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Some Lie Algebras of Vector Fields and their Derivations Case of Strictly Partially Classical Type

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1. This is another attempt to define structures on foliated manifolds and Lie algebras associated to those structures (c.f. [K-3, K-4]).

Fix a coordinate system $v_1, \ldots, v_p$ in a p-dimensional Euclidean space $U = \mathbb{R}^p$, and $w_1, \ldots, w_q$ in a q-dimensional $W = \mathbb{R}^q$. We consider smooth vector fields on the $(p+q)$-dimensional space $V = U \oplus W = \mathbb{R}^{p+q}$, and the Lie algebra $\mathfrak{A}(V)$ of all vector fields on V. Denote $\frac{\partial}{\partial v_i}$ by ∂_i $(i = 1, \ldots, p)$, and $\frac{\partial}{\partial w_\alpha}$ by ∂_α $(\alpha = 1, \ldots, q)$. Use Latin indices $i, j, k, \ldots$ for variables in U and Greek indices $\alpha, \beta, \ldots$ for variables in W, otherwise stated.

Consider the standard codimension q foliation $\mathcal{F}$ on V, defined by parallel p-planes: π_W^{-1}(a point), where π_W is a canonical projection of V onto W. Let $\mathcal{T}$ be the Lie algebra of all leaf-tangent vector fields on V, then by [K-2], the derivation algebra $\mathcal{D}er(\mathcal{T})$ of $\mathcal{T}$ is naturally isomorphic to the Lie algebra $\mathcal{L}$ of foliation-preserving vector fields, and $\mathcal{L}$ is decomposed as

$$\mathcal{L} = \mathcal{T} + \mathcal{L}',$$

where $\mathcal{L}'$ is naturally isomorphic to the Lie algebra $\mathfrak{A}(W)$.

Let $\mathcal{T}(v)$ or $\mathcal{T}(w)$ be the Lie subalgebra of $\mathcal{T}$, consisting of vector fields whose coefficient functions are independent of variables in W or U respectively, that is,

$$\mathcal{T}(v) = \{X \in \mathcal{T};\ [\partial_\alpha, X] = 0 \quad (1 \leqq \alpha \leqq q)\},$$
$$\mathcal{T}(w) = \{X \in \mathcal{T};\ [\partial_i, X] = 0 \quad (1 \leqq i \leqq p)\}.$$

2. Unimodular Structure. Put $p = n$, $x_i = v_i$ $(i = 1, \ldots, n)$, and $\tau = dx_1 \wedge \ldots \wedge dx_n$.

Lemma 1. *Write* $X \in \mathcal{T}$ *as* $X = \sum_{i=1}^{n} f_i(x, w)\, \partial_i$. *Then the following conditions are equivalent.*

(i) $L_X \tau = \phi(w)\tau$ *for some* C^∞*-function* $\phi(w) \in C^\infty(W)$, *where* L_X *means the Lie derivative with respect to* X;

(ii) $X \in \mathcal{T}(v)$ *and* $\sum_{i=1}^{n} \partial_i f_i = c$ *for some constant* c.

Proof. Put $\tau_i = dx_1\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_n$. Then we get

$$\begin{aligned} L_X\tau &= di_X\tau = d\,\{\sum_{i=1}^{n}(-1)^{i-1} f_i\tau_i\} \\ &= \sum_{i=1}^{n}(-1)^{i-1}\{\sum_{j=1}^{n}(\partial_j f_i)\,dx_j\wedge\tau_i + \sum_{\alpha=1}^{q}(\partial_\alpha f_i)\,dw_\alpha\wedge\tau_i\} \\ &= \{\sum_{i=1}^{n}(\partial_i f_i)\}\,\tau + \sum_{i=1}^{n}\sum_{\alpha=1}^{q}(-1)^{i-1}(\partial_\alpha f_i)\,dw_\alpha\wedge\tau_i. \end{aligned}$$

Here, assume that $L_X\tau = \phi(w)\tau$ for some $\phi(w)\in C^\infty(W)$. Then we get

$$\sum_{i=1}^{n}\partial_i f_i = \phi(w) \qquad \text{and} \qquad \partial_\alpha f_i = 0$$

for all i and α, because the differential n-forms τ and $dw_\alpha\wedge\tau_i$ are independent. Hence, $X\in\mathcal{T}(v)$ and the function $\phi(w)$ is constant. Thus, *(i)* implies *(ii)*.

