - \tfrac{1}{\beta^i}\right)\mathbf{1}_{[0,T_\beta^i(1))}(x) - \sum_{n=k+1}^\infty \tfrac{1}{\beta^n}\mathbf{1}_{[0,T^n(1))}(x),$$

and so by (1) of this theorem

$$|f_k(x) - h_\beta(x)| \le A^{k-1}\textstyle\sum_{i=1}^k C_i + \sum_{n=k+1}^\infty \tfrac{1}{\beta^n}.$$

Since $\frac{1-A^j}{1-A} < \frac{1}{1-A}$, the C_j satisfies

$$C_j < \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\tfrac{1}{1-A}\left(\tfrac{\lfloor\beta\rfloor-1}{2\lfloor\beta\rfloor-1}\right)^{j-1} \equiv D(B^{j-1}) \quad (D = \tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\tfrac{1}{1-A} \text{ and } B = \tfrac{\lfloor\beta\rfloor-1}{2\lfloor\beta\rfloor-1}).$$

Then it holds that

$$\begin{aligned}
|f_k(x) - h_\beta(x)| &< A^{k-1}D\textstyle\sum_{i=0}^{k-1} B^i + \tfrac{1}{\lfloor\beta\rfloor^{k+1}}\sum_{n=0}^\infty \tfrac{1}{\lfloor\beta\rfloor^n} \le A^{k-1}D\tfrac{1}{1-B} + \tfrac{1}{\lfloor\beta\rfloor^k(\lfloor\beta\rfloor-1)} \\
&< A^{k-1}D\tfrac{1}{1-B} + A^k\tfrac{1}{\lfloor\beta\rfloor-1} \qquad (\text{since } \tfrac{1}{\lfloor\beta\rfloor} < \tfrac{1}{\lfloor\beta\rfloor}\left(\tfrac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor-1}\right) = A) \\
&= A^{k-1}\left(\tfrac{1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\tfrac{1}{1-A}\tfrac{1}{1-B} + \tfrac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)^2}\right). \quad \square
\end{aligned}$$

§3. Convergence of the distribution of $T^i_{\beta,\alpha}([0,1))$

The proof of Theorem B ($T_{\beta,\alpha}$) is almost the same as that of Theorem A (T_β), and so in the following we imitate the proof of Theorem A. In the proof we sometimes use the symbol T instead of $T_{\beta,\alpha}$ for brevity.

3.1 The rule of generating *f_B* $\mathbf{F}$, *l_B* $\mathbf{L}_i$ and *h_B* $\mathbf{H}_i$ in the linear mod 1 transformation is as follows:

(i) If we apply the next $T_{\beta,\alpha}$ to $\mathbf{F}$ (i.e., in $T_{\beta,\alpha}(\mathbf{F})$, intuitively), we have $\lfloor\beta+\alpha\rfloor - 1$ $\mathbf{F}$s, one $\mathbf{L}_1$ and one $\mathbf{H}_1$.

(ii) If we apply the next $T_{\beta,\alpha}$ to $\mathbf{L}_i$ (i.e., in $T_{\beta,\alpha}(\mathbf{L}_i)$, intuitively), we have $\lfloor \beta T^i_{\beta,\alpha}(1)+\alpha\rfloor-1$ $\mathbf{F}$s, one $\mathbf{L}_{i+1}$ and one $\mathbf{H}_1$. (There may be a case that no $\mathbf{F}$, $\mathbf{L}_{i+1}$ and $\mathbf{H}_1$ appear, and, instead, only a line(branch) which starts from α and ends at $T^{i+1}_{\beta,\alpha}(1)=\beta T^i_{\beta,\alpha}(1)+\alpha<1$ appears. In this case, since the contribution of $T_{\beta,\alpha}(\mathbf{L}_i)$ to the density function $\tilde{f}_k$ is represented such as

$$\mathbf{1}_{[\alpha,T^{i+1}_{\beta,\alpha}(1))}(x)=\mathbf{1}_{[\alpha,1)}(x)+\mathbf{1}_{[0,T^{i+1}_{\beta,\alpha}(1))}(x)-\mathbf{1}_{[0,1)}(x),$$

we consider that $T_{\beta,\alpha}(\mathbf{L}_i)$ consists of one $\mathbf{L}_{i+1}$, one $\mathbf{H}_1$ and minus-one $\mathbf{F}$.)

(iii) If we apply the next $T_{\beta,\alpha}$ to $\mathbf{H}_i$ (i.e., in $T_{\beta,\alpha}(\mathbf{H}_i)$, intuitively), we have $\lfloor\beta+\alpha\rfloor-\lfloor\beta T^i_{\beta,\alpha}(0)+\alpha\rfloor-1$ $\mathbf{F}$s, one $\mathbf{L}_1$ and one $\mathbf{H}_{i+1}$. (Similarly to (ii), there may be a case that no $\mathbf{F}$, $\mathbf{L}_1$ and $\mathbf{H}_{i+1}$ appear, and, instead, only a line(branch) which starts from $T^{i+1}_{\beta,\alpha}(0)$ and ends at $T_{\beta,\alpha}(1)$ satisfying $\lfloor\beta T^i_{\beta,\alpha}(0)+\alpha\rfloor=\lfloor\beta+\alpha\rfloor$ appears. In this case, since the contribution of $T_{\beta,\alpha}(\mathbf{H}_i)$ to the density function $\tilde{f}_k$ is represented such as

$$\mathbf{1}_{[T^{i+1}_{\beta,\alpha}(0),T_{\beta,\alpha}(1))}(x)=\mathbf{1}_{[T^{i+1}_{\beta,\alpha}(0),1)}+\mathbf{1}_{[0,T_{\beta,\alpha}(1))}(x)-\mathbf{1}_{[0,1)}(x),$$

we consider that $T_{\beta,\alpha}(\mathbf{H}_i)$ consists of one $\mathbf{L}_1$, one $\mathbf{H}_{i+1}$ and minus-one $\mathbf{F}$.)

