

Notes on Partially Unimodular Structures (I)

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§1. Introduction and Preliminaries. In [K-4], we defined partially classical structures on foliated manifolds, and Lie algebras of partially classical type, and determined their derivation algebras. However, if the dimension of leaves is one, Lie algebras of partially unimodular type have not property (A) in general, so we dealt there only with the standard foliations on Euclidean spaces. Here, we restrict ourselves to partially unimodular (p.u.) structures for I -dimensional foliations, and investigate in global Lie algebras of p.u. vector fields and their derivation algebras.

In this article, we consider connected and orientable C^∞ -manifolds M of dimension $q+1$, and I -dimensional foliations \mathcal{F} on M . Denote by $\mathcal{I}(M, \mathcal{F})$ the Lie algebra of all leaf-tangent vector fields on (M, \mathcal{F}) , and by $\mathcal{J}(M, \mathcal{F})$ the ideal of the exterior algebra $\Omega(M)$ of all differential forms on M , defined as

$$\mathcal{J}(M, \mathcal{F}) = \{ \alpha \in \Omega(M) ; \iota_L^* \alpha = 0 \text{ for every leaf } L \text{ of } \mathcal{F} \},$$

where ι_L^* is the inclusion mapping of L into M .

A I -form τ on M is called p.u. structure, if $\iota_L^* \tau \neq 0$ for every leaf L of \mathcal{F} . By Proposition 4.5 of [K-4], we can take distinguished coordinates $(V; x, w_1, \dots, w_q)$ around any point of M such that τ is written on V as

$$\tau \equiv dx \pmod{\mathcal{J}(M, \mathcal{F})}.$$

Then, a vector field $X \in \mathcal{I}(M, \mathcal{F})$ is called partially conformally unimodular (p.c.u.), if $L_X \tau$ is congruent to $\phi \tau$ modulo $\mathcal{J}(M, \mathcal{F})$ for some function $\phi \in C^\infty(M)^{\mathcal{F}}$, where L_X means the Lie derivative and $C^\infty(M)^{\mathcal{F}}$ is the space of smooth functions on M which are constant on each leaves of \mathcal{F} . Moreover, if the function ϕ is zero, X is called partially unimodular (p.u.). Then we get two natural Lie subalgebras of $\mathcal{I}(M, \mathcal{F})$:

$$\mathcal{I}_\tau(M, \mathcal{F}) = \{ X \in \mathcal{I}(M, \mathcal{F}) ; L_X \tau \in \mathcal{J}(M, \mathcal{F}) \} ,$$

$$\begin{aligned} \mathcal{I}_{c\tau}(M, \mathcal{F}) = \{ X \in \mathcal{I}(M, \mathcal{F}) ; L_X \tau \equiv \phi \tau \pmod{\mathcal{J}(M, \mathcal{F})} \\ \text{for some } \phi \in C^\infty(M)^{\mathcal{F}} \} . \end{aligned}$$

A p.u. structure τ is called strictly partially unimodular (s.p.u.), if there are distinguished coordinates $(V ; x, w_1, \dots, w_q)$ around any point of M such that τ is written as $\tau = dx$ on V . Then, a vector field $X \in \mathcal{I}(M, \mathcal{F})$ is called strictly partially conformally unimodular (s.p.c.u.), if $L_X \tau$ equals to $c\tau$ for some constant c (c.f. Lemma 1 of [K-5]). Moreover, if the constant c is zero, X is called strictly partially unimodular (s.p.u.). Then we get two Lie subalgebras of $\mathcal{I}(M, \mathcal{F})$:

$$\mathcal{I}_{s\tau}(M, \mathcal{F}) = \{ X \in \mathcal{I}(M, \mathcal{F}) ; L_X \tau = 0 \} \subset \mathcal{I}_\tau(M, \mathcal{F}),$$

$$\mathcal{I}_{sc\tau}(M, \mathcal{F}) = \{ X \in \mathcal{I}(M, \mathcal{F}) ; L_X \tau = c\tau \text{ for some } c \in \mathbb{R} \} \subset \mathcal{I}_{c\tau}(M, \mathcal{F}).$$