The converse implication is obvious. Q. E. D.

A leaf-tangent vector field X is called *strictly partially conformally unimodular* (s. p. c. u.), if $L_X\tau$ equals to $c\tau$ for some constant c. Moreover, if the constant c is zero, X is called *strictly partially unimodular* (s. p. u.). Then we get two Lie subalgebras of $\mathcal{T}$:

$$\begin{aligned} \mathcal{T}_{s\tau} &= \{X\in\mathcal{T};\ L_X\tau = 0\}, \\ \mathcal{T}_{sc\tau} &= \{X\in\mathcal{T}';\ L_X\tau = c\,\tau \ \text{ for some constant } c\}. \end{aligned}$$

Similarly, we get two Lie subalgebras of $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}_{s\tau} &= \{X\in\mathcal{L};\ L_X\tau = 0\}, \\ \mathcal{L}_{sc\tau} &= \{X\in\mathcal{L};\ L_X\tau = c\,\tau \ \text{ for some constant } c\}. \end{aligned}$$

Since $L_X\tau = 0$ for $X\in\mathcal{L}'$, these are decomposed as

$$\mathcal{L}_{s\tau} = \mathcal{T}_{s\tau} + \mathcal{L}' \quad \text{and} \quad \mathcal{L}_{sc\tau} = \mathcal{T}_{sc\tau} + \mathcal{L}.$$

From Lemma 5. 2 in [K-1] and Lemma 1, we get easily

Proposition 2. *(i)* $\mathcal{T}_{s\tau}$ *and* $\mathcal{T}_{sc\tau}$ *are subalgebras of* $\mathcal{T}(v)$. *Hence,*

$$[\mathcal{T}_{sc\tau}, \mathcal{L}'] = [\mathcal{T}_{s\tau}, \mathcal{L}'] = 0.$$

(ii) $\mathcal{T}_{s\tau}$ *and* $\mathcal{T}_{sc\tau}$ *are naturally isomorphic to the Lie algebras* $\mathfrak{A}_\tau(U)$ *and* $\mathfrak{A}_{c\tau}(U)$ *respectively (see [K-1] for definitions of* $\mathfrak{A}_\tau(U)$ *and* $\mathfrak{A}_{c\tau}(V)$*).*

(iii) $\mathcal{T}_{s\tau}$ *is a codimension 1 ideal of* $\mathcal{T}_{sc\tau}$.

(iv) The decompositions of $\mathcal{L}_{s\tau}$ *and* $\mathcal{L}_{sc\tau}$ *are direct :*

$$\mathcal{L}_{sc\tau} \cong \mathfrak{A}_{c\tau}(U) \oplus \mathfrak{A}(W),$$
$$\mathcal{L}_{s\tau} \cong \mathfrak{A}_{\tau}(U) \oplus \mathfrak{A}(W).$$

(v) For $p \geqq 2$, $\mathcal{G}_{s\tau} = [\mathcal{G}_{s\tau}, \mathcal{G}_{s\tau}] = [\mathcal{G}_{sc\tau}, \mathcal{G}_{sc\tau}] = [\mathcal{L}_{sc\tau}, \mathcal{G}_{sc\tau}]$.

3. Symplectic Structure. Put $p = 2n$, $x_i = v_i$, $y_i = v_{i+n}$ $(1 \leqq i \leqq n)$, and $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Lemma 3. *Write* $X \in \mathcal{G}$ *as* $X = \sum_{i=1}^{2n} f_i(x, y, w)\, \partial_i$. *Then, the following conditions are equivalent* :

(i) $L_X \omega = \phi(w)\, \omega$ *for some* C^∞*-function* $\phi(w) \in C^\infty(W)$;

(ii) $X \in \mathcal{G}(v)$ *and for any* $1 \leqq i, j \leqq n$,

$$\partial_{i+n} f_j = \partial_{j+n} f_i, \quad \partial_i f_{j+n} = \partial_j f_{i+n}, \quad \partial_i f_j + \partial_{j+n} f_{i+n} = \delta_{ij}\, c$$