Let us put

$$F_0=\lfloor\beta+\alpha\rfloor-1 \quad \text{and}$$

$$L_i=\lfloor\beta T^i_{\beta,\alpha}(1)+\alpha\rfloor-1,\quad i=1,2,3,\cdots,$$

$$H_i=\lfloor\beta+\alpha\rfloor-\lfloor\beta T^i_{\beta,\alpha}(0)+\alpha\rfloor-1,\quad i=1,2,3,\cdots.$$

Remember that L_i and H_i are the number of $\mathbf{F}$s contained in $T_{\beta,\alpha}(\mathbf{L}_i)$ and $T_{\beta,\alpha}(\mathbf{H}_i)$ respectively. It is clear that

$$-1\le L_i\le F_0 \quad\text{and}\quad -1\le H_i\le F_0 \quad\text{for}\quad i=1,2,3,\cdots.$$

The next lemma is obvious from the rule:

Lemma 0B ($T_{\beta,\alpha}$).

In the graph of $T^k_{\beta,\alpha}([0,1))$ it holds that $\#\mathbf{L}_i=\#\mathbf{H}_i$ $\;i=1,2,\cdots,k$.

The next proposition follows immediately from Lemma 0B and the rule above:

Proposition 1B($T_{\beta,\alpha}$).

Suppose that in $T^{k-1}_{\beta,\alpha}([0,1))$ the number of f_Bs, l_Bs and h_Bs are

$$\#\mathbf{F}=\pi_0,\ \#\mathbf{L}_1=\#\mathbf{H}_1=q_1,\ \#\mathbf{L}_2=\#\mathbf{H}_2=q_2,\ \cdots,\ \#\mathbf{L}_{k-1}=\#\mathbf{H}_{k-1}=q_{k-1}.$$

Then in the next $T^k_{\beta,\alpha}([0,1))$ we have

$$\#\mathbf{F}=\pi_0F_0+\textstyle\sum_{i=1}^{k-1}q_i(L_i+H_i),$$

$$\#\mathbf{L}_1=\#\mathbf{H}_1=\pi_0+q_1+q_2+\cdots+q_{k-1}\quad\text{and}$$

$$\#\mathbf{L}_i = \#\mathbf{H}_i = q_{i-1}, \quad i = 2, 3, \cdots, k.$$

For the numbers $\#\mathbf{F}$ and $\#\mathbf{L}_i(= \#\mathbf{H}_i)$, $i = 1, 2, \cdots, k$, of $T^k_{\beta,\alpha}([0, 1))$, let us define

$$\sigma_k = \frac{\#\mathbf{L}_1}{\#\mathbf{F} + \sum_{i=1}^{k} \#\mathbf{L}_i}, \quad k = 1, 2, 3, \cdots,$$

and put

$$\sigma_{i \rhd j} = \sigma_i \sigma_{i-1} \cdots \sigma_j.$$

Applying the proposition above repeatedly to $k = 1, 2, 3, \cdots$, we obtain

Corollary of Proposition 1B$(T_{\beta,\alpha})$.

If

$$\left(\#\mathbf{F} + \sum_{i=1}^{k-1} \#\mathbf{L}_i\right) : \#\mathbf{L}_1 : \#\mathbf{L}_2 : \#\mathbf{L}_3 : \cdots : \#\mathbf{L}_{k-1}$$
$$= 1 : \sigma_{k-1} : \sigma_{k-1}\sigma_{k-2} : \sigma_{k-1}\sigma_{k-2}\sigma_{k-3} : \cdots : \sigma_{k-1 \rhd 1}$$

for $T^{k-1}_{\beta,\alpha}([0, 1))$, then for $T^k_{\beta,\alpha}([0, 1))$ it holds that

$$\left(\#\mathbf{F} + \sum_{i=1}^{k} \#\mathbf{L}_i\right) : \#\mathbf{L}_1 : \#\mathbf{L}_2 : \#\mathbf{L}_3 : \cdots : \#\mathbf{L}_{k-1} : \#\mathbf{L}_k$$
$$= \left\{\left((1 - \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i})F_0 + \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i}(L_i + H_i)\right) + (1 + \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i})\right\}$$
$$: 1 : \sigma_{k-1} : \sigma_{k-1}\sigma_{k-2} : \cdots : \sigma_{k-1 \rhd 2} : \sigma_{k-1 \rhd 1}$$
$$= 1 : \sigma_k : \sigma_k\sigma_{k-1} : \sigma_k\sigma_{k-1}\sigma_{k-2} : \cdots : \sigma_{k \rhd 2} : \sigma_{k \rhd 1}.$$

In particular for $T^k_{\beta,\alpha}([0, 1))$,

$$\sigma_k = \left\{\left((1 - \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i})F_0 + \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i}(L_i + H_i)\right) + (1 + \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i})\right\}^{-1}$$
$$= \left((F_0 + 1) + \sum_{i=1}^{k-1} \sigma_{k-1 \rhd k-i}(L_i + H_i - F_0 + 1)\right)^{-1},$$

and

$$\frac{\#\mathbf{L}_j}{\#\mathbf{F} + \sum_{i=1}^{k} \#\mathbf{L}_i} = \sigma_{k \rhd k-(j-1)}, \quad j = 1, 2, \cdots k.$$

In the following we frequently use the value $L_i + H_i - F_0 + 1$ appeared in σ_k, and so we give here a new symbol Γ_i to $L_i + H_i - F_0 + 1$, that is, for $= 1, 2, 3, \cdots$,

$$\Gamma_i \equiv L_i + H_i - F_0 + 1, \text{ and}$$
$$\Gamma_i \text{ satisfies } -F_0 - 1 \le \Gamma_i \le F_0 + 1.$$