From [K-2], the derivation algebra of $\mathcal{I}(M, \mathcal{F})$ is naturally isomorphic to the Lie algebra $\mathcal{L}(M, \mathcal{F})$ of all foliation-preserving vector fields on (M, \mathcal{F}) . Similarly we get the following Lie subalgebras of $\mathcal{L}(M, \mathcal{F})$:

$$\mathcal{L}_\tau(M, \mathcal{F}) = \{ X \in \mathcal{L}(M, \mathcal{F}) ; L_X \tau \in \mathcal{J}(M, \mathcal{F}) \} ,$$

$$\begin{aligned} \mathcal{L}_{c\tau}(M, \mathcal{F}) = \{ X \in \mathcal{L}(M, \mathcal{F}) ; L_X \tau \equiv \phi \tau \pmod{\mathcal{J}(M, \mathcal{F})} \\ \text{for some } \phi \in C^\infty(M)^{\mathcal{F}} \} , \end{aligned}$$

$$\mathcal{L}_{s\tau}(M, \mathcal{F}) = \{ X \in \mathcal{L}(M, \mathcal{F}) ; L_X \tau = 0 \} ,$$

$$\mathcal{L}_{sc\tau}(M, \mathcal{F}) = \{ X \in \mathcal{L}(M, \mathcal{F}) ; L_X \tau = c\tau \text{ for some } c \in \mathbb{R} \} .$$

Here, note that $\mathcal{I}_\tau(M, \mathcal{F})$ is abelian, since the commutator operation is a local operator, and $\mathcal{I}_\tau(\mathbb{R}^{q+1})$ is abelian (Proposition 1.8 in [K-4]).

Since every leaf L is a l -dimensional manifold, L is either a circle S^1 or a line \mathbb{R}^1 . Here, we get the following by similar arguments as in §5.3 of [K-4].

Proposition 1. *If all leaves are circles, or every leaf of \mathcal{F} is a dense leaf, then p.c.u. (or s.p.c.u.) vector fields are p.u. (s.p.u., respectively), that is,*

$$\begin{aligned} \mathcal{I}_\tau(M, \mathcal{F}) &= \mathcal{I}_{c\tau}(M, \mathcal{F}), & \mathcal{I}_{s\tau}(M, \mathcal{F}) &= \mathcal{I}_{sc\tau}(M, \mathcal{F}), \\ \mathcal{L}_\tau(M, \mathcal{F}) &= \mathcal{L}_{c\tau}(M, \mathcal{F}) & \text{and} & \quad \mathcal{L}_{s\tau}(M, \mathcal{F}) = \mathcal{L}_{sc\tau}(M, \mathcal{F}). \end{aligned}$$

§2. Bundle Foliation (I). In this section, we consider bundle foliations on orientable fiber bundles. Since their standard fibers are S^1 or \mathbb{R}^1 , then we can assume that $M = (M, \pi, B)$

is a principal bundle, and \mathcal{F} is given by fibers $\pi^{-1}(b)$, where b is a point of the base space B . In fact, S^1 and \mathbb{R}^1 are strongly deformation retracts of $\text{Diff}_+(S^1)$ and $\text{Diff}_+(\mathbb{R}^1)$, respectively (see e.g. [S-1]). Moreover, since $\{0\}$ is a deformation retract of \mathbb{R}^1 , any \mathbb{R}^1 -bundle is trivial, that is, M is a product manifold of \mathbb{R}^1 and the base manifold B .

A vector field X on (M, \mathcal{F}) is foliation-preserving (or leaf-tangent), if and only if X is projectable (or vertical, respectively) on (M, π, B) . Since M has a connection, $\mathcal{L}(M) = \mathcal{L}(M, \mathcal{F})$ is decomposed as

$$\mathcal{L}(M) = \mathcal{I}(M) + \mathfrak{A}(B),$$

where $\mathcal{I}(M)$ is an ideal, and $\mathfrak{A}(B)$ is the Lie algebra of all vector fields on the base space B . Here, $\mathfrak{A}(B)$ is identified with its horizontal lift with respect to the connection (c.f. Chap. VII of [S-2]). Moreover by the projection π , $C^\infty(M)^{\mathcal{F}}$ can be identified with the space $C^\infty(B)$ of all C^∞ -functions on B .

Let $G = S^1$ or \mathbb{R}^1 . We can consider that the G -action on the standard fiber G is given by volume-preserving diffeomorphisms of (G, τ_0) with some volume form τ_0 on G . Then we can construct a s.p.u. structure τ on M . In fact, put $\tau = \tau_0$ on \tilde{W} , where $\tilde{W} = \pi^{-1}(W)$ and W is a coordinate neighborhood of B . Then, we get

Proposition 2. (i) $i_X \tau = 0$ for any $X \in \mathfrak{A}(B)$, where $i_X \tau$ is the interior product of X and τ .

(ii) $\mathcal{L}_\tau(M) = \mathcal{I}_\tau(M) + \mathfrak{A}(B)$ and $\mathcal{L}_{c\tau}(M) = \mathcal{I}_{c\tau}(M) + \mathfrak{A}(B)$.

(iii) $\mathcal{I}_\tau(M)$ and $\mathcal{L}_\tau(M)$ have the property (A) for any \tilde{W} with an open set W of B .

Proof. (ii) Let $X \in \mathfrak{A}(B)$, then $L_X \tau = di_X \tau = 0$.