for some constant c, *where* δ_{ij} *is the Kronecker's delta.*

Proof. $L_X \omega = d i_X \omega$

$$= d \sum_{i=1}^{n} \{ f_i(x, y, w)\, dv_i + f_{i+n}(x, y, w)\, dv_{i+n} \}$$
$$= \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} \{ (\partial_j f_i)\, dx_j \wedge dy_i + (\partial_{j+n} f_i)\, dy_j \wedge dy_i \} + \sum_{\alpha=1}^{q} (\partial_\alpha f_i)\, dw_\alpha \wedge dy_i \Big]$$
$$- \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} \{ (\partial_j f_{i+n})\, dx_j \wedge dx_i + (\partial_{j+n} f_{i+n})\, dy_j \wedge dx_i \} + \sum_{\alpha=1}^{q} (\partial_\alpha f_{i+n})\, dw_\alpha \wedge dx_i \Big]$$
$$= \sum_{i,j} \{ (\partial_i f_j) + (\partial_{j+n} f_{i+n}) \}\, dx_i \wedge dy_j$$
$$+ \sum_{i<j} \{ (\partial_{j+n} f_i) - (\partial_{i+n} f_j) \}\, dy_i \wedge dy_j + \sum_{i,\alpha} (\partial_\alpha f_i)\, dw_\alpha \wedge dy_i$$
$$+ \sum_{i<j} \{ (\partial_j f_{i+n}) - (\partial_i f_{j+n}) \}\, dx_i \wedge dx_j - \sum_{i,\alpha} (\partial_\alpha f_{i+n})\, dw_\alpha \wedge dx_i.$$

Here, assume that $L_X \omega = \phi(w)\, \omega$ for some $\phi \in C^\infty(W)$. From the independence of 2-forms $dv_i \wedge dv_j$ $(i<j)$ and $dv_i \wedge dw_\alpha$, we get

$$\partial_\alpha f_i = 0 \qquad (1 \leqq i \leqq 2n,\ 1 \leqq \alpha \leqq q),$$
$$\partial_i f_{j+n} = \partial_j f_{i+n}, \qquad \partial_{i+n} f_i = \partial_{j+n} f_i \qquad (1 \leqq i,\ j \leqq n),$$
$$\partial_i f_j + \partial_{j+n} f_{i+n} = \delta_{ij}\, \phi(w) \qquad (1 \leqq i,\ j \leqq n).$$

Hence, $X \in \mathcal{G}(v)$ and the function $\phi(w)$ is constant. Thus, *(i)* implies *(ii)*.

The converse implication is obvious. Q. E. D.

A leaf-tangent vector field X is called *strictly partially conformally symplectic* (s. p. c. s.), if $L_X \omega$ equals to $c\omega$ for some constant c. Moreover, if the constant c is zero, X is called *strictly partially symplectic* (s. p. s). Then, we get two Lie subalgebras of $\mathcal{T}$:

$$\mathcal{T}_{s\omega} = \{X \in \mathcal{T} \ ; \quad L_X \omega = 0\},$$
$$\mathcal{T}_{sc\omega} = \{X \in \mathcal{T} \ ; \quad L_X \omega = c\,\omega \quad \text{for some constant} \quad c\}.$$

Similarly, we get two Lie subalgebras of $\mathcal{L}$:

$$\mathcal{L}_{s\omega} = \{X \in \mathcal{L} \ ; \quad L_X \omega = 0\},$$
$$\mathcal{L}_{sc\omega} = \{X \in \mathcal{L} \ ; \quad L_X \omega = c\,\omega \quad \text{for some constant} \quad c\}.$$

Since $L_X \omega = 0$ for $X \in \mathcal{L}'$, these are decomposed as

$$\mathcal{L}_{s\omega} = \mathcal{T}_{s\omega} + \mathcal{L}' \quad \text{and} \quad \mathcal{L}_{sc\omega} = \mathcal{T}_{sc\omega} + \mathcal{L}'.$$

From Lemma 6. 5 in [K-1] and Lemma 3, we get easily

Proposition 4. *(i)* $\mathcal{T}_{s\omega}$ *and* $\mathcal{T}_{sc\omega}$ *are subalgebras of* $\mathcal{T}(v)$. *Hence,*

$$[\mathcal{T}_{sc\omega}, \mathcal{L}'] = [\mathcal{T}_{s\omega}, \mathcal{L}'] = 0.$$

(ii) $\mathcal{T}_{s\omega}$ *and* $\mathcal{T}_{sc\omega}$ *are naturally isomorphic to the Lie algebras* $\mathfrak{A}_{\omega}(V)$ *and* $\mathfrak{A}_{c\omega}(U)$ *respectively (see [K-1] for definitions of these algebras).*

(iii) $\mathcal{T}_{s\omega}$ *is a codimension 1 ideal of* $\mathcal{T}_{sc\omega}$.