3.2 We prepare here several lemmas for the proof of Theorem B $(T_{\beta,\alpha})$. In order to evaluate σ_k we define new variables $\bar{\sigma}_0$, $\bar{\sigma}_k$ and $\underline{\sigma}_k$, $k = 1, 2, 3, \cdots$, recursively. Indeed let

$\bar{\sigma}_0 = (F_0 - 1)^{-1}$, and then

$\underline{\sigma}_1 = \{(F_0+1) + \sum_{i=1}^{\infty} \bar{\sigma}_0^i (F_0+1)\}^{-1}$, $\bar{\sigma}_1 = \{(F_0+1) - \sum_{i=1}^{\infty} \bar{\sigma}_0^i (F_0+1)\}^{-1}$,

and for $k = 2, 3, 4, \cdots$,

$\underline{\sigma}_k = \{(F_0+1) + \sum_{i=1}^{k-1} \check{\sigma}_{k-1\triangleright k-i}\Gamma_i + \sum_{i=k}^{\infty} \bar{\sigma}_0^i (F_0+1)\}^{-1}$ where

$\check{\sigma}_{k-1\triangleright k-i} = \bar{\sigma}_{k-1\triangleright k-i}$ if $\Gamma_i \geq 0$, and $= \underline{\sigma}_{k-1\triangleright k-i}$ if $\Gamma_i < 0$,

$\bar{\sigma}_k = \{(F_0+1) + \sum_{i=1}^{k-1} \hat{\sigma}_{k-1\triangleright k-i}\Gamma_i - \sum_{i=k}^{\infty} \bar{\sigma}_0^i (F_0+1)\}^{-1}$ where

$\hat{\sigma}_{k-1\triangleright k-i} = \underline{\sigma}_{k-1\triangleright k-i}$ if $\Gamma_i \geq 0$, and $= \bar{\sigma}_{k-1\triangleright k-i}$ if $\Gamma_i < 0$.

Note: Below we frequently use the notation $\check{\sigma}_{k-1\triangleright k-i}$ and $\hat{\sigma}_{k-1\triangleright k-i}$, $k = 2, 3, 4, \cdots$.

Lemma 1B ($T_{\beta,\alpha}$).

(a) $\underline{\sigma}_k < \sigma_k < \bar{\sigma}_k$, $k = 1, 2, 3, \cdots$.

(b) If $F_0 \geq 5$, then

$$\bar{\sigma}_0 \geq \bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_k \geq \bar{\sigma}_{k+1} \geq \cdots \quad \text{and} \quad \underline{\sigma}_1 \leq \underline{\sigma}_2 \leq \cdots \leq \underline{\sigma}_k \leq \underline{\sigma}_{k+1} \leq \cdots.$$

Proof. (a) It is easily seen that $\underline{\sigma}_1 < \sigma_1 < \bar{\sigma}_1$. Then it is elementary to show $\underline{\sigma}_k^{-1} > \sigma_k^{-1} > \bar{\sigma}_k^{-1}$ under the assumption that $\underline{\sigma}_i < \sigma_i \leq \bar{\sigma}_i$, $i = 1, 2, \cdots, k-1$. In fact

$$\underline{\sigma}_k^{-1} - \sigma_k^{-1} = \sum_{i=1}^{k-1} (\check{\sigma}_{k-1\triangleright k-i} - \sigma_{k-1\triangleright k-i})\Gamma_i + \sum_{i=k}^{\infty} \bar{\sigma}_0^i (F_0+1) > 0 \quad \text{and}$$
$$\sigma_k^{-1} - \bar{\sigma}_k^{-1} = \sum_{i=1}^{k-1} (\sigma_{k-1\triangleright k-i} - \hat{\sigma}_{k-1\triangleright k-i})\Gamma_i + \sum_{i=k}^{\infty} \bar{\sigma}_0^i (F_0+1) > 0.$$

(b) We first show $\bar{\sigma}_0 \geq \bar{\sigma}_1$: Since $\bar{\sigma}_1^{-1} = (F_0+1)\left(1 - \frac{\sigma_0}{1-\bar{\sigma}_0}\right) = \cdots = (F_0+1)\frac{F_0-3}{F_0-2}$, we have

$$\bar{\sigma}_1^{-1} \geq \bar{\sigma}_0^{-1} \Leftrightarrow (F_0+1)\frac{F_0-3}{F_0-2} \geq F_0 - 1 \;\Leftrightarrow (F_0+1)(F_0-3) \geq (F_0-1)(F_0-2) \Leftrightarrow F_0 \geq 5.$$

Next we show $\bar{\sigma}_1 \geq \bar{\sigma}_2$ and $\underline{\sigma}_1 \leq \underline{\sigma}_2$:

$$\begin{aligned}
\bar{\sigma}_2^{-1} - \bar{\sigma}_1^{-1} &= \hat{\sigma}_1\Gamma_1 + \bar{\sigma}_0(F_0+1) \\
(\text{if } \Gamma_1 \geq 0) &= \underline{\sigma}_1\Gamma_1 + \bar{\sigma}_0(F_0+1) > 0, \\
(\text{if } \Gamma_1 < 0) &= \bar{\sigma}_1\Gamma_1 + \bar{\sigma}_0(F_0+1) \geq \bar{\sigma}_0\Gamma_1 + \bar{\sigma}_0(F_0+1) \geq 0,
\end{aligned}$$

$$\begin{aligned}
\underline{\sigma}_1^{-1} - \underline{\sigma}_2^{-1} &= \bar{\sigma}_0(F_0+1) - \check{\sigma}_1\Gamma_1 \\
(\text{if } \Gamma_1 \geq 0) &= \bar{\sigma}_0(F_0+1) - \bar{\sigma}_1\Gamma_1 \geq \bar{\sigma}_0(F_0+1) - \bar{\sigma}_0\Gamma_1 \geq 0, \\
(\text{if } \Gamma_1 < 0) &= \bar{\sigma}_0(F_0+1) + \underline{\sigma}_1(-\Gamma_1) > 0.
\end{aligned}$$