(iii) Let $L = \mathcal{L}_\tau(M)$ or $\mathcal{I}_\tau(M)$. If $X \in L$ and $\phi \in C^\infty(B)$, then $\phi X \in L$.

Q.E.D.

Take the vector field $Z \in \mathcal{I}(M)$ defined by the equation $i_Z \tau = I$. Then, Z is s.p.u., since $L_Z \tau = d(i_Z \tau) = d(I) = 0$. Denote by \mathfrak{A} the space of solutions of equations $Z\phi = \psi$ for some $\psi \in C^\infty(B)$:

$$\mathfrak{A} = \{ \phi \in C^\infty(M), Z\phi \in C^\infty(B) \}.$$

Proposition 3. (i) If M is an S^1 -bundle, then $\mathfrak{A} = C^\infty(B)$,

$$\mathcal{I}_\tau(M) = \mathcal{I}_{c\tau}(M) = C^\infty(B)Z;$$

$$\mathcal{I}_{s\tau}(M) = \mathcal{I}_{sc\tau}(M) = \mathbb{R}Z.$$

(ii) If M is an \mathbb{R}^1 -bundle $M = \mathbb{R}^1 \times B$, then

$$\mathfrak{A} = C^\infty(B) + C^\infty(B)x,$$

where x is the standard coordinate of \mathbb{R}^1 , that is, x is the C^∞ -function on M such that $Zx = 1$ and $Yx = 0$ for any $Y \in \mathfrak{A}(B)$. In this case, $Z = \partial/\partial x$ and $\tau = dx$. Moreover,

$$\begin{aligned}\mathcal{I}_\tau(M) &= C^\infty(B)Z, & \mathcal{I}_{c\tau}(M) &= \mathcal{A}Z, \\ \mathcal{I}_{s\tau}(M) &= \mathbb{R}Z & \text{and} & \mathcal{L}_{sc\tau}(M) = \mathbb{R}Z + \mathbb{R}xZ.\end{aligned}$$

Proof. Let $\phi \in C^\infty(B)$. Then, we get

$$L_{\phi Z}\tau = \phi L_Z\tau + d\phi \wedge i_Z\tau = d\phi \in \mathcal{I}(M, \mathcal{F}).$$

Hence, $C^\infty(B)Z \subset \mathcal{I}_\tau(M)$ and $\mathbb{R}Z \subset \mathcal{I}_{s\tau}(M)$.

Conversely, let $X \in \mathcal{I}_\tau(M)$. Put $\psi = i_X\tau \in C^\infty(B)$, then $X = \psi Z$ and $d\psi = L_X\tau \in \mathcal{I}(M, \mathcal{F})$, so $\psi \in C^\infty(B)$. Hence, $\mathcal{I}_\tau(M)$ coincides with $C^\infty(B)Z$. Similarly, $\mathcal{I}_{s\tau}(M)$ coincides with $\mathbb{R}Z$.

By using the formula

$$L_{\psi Z}\tau = d\psi \equiv (Z\phi)\tau \pmod{\mathcal{I}(M, \mathcal{F})},$$

We can get similarly as above that $\mathcal{I}_{c\tau}(M)$ coincides with $\mathcal{A}Z$

Consider the set $\mathcal{A}' = \{ \psi \in C^\infty(M) ; Z\psi \in \mathbb{R} \}$, then we get that $\mathcal{I}_{sc\tau}(M)$ coincides with $\mathcal{A}'Z$.

By similar arguments as in §5.3 of [K-4], we can show that $\mathcal{A} = C^\infty(B)$ and $\mathcal{A}' = \mathbb{R}$ in the case that M is an S^1 -bundle.

Q.E.D.

Assume that (M, \mathcal{F}) is a product foliation $M = G \times B$ with $G = S^1$ or \mathbb{R}^1 , then we get

Proposition 4. (i) $[Z, \mathfrak{X}(B)] = 0$, and for $G = \mathbb{R}^1$, $[xZ, \mathfrak{X}(B)] = 0$.

(ii) Let $X \in \mathcal{L}_\tau(M)$. Assume that $[X, Y] = 0$ for any $Y \in \mathfrak{X}(B)$.

Then, $X = aZ$ for some constant $a \in \mathbb{R}$. Hence, the center of $\mathcal{L}_\tau(M)$ coincides with $\mathbb{R}Z$.

(iii) Let $X \in \mathcal{L}_\tau(M)$. Assume that $[X, Y] \in \mathcal{I}_\tau(M)$ for any $Y \in \mathfrak{X}(B)$. Then, X is vertical, hence $X \in \mathcal{I}_\tau(M)$.

(iv) Let $G = \mathbb{R}^1$ and $X \in \mathcal{L}_{c\tau}(M)$. Assume that $[X, Y] = 0$ for any $Y \in \mathcal{I}_{c\tau}(M)$. Then, $X = 0$.