(iv) The decompositions of $\mathcal{L}_{s\omega}$ *and* $\mathcal{L}_{sc\omega}$ *are direct* :

$$\mathcal{L}_{sc\omega} \cong \mathfrak{A}_{c\omega}(U) \oplus \mathfrak{A}(W),$$
$$\mathcal{L}_{s\omega} \cong \mathfrak{A}_{\omega}(U) \oplus \mathfrak{A}(W).$$

(v) $\mathcal{T}_{s\omega} = [\mathcal{T}_{s\omega}, \mathcal{T}_{s\omega}] = [\mathcal{T}_{sc\omega}, \mathcal{T}_{sc\omega}] = [\mathcal{L}_{sc\omega}, \mathcal{T}_{sc\omega}]$.

4. Contact Structure. Put $p = 2n+1$, $x_i = v_i$, $y_i = v_{i+n}$ $(1 \leqq i \leqq n)$, $z = v_{2n+1}$, and $\theta = dz - \sum_{i=1}^{n} y_i dx_i$.

Lemma 5. *Write* $X \in \mathcal{T}$ *as* $X = \sum_{i=1}^{2n+1} f_i(x, y, z, w)\partial_i$. *Then, the following conditions are equivalent:*

(i) $L_X \theta = \phi(x, y, z, w)\,\theta$ *for some function* $\phi(x, y, z, w) \in C^\infty(V)$;

(ii) $L_X \theta = \phi(x, y, z)\,\theta$ *for some function* $\phi(x, y, z) \in C^\infty(U)$;

(iii) There is a unique function $k(x, y, z) \in C^\infty(U)$ *such that for any* i $(1 \leqq i \leqq n)$,

$$f_i = -\partial_{i+n} k, \quad f_{i+n} = (\partial_i k) + y_i(\partial_{2n+1} k), \quad \text{and} \quad f_{2n+1} = k - \sum_{i=1}^{n} y_i(\partial_{i+n} k).$$

Here, k *is obtained as* $k = i_X\,\theta = f_{2n+1} - \sum_{i=1}^{n} y_i f_i$.

Proof. Using Cartan's formula $L_X = di_X + i_X d$, we get

$$di_X\theta = dk, \qquad i_X d\theta = \sum_{i=1}^{n} (f_i dy_i - f_{i+n}\, dx_i),$$

$$L_X\theta = \sum_{i=1}^{n} [\{(\partial_i k) - f_{i+n}\}\, dx_i + \{(\partial_{i+n} k) + f_i\}\, dy_i]$$

$$+ (\partial_{2n+1} k)\, dz + \sum_{\alpha=1}^{q} (\partial_\alpha k)\, dw.$$

Here, assume that $L_X\,\theta = \phi(x, y, z, w)\,\theta$ for some $\phi \in C^\infty(V)$. Then, we get

$$\phi = \partial_{2n+1}k, \qquad \partial_i k - f_{i+n} = -y_i\phi, \qquad \partial_{i+n}k + f_i = 0, \qquad \partial_\alpha k = 0$$

for all i and α, because the differential 1-forms dv_i $(1 \leqq i \leqq 2n+1)$ and dw_α $(1 \leqq \alpha \leqq q)$ are independent on V. Hence, the function k and ϕ are independent of variables in W, and further the coefficient functions of X are given as

$$f_i = -(\partial_{i+n} k), \quad f_{i+n} = (\partial_i k) + y_i\phi, \text{ and } f_{2n+1} = k + \sum_{i=1}^{n} y_i f_i.$$

Thus, *(i)* implies *(ii)*, and *(ii)* implies *(iii)*.

Now, it is obvious that *(iii)* implies *(i)*. Q. E. D.