Finally we show $\bar{\sigma}_{k-1} \geq \bar{\sigma}_k$ and $\underline{\sigma}_{k-1} \leq \underline{\sigma}_k$ under the assumption that

$$\bar{\sigma}_0 \geq \bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_{k-2} \geq \bar{\sigma}_{k-1} \quad \text{and} \quad \underline{\sigma}_1 \leq \underline{\sigma}_2 \leq \cdots \leq \underline{\sigma}_{k-2} \leq \underline{\sigma}_{k-1}:$$

$$\begin{aligned}
\bar{\sigma}_k^{-1} - \bar{\sigma}_{k-1}^{-1} &= \sum_{i=1}^{k-2} (\hat{\sigma}_{k-1\triangleright k-i} - \hat{\sigma}_{(k-1)-1\triangleright(k-1)-i})\Gamma_i + \hat{\sigma}_{k-1\triangleright 1}\Gamma_{k-1} + \bar{\sigma}_0^{k-1}(F_0+1) \\
&\geq \hat{\sigma}_{k-1\triangleright 1}\Gamma_{k-1} + \bar{\sigma}_0^{k-1}(F_0+1) \\
(\text{if } \Gamma_{k-1} \geq 0) &= \underline{\sigma}_{k-1\triangleright 1}\Gamma_{k-1} + \bar{\sigma}_0^{k-1}(F_0+1) > 0,
\end{aligned}$$

$$(\text{if } \Gamma_{k-1}<0) = \bar\sigma_{k-1\rhd 1}\Gamma_{k-1} + \bar\sigma_0^{k-1}(F_0+1) \ge \bar\sigma_0^{k-1}\Gamma_{k-1} + \bar\sigma_0^{k-1}(F_0+1) \ge 0,$$

$$\underline{\sigma}_{k-1}^{-1} - \underline{\sigma}_k^{-1} = \textstyle\sum_{i=1}^{k-2}(\check\sigma_{(k-1)-1\rhd(k-1)-i} - \check\sigma_{k-1\rhd k-i})\Gamma_i + \bar\sigma_0^{k-1}(F_0+1) - \check\sigma_{k-1\rhd 1}\Gamma_{k-1}$$
$$\ge \bar\sigma_0^{k-1}(F_0+1) - \check\sigma_{k-1\rhd 1}\Gamma_{k-1}$$
$$(\text{if } \Gamma_{k-1}\ge 0) = \bar\sigma_0^{k-1}(F_0+1) - \bar\sigma_{k-1\rhd 1}\Gamma_{k-1} \ge \bar\sigma_0^{k-1}(F_0+1) - \bar\sigma_0^{k-1}\Gamma_{k-1} \ge 0,$$
$$(\text{if } \Gamma_{k-1}<0) = \bar\sigma_0^{k-1}(F_0+1) + \underline{\sigma}_{k-1\rhd 1}(-\Gamma_{k-1}) > 0. \quad \square$$

Lemma 2B ($T_{\beta,\alpha}$).
Assume $F_0 \ge 5$.

(a) $\bar\sigma_{k-1\rhd k-i} - \underline{\sigma}_{k-1\rhd k-i} \le \bar\sigma_0^{i-1}\sum_{j=1}^{i}(\bar\sigma_{k-j} - \underline{\sigma}_{k-j}), \quad i=1,2,\cdots,k-1, \quad k=2,3,\cdots.$

(b) $\underline{\sigma}_k^{-1} - \bar\sigma_k^{-1} \le (F_0+1)\,\frac{\bar\sigma_0^2}{1-\bar\sigma_0}\left\{\left(\sum_{i=1}^{k-1}\bar\sigma_0^{i-1}(\underline{\sigma}_{k-i}^{-1} - \bar\sigma_{k-i}^{-1})\right) + 2\bar\sigma_0^{k-2}\right\} \equiv S_k, \quad k=2,3,\cdots.$

(c) $\bar\sigma_k - \underline{\sigma}_k \le 2(F_0+1)\frac{\bar\sigma_0^3}{1-\bar\sigma_0}\left((F_0+1)\frac{\bar\sigma_0^2}{1-\bar\sigma_0} + \bar\sigma_0\right)^{k-1}, \quad k=1,2,3,\cdots.$

Proof. (a) This is the same as that of Lemma 2A.