Moreover, let D be a derivation of $\mathcal{L}_{c\tau}(M)$. If $X \in \mathcal{L}_{c\tau}(M)$ vanishes on \tilde{W} for an open set W of B , then $D(X)$ vanishes on \tilde{W} .

Proof. (i) Since M is a product manifold, Z is given by the standard unit vector field on G .

(iii) Let p be a point of M . Take a coordinate neighborhood $(W; w_1, \dots, w_q)$ of B around $\pi(p)$ and an open set W' of B such that $p \in \bigcup_{\alpha=1}^q W' \subset \bar{W}' \subset W$. Take vector fields Y_α ($1 \leq \alpha \leq q$), $J \in \mathfrak{X}(B) \subset \mathcal{I}_\tau(M)$ such that $Y_\alpha = \partial_\alpha$, $J = \sum_{\alpha=1}^q w_\alpha \partial_\alpha$ on \tilde{W}' , and the supports of Y_α and J are contained in \tilde{W} .

Write X on \tilde{W}' as $X = \phi(w)Z + \sum_{\alpha=1}^q \phi_\alpha(w) \partial_\alpha$ with ϕ and $\phi_\alpha \in C^\infty(W)$. Then we get that on \tilde{W}'

$$0 \equiv [Y_\alpha, X] \equiv \sum_{\beta=1}^q (\partial_\alpha \phi_\beta) \partial_\beta,$$

$$0 \equiv [X, J] \equiv \sum_{\beta=1}^q \{ \phi_\beta - \sum_{\alpha=1}^q (\partial_\alpha \phi_\beta) \} \partial_\beta$$

modulo $\mathcal{I}_\tau(M)$, and hence $\partial_\alpha \phi_\beta = 0$ and $\phi_\beta = 0$. Thus, X is written on \tilde{W}' as $X = \phi(w)Z$. Therefore, X is expressed on M as $X = \phi(b)Z$ for some $\phi \in C^\infty(B)$, that is, $X \in \mathcal{I}_\tau(M)$.

(ii) Moreover, we get that $0 = [Y_\alpha, X] = (\partial_\alpha \phi)Z$ on \tilde{W}' , and so ϕ is constant on \tilde{W}' , hence on M , since the base space B is connected.

Since $[Z, \mathfrak{A}(B)] = 0$ and $\mathcal{I}_\tau(M)$ is abelian, the center of $\mathcal{L}_\tau(M)$ coincides with $\mathbb{R}Z$.

(iv) Write X as $X \equiv \phi Z + \psi xZ \pmod{\mathfrak{A}(B)}$. Then we get

$$0 = [Z, X] = \psi Z \quad \text{and} \quad 0 = [X, xZ] = \phi Z,$$

hence $\phi = \psi = 0$, that is, $X \in \mathfrak{A}(B)$. For any $\phi \in C^\infty(B)$,

$$0 = [X, \phi Z] = (X\phi)Z,$$

so $X\phi = 0$. This implies that $X = 0$, since $\phi \in C^\infty(B)$ is arbitrary taken.

From this, we can prove the last statement by similar arguments as in the proof of Proposition 2.4 of [K-1].

Q.E.D.

Since $\mathcal{I}_\tau(M) = C^\infty(B)Z$ and $\mathcal{I}_{s\tau}(M) = \mathbb{R}Z$ are abelian, we can easily determine their derivation algebras:

Theorem 5. (i) $H^1(\mathcal{I}_{s\tau}(M); \mathcal{I}_{s\tau}(M)) \cong \mathcal{D}er(\mathcal{I}_{s\tau}(M)) \cong \mathbb{R}$.

$$H^1(\mathcal{I}_\tau(M); \mathcal{I}_\tau(M)) \cong \mathcal{D}er(\mathcal{I}_\tau(M)) \cong \mathcal{L}in(C^\infty(B), C^\infty(B)),$$

where $\mathcal{L}in(C^\infty(B), C^\infty(B))$ is the space of all linear mappings of $C^\infty(B)$ to itself.

(ii) Let D be a derivation of $\mathcal{I}_\tau(M)$ given as $D = \text{ad}X$ for some vector field X on M . Then, $X \in \mathcal{L}_{c\tau}(M)$ and X is uniquely determined modulo $\mathcal{I}_\tau(M)$. Hence, the algebra of natural outer derivations of $\mathcal{I}_\tau(M)$ (see §6.5 of [K-4]) is isomorphic to $\mathcal{L}_{c\tau}(M)/\mathcal{I}_\tau(M)$.