A leaf-tangent or foliation-preserving vector field X is called *strictly partially contact* (s. p. c.), if X satisfies one of the conditions of Lemma 5. Then we get Lie subalgebras $\mathcal{T}_{s\theta}$ and $\mathcal{L}_{s\theta}$ of $\mathcal{T}$ and $\mathcal{L}$, respectively:

$$\mathcal{T}_{s\theta} = \{X \in \mathcal{T};\ L_X\theta = \phi(x, y, z)\theta \quad \text{for some } \phi \in C^\infty(U)\},$$
$$\mathcal{L}_{s\theta} = \{X \in \mathcal{L};\ L_X\theta = \phi(x, y, z)\theta \quad \text{for some } \phi \in C^\infty(U)\}.$$

Here, note that $L_X\theta = 0$ for $X \in \mathcal{L}'$, and $\mathcal{L}_{s\theta}$ is decomposed as

$$\mathcal{L}_{s\theta} = \mathcal{T}_{s\theta} + \mathcal{L}'.$$

From Lemma 5 and by the similar arguments for the proof of Proposition 1.8 *(i)* of [K-4], we get easily

Proposition 6. *(i)* $\mathcal{T}_{s\theta}$ *is a subalgebra of* $\mathcal{T}(v)$. *Hence*

$$[\mathcal{T}_{s\theta}, \mathcal{L}'] = 0.$$

(ii) $\mathcal{T}_{s\theta}$ *is naturally isomorphic to the Lie algebra* $\mathfrak{A}_\theta(U)$ *of all contact vector fields on* U *(see [K-1] for definitions).*

(iii) The decomposition of $\mathcal{L}_{s\theta}$ *is direct:*

$$\mathcal{L}_{s\theta} \cong \mathfrak{A}_\theta(U) \oplus \mathfrak{A}(W).$$

(iv) $\mathcal{T}_{s\theta} = [\mathcal{T}_{s\theta}, \mathcal{T}_{s\theta}] = [\mathcal{L}_{s\theta}, \mathcal{T}_{s\theta}]$.

5. Derivations (General theory). Consider two Lie algebras L and M. Assume that M is a Lie subalgebra of L, or L is an ideal of M. A linear map D of M to L is called a derivation, if D satisfies the equality

$$D([X_1, X_2]) = [D(X_1), X_2] + [X_1, D(X_2)]$$

for any X_i $(i = 1, 2)$ of M. Define the adjoint operator $\mathrm{ad}X$ for $X \in L$ as

$$(\mathrm{ad}X)\,Y = [X, Y] \qquad (Y \in M),$$

then $\mathrm{ad}X$ is a derivation of M to L.

Denote by $\mathcal{D}er(M; L)$ the set of all derivations of M to L. If M coincides with L, the set $\mathcal{D}er(L) = \mathcal{D}er(L; L)$ is an associative algebra, and the set $\mathrm{ad}(L)$ of all adjoint operators is an ideal of $\mathcal{D}er(L)$ as a Lie algebra.

Proposition 7. *Let M be an ideal of a Lie algebra L such that $M = [M, M]$. Then, any derivation D of M to L is a derivation of M to itself, that is,*

$$\mathcal{D}er(M; L) = \mathcal{D}er(M).$$

Proof. Let $X \in M$. Write X as $X = \sum_{i=1}^{r} [X_i, X_{i+r}]$ with $X_i \in M$ $(1 \leqq i \leqq 2r)$. Then, we get

$$D(X) = D(\sum_i [X_i, X_{i+r}]) = \sum_i \{[D(X_i), X_{i+r}] + [X_i, D(X_{i+r})]\},$$

because M is an ideal of L. Q. E. D.

From this, easily we obtain

Proposition 8. *Let M_1 and M_2 be ideals of a Lie algebra L, such that $L = M_1 \oplus M_2$, $[M_1, M_2] = 0$ and $M_i = [M_i, M_i]$ $(i = 1, 2)$. Then, any derivation D of L is uniquely decomposed as*

$$D = D_1 \oplus D_2$$

where D_i $(i = 1, 2)$ is a derivation of M_i and is given as the restriction of D to M_i. Hence,

$$\mathcal{D}er(L) \cong \mathcal{D}er(M_1) \oplus \mathcal{D}er(M_2).$$

As is well-known, the first cohomology $H^1(L;\ L)$ of a Lie algebra L with coefficents in its adjoint representation is isomorphic to the quotient of its derivation algebra by the ideal of all inner derivations:

$$H^1(L;\ L) \cong \mathscr{D}e\varkappa(L)/\mathrm{ad}(L)$$

(see [C-1] for definitions).

6. Now we go back to our situation. By Theorems 3.3, 5.7, 5.8 and 6.8 in [K-1], we get

Theorem9. *(i) Let* $\sigma = c\tau\ (\dim U \neq 1)$, $c\omega$ *or* θ. *Then all derivations of* $\mathfrak{A}_\sigma(U)$ *are inner, that is,* $\mathscr{D}e\varkappa(\mathfrak{A}_\sigma(U)) = \mathrm{ad}(\mathfrak{A}_\sigma(U)) \cong \mathfrak{A}_\sigma(U)$.