(b) From the definition of $\underline{\sigma}_k$ and $\bar\sigma_k$,

$$\underline{\sigma}_k^{-1} - \bar\sigma_k^{-1} = \left(\sum_{i=1}^{k-1}(\check\sigma_{k-1\rhd k-i} - \hat\sigma_{k-1\rhd k-i})\Gamma_i\right) + 2(F_0+1)\bar\sigma_0^k\sum_{i=0}^{\infty}\bar\sigma_0^i$$
$$\le \left(\sum_{i=1}^{k-1}(\bar\sigma_{k-1\rhd k-i} - \underline{\sigma}_{k-1\rhd k-i})(F_0+1)\right) + 2(F_0+1)\bar\sigma_0^k\frac{1}{1-\bar\sigma_0}$$
$$\le (F_0+1)\left[(\bar\sigma_{k-1} - \underline{\sigma}_{k-1}) + \bar\sigma_0\{(\bar\sigma_{k-1} - \underline{\sigma}_{k-1}) + (\bar\sigma_{k-2} - \underline{\sigma}_{k-2})\} + \cdots\right.$$
$$\left. + \bar\sigma_0^{k-2}\{(\bar\sigma_{k-1} - \underline{\sigma}_{k-1}) + (\bar\sigma_{k-2} - \underline{\sigma}_{k-2}) + \cdots + (\bar\sigma_1 - \underline{\sigma}_1)\}\right] + 2(F_0+1)\bar\sigma_0^k\frac{1}{1-\bar\sigma_0} \quad \text{(by (a))}$$
$$= (F_0+1)\left(\sum_{i=1}^{k-1}\bar\sigma_0^{i-1}(\bar\sigma_{k-i} - \underline{\sigma}_{k-i})(1 + \bar\sigma_0 + \bar\sigma_0^2 + \cdots + \bar\sigma_0^{k-i-1})\right) + 2(F_0+1)\bar\sigma_0^k\frac{1}{1-\bar\sigma_0}$$
$$\le (F_0+1)\frac{1}{1-\bar\sigma_0}\left\{\left(\sum_{i=1}^{k-1}\bar\sigma_0^{i-1}(\bar\sigma_{k-i} - \underline{\sigma}_{k-i})\right) + 2\bar\sigma_0^k\right\}$$
$$= (F_0+1)\frac{1}{1-\bar\sigma_0}\left\{\left(\sum_{i=1}^{k-1}\bar\sigma_0^{i-1}\underline{\sigma}_{k-i}\bar\sigma_{k-i}(\underline{\sigma}_{k-i}^{-1} - \bar\sigma_{k-i}^{-1})\right) + 2\bar\sigma_0^k\right\}$$
$$\le (F_0+1)\frac{\bar\sigma_0^2}{1-\bar\sigma_0}\left\{\left(\sum_{i=1}^{k-1}\bar\sigma_0^{i-1}(\underline{\sigma}_{k-i}^{-1} - \bar\sigma_{k-i}^{-1})\right) + 2\bar\sigma_0^{k-2}\right\}.$$

(c) Let us write the r.h.s.of (b) by S_k. Since S_k can be rewritten such as

$$S_k = (F_0+1)\frac{\bar\sigma_0^2}{1-\bar\sigma_0}(\underline{\sigma}_{k-1}^{-1} - \bar\sigma_{k-1}^{-1}) + \bar\sigma_0(F_0+1)\frac{\bar\sigma_0^2}{1-\bar\sigma_0}\left\{\left(\sum_{i=1}^{k-2}\bar\sigma_0^{i-1}(\underline{\sigma}_{k-1-i}^{-1} - \bar\sigma_{k-1-i}^{-1})\right) + 2\bar\sigma_0^{k-3}\right\},$$

the S_k satisfies the relation

$$S_k = (F_0+1)\frac{\bar\sigma_0^2}{1-\bar\sigma_0}(\underline{\sigma}_{k-1}^{-1} - \bar\sigma_{k-1}^{-1}) + \bar\sigma_0 S_{k-1}, \quad k=2,3,\cdots, \quad \text{where } \ S_1 = 2(F_0+1)\frac{\bar\sigma_0}{1-\bar\sigma_0}.$$

Then applying (b) to $\underline{\sigma}_{k-1}^{-1} - \bar\sigma_{k-1}^{-1}$ we have

$$S_k \le \left((F_0+1)\frac{\bar\sigma_0^2}{1-\bar\sigma_0} + \bar\sigma_0\right)S_{k-1}, \quad k=2,3,\cdots,$$

which yields the inequality

$$S_k \leq \left((F_0+1)\tfrac{\bar\sigma_0^2}{1-\bar\sigma_0}+\bar\sigma_0\right)^{k-1} S_1, \quad k=2,3,\cdots.$$

Now we obtain

$$\bar\sigma_k-\underline{\sigma}_k=\underline{\sigma}_k\bar\sigma_k(\underline{\sigma}_k^{-1}-\bar\sigma_k^{-1})\leq\bar\sigma_0^2S_k\leq 2(F_0+1)\tfrac{\bar\sigma_0^3}{1-\bar\sigma_0}\left((F_0+1)\tfrac{\bar\sigma_0^2}{1-\bar\sigma_0}+\bar\sigma_0\right)^{k-1}. \ \square$$

Proposition 2B($T_{\beta,\alpha}$).

The β is expressed using F_0, L_i and H_i such as

$$\beta=(F_0+1)+\textstyle\sum_{i=1}^{\infty}\frac{1}{\beta^i}(L_i+H_i-F_0+1)$$
$$\equiv(F_0+1)+\textstyle\sum_{i=1}^{\infty}\frac{1}{\beta^i}\Gamma_i.$$

This proposition is easily verified if we replace F_0, L_i and H_i in the r.h.s. by definitions, then

$$\begin{aligned}\text{r.h.s.}&=\cdots=\lfloor\beta+\alpha\rfloor+\textstyle\sum_{i=1}^{\infty}\frac{1}{\beta^i}(\lfloor\beta T^i(1)+\alpha\rfloor-\lfloor\beta T^i(0)+\alpha\rfloor)\\&=(\beta+\alpha-T^1(1))+\textstyle\sum_{i=1}^{\infty}\frac{1}{\beta^i}\left((\beta T^i(1)+\alpha-T^{i+1}(1))-(\beta T^i(0)+\alpha-T^{i+1}(0))\right)\\&=\beta+\alpha-T^1(0)=\beta.\end{aligned}$$

The next lemma is proved using the expression of β above.

Lemma 3B($T_{\beta,\alpha}$).