Proof. (ii) Decompose the vector field X as $X = X_1 + X_2$, where $X_1 \in \mathfrak{A}(B)$ and $X_2 = fZ$ for some $f \in C^\infty(M)$. Define $g \in C^\infty(B)$ as $D(Z) = gZ$, then $gZ = [X, Z] = -(Zf)Z$, so $Zf = -g$. This means that $f \in \mathcal{A}$, that is, $X_2 \in \mathcal{I}_{c\tau}(M)$.

Moreover, since $\mathcal{I}_\tau(M)$ is an ideal of $\mathcal{L}_{c\tau}(M) = \mathcal{I}_{c\tau}(M) + \mathfrak{A}(B)$, X must be in $\mathcal{L}_{c\tau}(M)$.

The uniqueness of the vector field X modulo $\mathcal{I}_\tau(M)$ follows from Proposition 4 (iii).

Q.E.D.

§3. Bundle Foliation (II). This section is devoted to determine derivation algebras of $\mathcal{I}_{c\tau}(M)$, $\mathcal{L}_{c\tau}(M)$, $\mathcal{I}_{s\tau}(M)$ and $\mathcal{L}_{s\tau}(M)$ in the case that M is a product manifold $M = \mathbb{R}^1 \times B$, and leaves of \mathcal{F} are given as $\{y\} \times B$ ($y \in \mathbb{R}^1$) (c.f. Proposition 1).

Theorem 6. Let $L = \mathcal{I}_{s\tau}(M)$ or $\mathcal{L}_{s\tau}(M)$. Then,

$$H^1(L; L) = 0.$$

Proof. Let D be a derivation of L . Define constants a, b, c and $e \in \mathbb{R}$ as

$$D(Z) \equiv aZ + b xZ \quad \text{and} \quad D(xZ) \equiv cZ + e xZ$$

modulo $\mathfrak{A}(B)$. Let $W_1 = cZ - axZ \in \mathcal{I}_{sc\tau}(M)$ and $D_1 = D - \text{ad}W_1$, then D_1 is a derivation of L such that $D_1(\mathcal{I}_{sc\tau}(M)) \subset \mathfrak{A}(B)$. In fact, apply D_1 to the equality $Z = [Z, xZ]$, and note that $[\mathcal{I}_{sc\tau}(M), \mathfrak{A}(B)] = 0$. This completes the proof for the case of $L = \mathcal{I}_{sc\tau}(M)$.

Let $L = \mathcal{L}_{sc\tau}(M)$. Since $\mathfrak{A}(B)$ is a perfect ideal of L , we get by Proposition 7 of [K-5] that the restriction of D_1 to $\mathfrak{A}(B)$ is a derivation of $\mathfrak{A}(B)$. Then, by F.Takens [T-1], there exists a unique vector field $W_2 \in \mathfrak{A}(B)$ such that $D_1 = \text{ad}W_2$ on $\mathfrak{A}(B)$.

Let $D_2 = D_1 - \text{ad}W_2$, then D_2 is a derivation of L such that $D_2(\mathfrak{A}(B)) = 0$ and $D_1(\mathcal{I}_{sc\tau}(M)) \subset \mathfrak{A}(B)$. Apply D_2 to $[\mathcal{I}_{sc\tau}(M), \mathfrak{A}(B)] = 0$, then we get $[D_2(\mathcal{I}_{sc\tau}(M)), \mathfrak{A}(B)] = 0$. This implies that $D_2(\mathcal{I}_{sc\tau}(M)) = 0$, from Proposition 4(ii).

Q.E.D.

Theorem 7. Let D be a derivation of $\mathcal{I}_{c\tau}(M)$ or $\mathcal{L}_{c\tau}(M)$. Then, there exists a unique vector field W on M such that $D = \text{ad}W$. Moreover, $W \in \mathcal{L}_{c\tau}(M)$. Hence

$$\begin{aligned} H^1(\mathcal{L}_{c\tau}(M); \mathcal{L}_{c\tau}(M)) &= 0, \\ H^1(\mathcal{I}_{c\tau}(M); \mathcal{I}_{c\tau}(M)) &\cong \mathcal{L}_{c\tau}(M) / \mathcal{I}_{c\tau}(M) \cong \mathfrak{A}(B). \end{aligned}$$

Proof. Let D be a derivation of $L = \mathcal{I}_{c\tau}(M)$ or $\mathcal{L}_{c\tau}(M)$. Apply D to equalities $\phi Z = [\phi Z, xZ]$ for $\phi \in C^\infty(B)$, then we get

$$D(\phi Z) = [D(\phi Z), xZ] + [\phi Z, D(xZ)] \in \mathcal{I}_\tau(M),$$

which implies that $D(\mathcal{I}_\tau(M)) \subset \mathcal{I}_\tau(M)$.