(ii) Let $\sigma = \tau\ (\dim U \neq 1)$ *or* ω. *Then, the derivation algebra of* $\mathfrak{A}_\sigma(U)$ *is naturally isomorphic to* $\mathfrak{A}_{c\sigma}(U)$, *that is,* $\mathscr{D}e\varkappa(\mathfrak{A}_\sigma(U)) = \{\mathrm{ad}Z \mid \mathfrak{A}_\sigma(U);\ Z \in \mathfrak{A}_{c\sigma}(U)\} \cong \mathfrak{A}_{c\sigma}(U)$.

Now, we can determine the structure of the derivation algebras of strictly partially classical Lie algebras:

Theorem10. *(i) Let* $\sigma = c\tau\ (p \neq 1)$, $c\omega$ *or* θ. *All derivations of* $\mathcal{T}_{s\sigma}$ *are inner, that is,* $\mathscr{D}e\varkappa(\mathcal{T}_{s\sigma}) = \mathrm{ad}(\mathcal{T}_{s\sigma}) \cong \mathcal{T}_{s\sigma}$. *Hence,*

$$H^1(\mathcal{T}_{s\sigma}\ ;\ \mathcal{T}_{s\sigma}) = 0.$$

(ii) Let $\sigma = \tau\ (p \neq 1)$, *or* ω. *The derivation algebra of* $\mathcal{T}_{s\sigma}$ *is naturally isomorphic to* $\mathcal{T}_{sc\sigma}$, *that is,* $\mathscr{D}e\varkappa(\mathcal{T}_{s\sigma}) = \{\mathrm{ad}Z \mid \mathcal{T}_{s\sigma}\ ;\ Z \in \mathcal{T}_{sc\sigma}\} \cong \mathcal{T}_{sc\sigma}$. *Hence,*

$$H^1(\mathcal{T}_{s\sigma}\ ;\ \mathcal{T}_{s\sigma}) \cong \mathcal{T}_{sc\sigma} / \mathcal{T}_{s\sigma} \cong \mathbb{R}.$$

(iii) Let $\sigma = c\tau\ (p \neq 1)$, $c\omega$ *or* θ. *All derivations of* $\mathcal{L}_{s\sigma}$ *are inner, that is,* $\mathscr{D}e\varkappa(\mathcal{L}_{s\sigma}) = \mathrm{ad}(\mathcal{L}_{s\sigma}) \cong \mathcal{L}_{s\sigma}$. *Hence,*

$$H^1(\mathcal{L}_{s\sigma}\ ;\ \mathcal{L}_{s\sigma}) = 0.$$

(iv) Let $\sigma = \tau\ (p \neq 1)$ *or* θ. *The derivation algebra of* $\mathcal{L}_{s\sigma}$ *is naturally isomorphic to* $\mathcal{L}_{sc\sigma}$, *that is,* $\mathscr{D}e\varkappa(\mathcal{L}_{s\sigma}) = \{\mathrm{ad}Z \mid \mathcal{L}_{s\sigma}\ ;\ Z \in \mathcal{L}_{sc\sigma}\} \cong \mathcal{L}_{sc\sigma}$. *Hence,*

$$H^1(\mathcal{L}_{s\sigma}\ ;\ \mathcal{L}_{s\sigma}) \cong \mathcal{L}_{sc\sigma} / \mathcal{L}_{s\sigma} \cong \mathbb{R}.$$

Proof. *(i)* and *(ii)* are obtained from Propositions 2, 4 and 6, and Theorem 9. *(iii)* and *(iv)* are obtained from the above *(i)*,*(ii)*, Propositions 2, 4, 6 and 8, and the fact that all derivations of $\mathfrak{A}(W)$ are also inner. Q. E. D.

7. Product Foliation. The Lie algebras of strictly classical type have not property (A) (see [K-1]), so it is difficult to investigate those Lie algebras on the whole manifolds in general. Here, we consider only product foliations. Let $E = Q \times P$ be a product manifold, and $\mathcal{F}$ the foliation defined by $\pi_Q^{-1}(y)$ $(y \in Q)$, where π_Q is the projection to the manifold Q. If P is equipped with a classical structure $\sigma_P = \tau_P$, ω_P or θ_P, then we get a strictly partially classical structure $\sigma = \pi_P^* \sigma_P$ on E, and Lie algebras $\mathcal{T}_{s\sigma}(E)$ and $\mathcal{L}_{s\sigma}(E)$ similarly as §2–§4.