(a) $\bar\sigma_0^{-1}<\beta,\quad \bar\sigma_k^{-1}<\beta<\underline{\sigma}_k^{-1}$, that is, $\bar\sigma_0>\frac{1}{\beta},\quad \underline{\sigma}_k<\frac{1}{\beta}<\bar\sigma_k,\quad k=1,2,3,\cdots.$

(b) $\sigma_{k\rhd k-(j-1)}-\frac{1}{\beta^j}=\left(\sum_{i=0}^{j-2}(\sigma_{k-i}-\frac{1}{\beta})\frac{1}{\beta^i}(\sigma_{k-i-1\rhd k-(j-1)})\right)+(\sigma_{k-(j-1)}-\frac{1}{\beta})\frac{1}{\beta^{j-1}}.$

Proof. (a) First let us see $\bar\sigma_0^{-1}<\beta$:

$$\beta-\bar\sigma_0^{-1}=\beta-(F_0-1)=\beta-\lfloor\beta+\alpha\rfloor+2\geq 2-\alpha>0.$$

Next let us see $\beta>\bar\sigma_1^{-1}$ and $\underline{\sigma}_1^{-1}>\beta$: Because $|\Gamma_i|\leq F_0+1$, we have

$$\beta-\bar\sigma_1^{-1}=\textstyle\sum_{i=1}^{\infty}\frac{1}{\beta^i}\Gamma_i+(F_0+1)\sum_{i=1}^{\infty}\bar\sigma_0^i\geq-(F_0+1)\sum_{i=1}^{\infty}\frac{1}{\beta^i}+(F_0+1)\sum_{i=1}^{\infty}\bar\sigma_0^i>0,$$
$$\underline{\sigma}_1^{-1}-\beta=(F_0+1)\textstyle\sum_{i=1}^{\infty}\bar\sigma_0^i-\sum_{i=1}^{\infty}\frac{1}{\beta^i}\Gamma_i\geq(F_0+1)\sum_{i=1}^{\infty}\bar\sigma_0^i-(F_0+1)\sum_{i=1}^{\infty}\frac{1}{\beta^i}>0.$$

Finally let us show $\bar\sigma_k^{-1}<\beta$ and $\underline{\sigma}_k^{-1}>\beta$ under the assumption that $\underline{\sigma}_k<\frac{1}{\beta}<\bar\sigma_k$ for $i=1,2,\cdots,k-1$:

$$\begin{aligned}\beta-\bar\sigma_k^{-1}&=\textstyle\sum_{i=1}^{k-1}(\frac{1}{\beta^i}-\hat\sigma_{k-1\rhd k-i})\Gamma_i+\sum_{i=k}^{\infty}\frac{1}{\beta^i}\Gamma_i+(F_0+1)\sum_{i=k}^{\infty}\bar\sigma_0^i\\&\geq\textstyle\sum_{i=k}^{\infty}\frac{1}{\beta^i}\Gamma_i+(F_0+1)\sum_{i=k}^{\infty}\bar\sigma_0^i\geq-(F_0+1)\sum_{i=k}^{\infty}\frac{1}{\beta^i}+(F_0+1)\sum_{i=k}^{\infty}\bar\sigma_0^i>0,\end{aligned}$$
$$\begin{aligned}\underline{\sigma}_k^{-1}-\beta&=\textstyle\sum_{i=1}^{k-1}(\check\sigma_{k-1\rhd k-i}-\frac{1}{\beta^i})\Gamma_i+(F_0+1)\sum_{i=k}^{\infty}\bar\sigma_0^i-\sum_{i=k}^{\infty}\frac{1}{\beta^i}\Gamma_i\\&\geq(F_0+1)\textstyle\sum_{i=k}^{\infty}\bar\sigma_0^i-\sum_{i=k}^{\infty}\frac{1}{\beta^i}\Gamma_i\geq(F_0+1)\sum_{i=k}^{\infty}\bar\sigma_0^i-(F_0+1)\sum_{i=k}^{\infty}\frac{1}{\beta^i}>0.\end{aligned}$$

(b) All terms cancel out in the r.h.s. except those of l.h.s. $\square$

3.3 Now we can complete the proof of Theorem B ($T_{\beta,\alpha}$).

Proof of Theorem B ($T_{\beta,\alpha}$). (0) This is Lemma 0B.

(1) We first note that the r.h.s. of Lemma 2B (c) is rewritten such as $\frac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\left(\frac{2F_0-1}{(F_0-1)(F_0-2)}\right)^{k-1}$. (Notice that $\frac{2F_0-1}{(F_0-1)(F_0-2)} < 1$ for $F_0 \geq 5$.) Then from Lemma 1B (a), Lemma 3B (a) and Lemma 2B (c) it holds that

$$|\sigma_k - \tfrac{1}{\beta}| \leq \bar{\sigma}_k - \underline{\sigma}_k \leq \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\left(\tfrac{2F_0-1}{(F_0-1)(F_0-2)}\right)^{k-1} \equiv \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\tilde{A}^{k-1}.$$

Hence from Corollary of Proposition 1B, Lemma 3B (b) and $\sigma_i \leq \bar{\sigma}_0$ (by Lemma 1B), $\frac{1}{\beta} \leq \bar{\sigma}_0$ (by Lemma 3B (a)) we have

$$\begin{aligned}\left|\tfrac{\#\mathbf{L}_j}{\#\mathbf{F}+\sum_{i=1}^{k}\#\mathbf{L}_i} - \tfrac{1}{\beta^j}\right| &= |\sigma_{k\rhd k-(j-1)} - \tfrac{1}{\beta^j}| \leq \bar{\sigma}_0^{j-1}\textstyle\sum_{i=0}^{j-1}|\sigma_{k-i} - \tfrac{1}{\beta}| \\ &\leq \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\bar{\sigma}_0^{j-1}\textstyle\sum_{i=0}^{j-1}\tilde{A}^{k-1-i} = \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\bar{\sigma}_0^{j-1}\tilde{A}^{k-j}\tfrac{1-\tilde{A}^j}{1-\tilde{A}} \\ &= \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}(\bar{\sigma}_0\tilde{A}^{-1})^{j-1}\tfrac{1-\tilde{A}^j}{1-\tilde{A}}\tilde{A}^{k-1} = \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\left(\tfrac{F_0-2}{2F_0-1}\right)^{j-1}\tfrac{1-\tilde{A}^j}{1-\tilde{A}}\tilde{A}^{k-1}.\end{aligned}$$