Define $\phi_1 \in C^\infty(B)$ as $D(Z) = \phi_1(b)Z$ and put $W_1 = -\phi_1 xZ \in \mathcal{I}_{c\tau}(M)$, then $D(Z) = [W_1, Z]$. Let $D_1 = D - \text{ad}W_1$, then $D_1(Z) = 0$, and D_1 is a derivation of L .

Define $\phi_2 \in \mathfrak{A}$ as $D_1(xZ) \equiv \phi_2 Z \pmod{\mathfrak{A}(B)}$, and apply D_1 to the equality $[Z, xZ] = Z$. Then we get that $Z\phi_2 = 0$, hence $\phi_2 \in C^\infty(B)$. Let $W_2 = \phi_2 Z \in \mathcal{I}_\tau(M)$ and $D_2 = D_1 - \text{ad}W_2$, then D_2 is a derivation of L such that $D_2(Z) = 0$ and $D_2(xZ) \in \mathfrak{A}(B)$.

For any $\psi \in C^\infty(B)$, define $\psi_i \in C^\infty(B)$ ($i = 1, 2, 3, 4$) as

$$D_2(\psi Z) \equiv \psi_1 Z + \psi_2 xZ \quad \text{and} \quad D_2(xZ) \equiv \psi_3 Z + \psi_4 xZ \pmod{\mathfrak{A}(B)}.$$

Apply D_2 to the equalities $[Z, \psi Z] = 0$, $\psi Z = [Z, \psi xZ]$ and $[xZ, \psi xZ] = 0$, then we get

$$0 \equiv [Z, \psi_1 Z + \psi_2 xZ] = \psi_2 Z,$$

$$\begin{aligned}\psi_1 Z + \psi_2 xZ &\equiv [Z, \psi_3 Z + \psi_4 xZ] = \psi_4 Z, \\ 0 &\equiv [D_2(xZ), \psi xZ] + [xZ, \psi_3 Z + \psi_4 xZ] = x(D_2(xZ)\psi)Z - \psi_3 Z\end{aligned}$$

modulo $\mathfrak{A}(B)$, hence we get that

$$\psi_2 = \psi_3 = 0, \quad \psi_1 = \psi_4 \quad \text{and} \quad D_2(xZ)\psi = 0.$$

The last equality implies that the vector field $D_2(xZ) \in \mathfrak{A}(B)$ vanishes, since $\psi \in C^\infty(B)$ is arbitrarily taken.

Now, denote by \bar{D} the linear map of $C^\infty(B)$ to itself, given by

$$D_2(\psi Z) \equiv \bar{D}(\psi)Z \quad (\text{mod } \mathfrak{A}(B)).$$

Apply D_2 to the equalities $[\phi Z, \psi xZ] = \phi \psi Z$ ($\phi, \psi \in C^\infty(B)$), then we get

$$\begin{aligned}\bar{D}(\phi \psi)Z &\equiv [\bar{D}(\phi)Z, \psi xZ] + [\phi Z, \bar{D}(\psi)xZ] \quad (\text{mod } \mathfrak{A}(B)) \\ &= \{ \bar{D}(\phi)\psi + \phi \bar{D}(\psi) \} Z.\end{aligned}$$

So,

$$\bar{D}(\phi \psi) = \bar{D}(\phi)\psi + \phi \bar{D}(\psi),$$

which means that \bar{D} is a derivation of $C^\infty(B)$. Hence, it is well-known that \bar{D} is realized by a vector field $W_3 \in \mathfrak{A}(B)$ as $\bar{D}(\psi) = W_3 \psi$ (c.f. Lemma 3.4. of [K-2]). Therefore we get the following equalities modulo $\mathfrak{A}(B)$:

$$\begin{aligned}D_2(\psi Z) &\equiv \bar{D}(\psi)Z = [W_3, \psi Z], \\ D_2(\psi xZ) &\equiv \bar{D}(\psi)xZ = [W_3, \psi xZ].\end{aligned}$$

Let $D_3 = D_2 - \text{ad}W_3$, then D_3 is a derivation of L such that $D_3(Z) = D_3(xZ) = 0$ and $D_3(\mathcal{I}_{CT}(M)) \subset \mathfrak{A}(B)$.

Let $W = W_1 + W_2 + W_3 \in \mathcal{L}_{CT}(M)$. Then D coincides with $\text{ad}W$ on L for the case $L = \mathcal{I}_{CT}(M)$. In the following, we show that $D = \text{ad}W$ on L also for $L = \mathcal{L}_{CT}(M)$.

For any $\psi \in C^\infty(B)$,

$$D_3(\psi Z) = D_3([Z, \psi xZ]) = [Z, D_3(\psi xZ)] = 0,$$

hence it follows that $D_3(\mathcal{I}_T(M)) = 0$.