By Propositions 2, 4 and 6, easily we get the following

Proposition 12. *Let* $\sigma = \tau, c\tau, \omega, c\omega$ *or* θ.

(i) The Lie algebra $\mathcal{T}_{s\sigma}(E)$ *is naturally isomorphic to* $\mathfrak{A}_{\sigma_P}(P)$.

(ii) The Lie algebra $\mathcal{L}_{s\sigma}(E)$ *is naturally isomorphic to the direct sum of* $\mathfrak{A}_{\sigma_P}(P)$ *and* $\mathfrak{A}(Q)$.

Now, we get the following similarly as Theorem 10:

Theorem 13. *(i) Let* $\sigma = c\tau$ $(p \neq 1)$, $c\omega$ *or* θ. *All derivations of* $\mathcal{T}_{s\sigma}(E)$ *and* $\mathcal{L}_{s\sigma}(E)$ *are inner. Hence,*

$$H^1(\mathcal{T}_{s\sigma}(E);\ \mathcal{T}_{s\sigma}(E)) = 0 \quad \text{and} \quad H^1(\mathcal{L}_{s\sigma}(E);\ \mathcal{L}_{s\sigma}(E)) = 0.$$

(ii) Let $\sigma = \tau$ $(p \neq 1)$ or ω. Assume that P is connected. The derivation algebra of $\mathcal{L}_{s\sigma}(E)$ and $\mathcal{T}_{s\sigma}(E)$ are naturally isomorphic to $\mathcal{L}_{sc\sigma}(E)$ and $\mathcal{T}_{sc\sigma}(E)$, respectively. Hence,

$$H^1(\mathcal{T}_{s\sigma}(E);\ \mathcal{T}_{s\sigma}(E)) = \mathcal{T}_{sc\sigma}(E) / \mathcal{T}_{s\sigma}(E) \cong \mathbb{R} \ \text{or} \ 0.$$

$$H^1(\mathcal{L}_{s\sigma}(E);\ \mathcal{L}_{s\sigma}(E)) = \mathcal{L}_{sc\sigma}(E) / \mathcal{L}_{s\sigma}(E) \cong \mathbb{R} \ \text{or} \ 0,$$

Moreover, $H^1 \cong \mathbb{R}$ *if and only if* σ_P *is an exact form on* P.

Proof. The last part of *(ii)* follows from Lemmata 4. 1 and 6. 1 in [K-1].

Q. E. D.

8. One Dimensional Unimodular Structure. Let $p = 1$. From Lemma 1, we get

$$\mathcal{T}_{s\tau} = \mathbb{R}\partial \cong \mathfrak{A}_\tau(\mathbb{R}) \quad \text{and} \quad \mathcal{T}_{sc\tau} = \mathbb{R}\partial + \mathbb{R}x\partial,$$

where we omit indices 1 of x_1 and ∂_1. And similarly as §6 in [K-4], we get

Theorem 14. *(i)* $\mathcal{D}_{ex}(\mathcal{T}_{s\tau}) = \mathrm{ad}\,\mathcal{T}_{sc\tau} \cong \mathbb{R}$; $\quad H^1(\mathcal{T}_{s\tau};\ \mathcal{T}_{s\tau}) \cong \mathbb{R}$.

(ii) $\mathcal{D}_{ex}(\mathcal{T}_{sc\tau}) = \mathrm{ad}\,\mathcal{T}_{sc\tau}$; $\quad H^1(\mathcal{T}_{sc\tau};\ \mathcal{T}_{sc\tau}) = 0$.

(iii) $\mathcal{D}_{ex}(\mathcal{L}_{s\tau}) = \mathrm{ad}\,\mathcal{L}_{sc\tau}$; $\quad H^1(\mathcal{L}_{s\tau};\ \mathcal{L}_{s\tau}) \cong \mathbb{R}$.

(iv) $\mathcal{D}er(\mathcal{L}_{scr}) = \mathrm{ad}\,\mathcal{L}_{scr}$; $H^1(\mathcal{L}_{scr} ; \mathcal{L}_{scr}) = 0$.

Remark. There are no properly outer derivations of $\mathcal{T}_{sr}$ and $\mathcal{L}_{sr}$ (c. f. §6 in [K-4]).

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