(2) A density function of $T_{\beta,\alpha}^k([0,1))$, say $\tilde{f}_k$, is given by

$$\begin{aligned}\tilde{f}_k(x) &= \tfrac{\#\mathbf{F}}{\#\mathbf{F}+\sum_{j=1}^{k}\#\mathbf{L}_j}\left\{\mathbf{1}_{[0,1)}(x) + \textstyle\sum_{i=1}^{k}\left((\tfrac{\#\mathbf{L}_i}{\#\mathbf{F}})\mathbf{1}_{[0,T^i(1))}(x) + (\tfrac{\#\mathbf{H}_i}{\#\mathbf{F}})\mathbf{1}_{[T^i(0),1)}(x)\right)\right\} \\ &= \left(1 - \tfrac{\sum_{j=1}^{k}\#\mathbf{L}_j}{\#\mathbf{F}+\sum_{j=1}^{k}\#\mathbf{L}_j}\right)\mathbf{1}_{[0,1)}(x) + \tfrac{\#\mathbf{F}}{\#\mathbf{F}+\sum_{j=1}^{k}\#\mathbf{L}_j}\textstyle\sum_{i=1}^{k}(\tfrac{\#\mathbf{L}_i}{\#\mathbf{F}})\left(\mathbf{1}_{[0,T^i(1))}(x) - \mathbf{1}_{[0,T^i(0))}(x) + \mathbf{1}_{[0,1)}(x)\right)\bigr) \\ &= \mathbf{1}_{[0,1)}(x) + \textstyle\sum_{i=1}^{k}(\tfrac{\#\mathbf{L}_i}{\#\mathbf{F}+\sum_{j=1}^{k}\#\mathbf{L}_j})\left(\mathbf{1}_{[0,T^i(1))}(x) - \mathbf{1}_{[0,T^i(0))}(x)\right).\end{aligned}$$

Using Corollary of Proposition 1B we have

$$\begin{aligned}\tilde{f}_k(x) - h_{\beta,\alpha}(x) = \textstyle\sum_{i=1}^{k}&\left(\sigma_{k\rhd k-(i-1)} - \tfrac{1}{\beta^i}\right)\left(\mathbf{1}_{[0,T^i(1))}(x) - \mathbf{1}_{[0,T^i(0))}(x)\right) \\ &- \textstyle\sum_{n=k+1}^{\infty}\tfrac{1}{\beta^n}\left(\mathbf{1}_{[0,T^n(1))}(x) - \mathbf{1}_{[0,T^n(0))}(x)\right),\end{aligned}$$

and so by (1) of this theorem

$$|\tilde{f}_k(x) - h_{\beta,\alpha}(x)| \leq \tilde{A}^{k-1}\textstyle\sum_{i=1}^{k}\tilde{C}_i + \sum_{n=k+1}^{\infty}\tfrac{1}{\beta^n}.$$

Since $\frac{1-\tilde{A}^j}{1-\tilde{A}} < \frac{1}{1-\tilde{A}}$, the $\tilde{C}_j$ satisfies

$$\tilde{C}_j < \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\left(\tfrac{F_0-2}{2F_0-1}\right)^{j-1}\tfrac{1}{1-\tilde{A}} \equiv \tilde{D}(\tilde{B}^{j-1}) \quad (\tilde{D} = \tfrac{2(F_0+1)}{(F_0-1)^2(F_0-2)}\tfrac{1}{1-\tilde{A}},\ \tilde{B} = \tfrac{F_0-2}{2F_0-1}).$$

Hence it holds that

$$\begin{aligned}|\tilde{f}_k(x) - h_{\beta,\alpha}(x)| &< \tilde{A}^{k-1}\tilde{D}\textstyle\sum_{i=0}^{k-1}\tilde{B}^i + \bar{\sigma}_0^{k+1}\sum_{n=0}^{\infty}\bar{\sigma}_0^n \quad (\text{by } \tfrac{1}{\beta} \leq \bar{\sigma}_0) \\ &< \tilde{D}\tfrac{1}{1-\tilde{B}}\tilde{A}^{k-1} + \tfrac{1}{(F_0-1)(F_0-2)}\tilde{A}^{k-1}\end{aligned}$$

because

$\bar{\sigma}_0^{k+1}\sum_{n=0}^{\infty}\bar{\sigma}_0^n = \bar{\sigma}_0^k \frac{1}{\bar{\sigma}_0^{-1}-1} = \left(\frac{1}{F_0-1}\right)^k\left(\frac{1}{F_0-2}\right) = \frac{1}{(F_0-1)(F_0-2)}\left(\frac{F_0-2}{2F_0-1}\tilde{A}\right)^{k-1} < \frac{1}{(F_0-1)(F_0-2)}\tilde{A}^{k-1}.$ □