Take a vector field $X \in \mathfrak{A}(B)$, and apply D_3 to $[X, Z] = [X, xZ] = 0$, then

$$[D_3(X), Z] = [D_3(X), xZ] = 0,$$

hence $D_3(X) \in \mathfrak{A}(B)$. Apply D_3 to $(X\psi)Z = [X, \psi Z]$, then we get

$$0 = [D_3(X), \psi Z] = (D_3(X)\psi)Z,$$

which implies that $D_3(X) = 0$, from the arbitrariness of $\psi \in C^\infty(B)$. Thus, $D_3(\mathfrak{A}(B)) = 0$.

It remains to show that $D_3(\psi xZ) = 0$ for any $\psi \in C^\infty(B)$. For any point p of M , take such open sets W, W' of B , coordinates $\{w_\alpha\}$ on W , and vector fields $Y_\alpha, J \in \mathfrak{A}(B)$ as in the proof of Proposition 4(iii). Take the functions $\bar{w}_\alpha \in C^\infty(B)$ such that $\bar{w}_\alpha = w_\alpha$ on \hat{W}' , and $\bar{w}_\alpha = 0$ on \hat{W}^c . Define the functions $\phi_{\alpha\beta} \in C^\infty(B)$ with their supports contained in \hat{W} as $D_3(\bar{w}_\alpha xZ) = \sum_\beta \phi_{\alpha\beta} Y_\beta$ on \hat{W}' . Then, we get that on \hat{W}'

$$[Y_\alpha, \bar{w}_\alpha xZ] = \phi_{\alpha\beta} Z \quad \text{and} \quad [J, \bar{w}_\alpha xZ] = \bar{w}_\alpha xZ.$$

Apply D_3 to $[Y_\beta, \bar{w}_\alpha xZ]$, then we get from Proposition 4(iv) that

$$0 = [Y_\beta, D_3(\bar{w}_\alpha xZ)] = \sum_\gamma (\partial_\gamma \phi_{\alpha\beta}) \quad \text{on } \hat{W}',$$

hence $\phi_{\alpha\beta}$ are constants on \hat{W}' . So, by applying D_3 to $[J, \bar{w}_\alpha xZ]$, we get

$$\sum_\beta \phi_{\alpha\beta} \partial_\beta = [J, \sum_\beta \phi_{\alpha\beta} \partial_\beta] = -\sum_\beta \phi_{\alpha\beta} \partial_\beta \quad \text{on } \hat{W}',$$

hence $\phi_{\alpha\beta}$ vanishes on \hat{W}' . Thus, $D_3(\bar{w}_\alpha xZ)$ vanishes on \hat{W}' . Therefore $D_3(\psi xZ)$ vanishes on \hat{W}' for any $\psi \in C^\infty(B)$, because the vector field $[\psi Y_\alpha, \bar{w}_\alpha xZ]$ coincides with ψxZ on \hat{W}' . Hence, $D_3(\psi xZ)$ vanishes everywhere on M . Finally we get that $D_3 = 0$, that is, $D = \text{ad} W$ on $\mathcal{L}_{c\tau}(M)$.

The uniqueness of the vector field W follows from Proposition 4(iv). Q.E.D.

§4. Foliations on Torus. Consider the foliation \mathcal{F}'_λ ($\lambda \in \mathbb{R}$) on $\mathbb{R}^2 = \{(x, y)\}$ given by parallel lines $\{y = \lambda x + c\}$ $c \in \mathbb{R}$ and identify the torus T^2 with the quotient space of \mathbb{R}^2 by \mathbb{Z}^2 , then we get the linear foliation \mathcal{F}_λ on T^2 with the slope λ and the s.p.u. structure τ_λ on $(T^2, \mathcal{F}_\lambda)$ as $\tau_\lambda = d_v$, where $v = (x + \lambda^{-1}y) / (1 + \lambda^{-2})^{1/2}$.

By similar arguments in Example 4 of §5.4. of [K-4], we get

Theorem 8. *Let $\mathcal{F} = \mathcal{F}_\lambda$ and $\tau = \tau_\lambda$, and assume that the slope λ is irrational. Then, $Z = \partial/\partial v = \partial_v$, $C^\infty(T^2)^\mathcal{F} = \mathbb{R}$,*

$$\hat{\mathcal{F}}_\tau(T^2) = \mathcal{F}_{c\tau}(T^2) = \mathcal{F}_{s\tau}(T^2) = \mathcal{F}_{sc\tau}(T^2) = \mathbb{R}Z \cong \mathbb{R},$$

$$\mathcal{L}_\tau(T^2) = \mathcal{L}_{c\tau}(T^2) = \mathcal{L}_{s\tau}(T^2) = \mathcal{L}_{sc\tau}(T^2) = \mathbb{R}Z \oplus \mathbb{R} \partial_y \cong \mathbb{R}^2.$$

These Lie algebras are abelian and their derivation algebras are of all endomorphisms:

$$\begin{aligned} H^1(\mathcal{I}_\tau(T^2, \mathcal{F}_\lambda); \mathcal{I}_\tau(T^2, \mathcal{F}_\lambda)) &\cong \mathbb{R} \\ H^1(\mathcal{L}_\tau(T^2, \mathcal{F}_\lambda); \mathcal{L}_\tau(T^2, \mathcal{F}_\lambda)) &\cong \mathcal{L}in(\mathbb{R}^2, \mathbb{R}^2). \end{aligned}$$

Moreover, any derivation which is not identically zero is properly outer.

Assume that the slope λ is rational and expressed as a fraction $\lambda = n/m$ of integers n and m with $m > 0$. Here we choose m and n as mutually prime, then there exists an integral solution (h, k) of the equation $km - hn = 1$ with $h > 0$. Denote by S_λ the set

$$\{(x, y) \in \mathbb{R}^2; nx \leq my \leq nx + 1, \quad kx - 1 \leq hy \leq kx\}.$$

Then the foliation \mathcal{F}'_λ restricted on the set S_λ is, obviously, a product foliation. Since the set S_λ is also a fundamental domain, the foliation \mathcal{F}_λ on T^2 is a product foliation $T^2 = S^1 \times S^1$. Now consider the product foliation \mathcal{F} on $T^2 = S^1_x \times S^1_y = \{(x, y) \in \mathbb{R}^2 / (2\pi\mathbb{Z})^2\}$ given by $\{y = \text{constant}\}$, and the s.p.u. structure $\tau = dy$. Then we get that $Z = \partial/\partial x = \partial_x$, $C^\infty(T^2)^{\mathcal{F}} = C^\infty(S^1_y)$,

$$\begin{aligned} \mathcal{I}_\tau(T^2) &= \mathcal{I}_{c\tau}(T^2) = C^\infty(S^1_y)Z, \\ \mathcal{L}_\tau(T^2) &= \mathcal{L}_{c\tau}(T^2) = \mathcal{I}_\tau(T^2) + C^\infty(S^1_y)\partial_y, \\ \mathcal{I}_{s\tau}(T^2) &= \mathcal{I}_{sc\tau}(T^2) = \mathbb{R}Z, \\ \mathcal{L}_{s\tau}(T^2) &= \mathcal{L}_{sc\tau}(T^2) = \mathcal{I}_{s\tau}(T^2) \oplus C^\infty(S^1_y). \end{aligned}$$

We already know the derivation algebras of $\mathcal{I}_{s\tau}(T^2)$ and $\mathcal{I}_\tau(T^2)$ (Theorem 5).

Theorem 9. (i) Denote by D_c the derivation of $\mathcal{L}_{s\tau}(T^2)$ such that $D_c(Z) = cZ$ and $D_c(\mathfrak{A}(S^1_y)) = 0$. Then, $D_c(c \neq 0)$ is properly outer.

(ii) Let D be a derivation of $\mathcal{L}_{s\tau}(T^2)$. Then, there exists a vector field W on M and a constant c such that $D = \text{ad}W + D_c$. Moreover, $W \in \mathcal{L}_{s\tau}(T^2) = \mathcal{I}_{s\tau}(T^2) + \mathfrak{A}(S^1_y)$ is unique modulo $\mathcal{I}_{s\tau}(T^2)$.

Proof. Let D be a derivation of $\mathcal{L}_{s\tau}(T^2)$. Since $\mathfrak{A}(S^1_y)$ is a perfect ideal, the restriction of D to $\mathfrak{A}(S^1_y)$ gives a derivation of $\mathfrak{A}(S^1_y)$, hence is given by a vector field $W \in \mathfrak{A}(S^1_y)$ as $D = \text{ad}W$ on $\mathfrak{A}(S^1_y)$.

Let $D_1 = D - \text{ad}W$, then D_1 is a derivation of $\mathcal{L}_{s\tau}(T^2)$ such that $D_1(\mathfrak{A}(S^1_y)) = 0$. Define the constant $c \in \mathbb{R}$ and the function $\phi(y) \in C^\infty(S^1_y)$ as $D(Z) = cZ + \phi(y)\partial_y$. Apply D_1 to $[\partial_y, Z] = 0$, then we get that $0 = [\partial_y, cZ + \phi(y)\partial_y] = (\partial_y\phi)(y)\partial_y$, hence ϕ is constant. Apply D_1 to $[Z, (\sin y)\partial_y] = 0$, then we get that $0 = [cZ + \phi\partial_y, (\sin y)\partial_y] = \phi(\cos y)\partial_y$,

hence ϕ is zero. Thus we get that D_1 coincides with D_c .

Q.E.D.

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