Concluding Remarks. In [1] Góra investigated the invariant density of the generalized β-transformation τ on [0,1]. The graph of $\tau([0,1])$ consists of $\lfloor\beta\rfloor$ full_Branches($\mathbf{F}$s) and one high_Branch $\mathbf{H}_1$ or one low_Branch $\mathbf{L}_1$ depending on τ. (The $\mathbf{F}$s in $\tau([0,1])$ may be a mixture of $\mathbf{F}=\mathbf{F}^+$ and $\mathbf{F}=\mathbf{F}^-$ whose slopes are up and down respectively.) Let $\mathbf{E}_1$ denote $\mathbf{H}_1$ or $\mathbf{L}_1$ appearing in $\tau([0,1])$. Then $\tau(\mathbf{E}_1)$ consists of several $\mathbf{F}$s, one or no $\mathbf{E}_1$, and one $\mathbf{H}_2$ or $\mathbf{L}_2$ which is denoted by $\mathbf{E}_2$. (Note that the 'up/down' of the slope of $\mathbf{E}_1$ appearing in $\tau(\mathbf{E}_1)$ may be different from that of $\tau([0,1])$. However when we treat the distribution of $\tau^2([0,1])$, it is not necessary for us to know whether branches $\mathbf{F}$s, $\mathbf{E}_1$'s and $\mathbf{E}_2$ are ascending or descending.) Let us apply τ to $\mathbf{E}_{i-1}$ generally. If $\mathbf{E}_{i-1}$ is $\mathbf{L}_{i-1}$, then $\tau(\mathbf{E}_{i-1})$ consists of several $\mathbf{F}$s and one $\mathbf{H}_i$ or $\mathbf{L}_i$ which is denoted by $\mathbf{E}_i$. If $\mathbf{E}_{i-1}$ is $\mathbf{H}_{i-1}$, then $\tau(\mathbf{E}_{i-1})$ consists of several $\mathbf{F}$s, one $\mathbf{E}_1$, and one $\mathbf{H}_i$ or $\mathbf{L}_i$ which is denoted by $\mathbf{E}_i$. There may be a case that no $\mathbf{H}_i$ and $\mathbf{L}_i$ appears and, instead, only a branch which extends, for example, from $\tau(1)$ to $\tau^i(1)(0<\tau(1)<\tau^i(1)<1)$ appears. In this case we consider that there appear one $\mathbf{H}_1$, one $\mathbf{L}_i$ and minus-one $\mathbf{F}$ (cf. rules (ii) and (iii) of Section 3.1). For $i=1,2,\cdots$, we define $\omega(i)=1$ if $\mathbf{E}_i$ is $\mathbf{H}_i$, and $=0$ otherwise.

Now suppose $\tau^k([0,1])$ contains $\#\mathbf{F}$ full_Branches and $\#\mathbf{E}_i(=\#\mathbf{H}_i$ or $\#\mathbf{L}_i)$, $i=1,2,\cdots,k$, high_ or low_Branches. Then we can show that

$$\lim_{k\to\infty}\frac{\#\mathbf{E}_j}{\#\mathbf{F}+\sum_{i=1}^k\omega(i)\#\mathbf{E}_i}=\frac{1}{\beta^j},\quad \text{and}$$

a density function of $\tau^k([0,1])$

$$\begin{aligned} f_k(x) &= \frac{\#\mathbf{F}}{\#\mathbf{F}+\sum_{j=1}^k\omega(j)\#\mathbf{E}_j}\left[\mathbf{1}_{[0,1)}(x)+\sum_{i=1}^k\left\{\left(\frac{\omega(i)\#\mathbf{E}_i}{\#\mathbf{F}}\right)\mathbf{1}_{[\tau^i(1),1)}(x)+\left(\frac{(1-\omega(i))\#\mathbf{E}_i}{\#\mathbf{F}}\right)\mathbf{1}_{[0,\tau^i(1))}(x)\right\}\right] \\ &= \cdots \\ &= \mathbf{1}_{[0,1)}(x)+\sum_{i=1}^k(-1)^{\omega(i)}\frac{\#\mathbf{E}_i}{\#\mathbf{F}+\sum_{j=1}^k\omega(j)\#\mathbf{E}_j}\mathbf{1}_{[0,\tau^i(1))}(x) \end{aligned}$$

approaches to the invariant density function of τ as $k\to\infty$. Details will be given subsequently.

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Appendix

1. Values of the r.h.s. of $\|f_k$ - $h_\beta\| < C\left(\frac{2\lfloor\beta\rfloor-1}{\lfloor\beta\rfloor(\lfloor\beta\rfloor-1)}\right)^{k-1}$ (Theorem A (2)):

$\lfloor\beta\rfloor$	$k=1$	2	3	4	5	6	8	10	$k=16$
3	2.0833	1.7361	1.4468	1.2056	1.0047	0.8372	0.5814	0.4038	0.1352
6	0.1698	0.0623	0.0228	0.0084	0.0031	0.0011	0.0002	0.0000	0.0000
9	0.0639	0.0151	0.0036	0.0008	0.0002	0.0000	0.0000	0.0000	0.0000
12	0.0334	0.0058	0.0010	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
15	0.0205	0.0028	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
18	0.0139	0.0016	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
21	0.0100	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
24	0.0076	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
27	0.0059	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

(0.0000 means < 0.0001)

2. Values of the r.h.s. of $\|\tilde{f}_k$ - $h_{\beta,\alpha}\| < \tilde{C}\left(\frac{2F_0-1}{(F_0-1)(F_0-2)}\right)^{k-1}$ (Theorem B (2), $F_0 = \lfloor\beta+\alpha\rfloor - 1$):

F_0	$k=1$	2	3	4	5	6	8	10	$k=16$
5	1.5833	1.1875	0.8906	0.6680	0.5010	0.3757	0.2113	0.1189	0.0212
6	0.5389	0.2964	0.1630	0.0897	0.0493	0.0271	0.0082	0.0025	0.0001
9	0.1268	0.0385	0.0117	0.0035	0.0011	0.0003	0.0000	0.0000	0.0000
12	0.0572	0.0120	0.0025	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000
15	0.0326	0.0052	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
18	0.0211	0.0027	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
21	0.0147	0.0016	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
24	0.0109	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
27	0.0084	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

(0.0000 means < 0.0001)