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On a Normal Form of Unimodal Maps of $[0,1]$

By

Hiroshi Ishitani *

We consider the topological conjugacy of a class of continuous single-humped mappings of $[0, 1]$ onto itself. H. Rüssmann and E. Zehnder ([1]) proved that f satisfying $f(x)=f(1-x)$ can be put into the normal form $N=\varphi^{-1}\circ f\circ\varphi$

(1) $$N(x)=\begin{cases} 2x & \text{if } 0\leqq x\leqq 1/2 \\ 2(1-x) & \text{if } 1/2<x\leqq 1 \end{cases}$$

by means of a homeomorphism φ of $[0, 1]$, under some additional conditions. In this note we generalize their results by using simpler method. The statement is as follows:

Theorem. *Let* $f: [0, 1] \to [0, 1]$ *be continuous and satisfying*

(2) $$f(0)=f(1)=0,\ f(a)=1$$

for some $0<a<1$, *and* $f_{|[0,a]}$ *is increasing and* $f_{|[a,1]}$ *deceasing. Assume that for* $x\neq y$ *there exists a nonnegative integer* n *such that* $f^n(x)\leqq a<f^n(y)$ *or* $f^n(y)\leqq a<f^n(x)$. *Then there is a strictly increasing continuous map* φ *of* $[0,1]$ *onto itself such that*

(3) $$N(\varphi(x))=\varphi(f(x))$$

with N *as defined by* (1). *Moreover, if* f *satisfies*

(4) $$|f(x)-f(y)|\leqq 2^{1/\sigma}|x-y|$$

for some σ *with* $0<\sigma<1$ *then* φ *is Hölder continuous with exponent* σ.

Proof. The idea to prove this statement is as follows. Roughly, the relation (3) implies $\varphi=N^{-1}\circ\varphi\circ f$. Therefore, if we could define the operator T by $T\alpha(x)=N^{-1}(\alpha(f(x)))$, then φ must be the fixed point of T.

Now let us prove our result precisely. First we introduce the complete

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* Depantment of Mathematics, Mie University.

metric space

$$M = \{\alpha\colon [0,\ 1] \to [0,\ 1],\quad \alpha \text{ continuous, increasing, and satisfying } \alpha(0)=0,\quad \alpha(1)=1\}$$

with the metric

$$\| \alpha - \beta \| = \max \{ | \alpha(x) - \beta(x) | : 0 \leqq x \leqq 1 \}$$

Let T be the operator on M defined by

$$T(\alpha)(x) = \begin{cases} 2^{-1}\alpha(f(x)) & \text{if } 0 \leqq x \leqq a \\ 2^{-1}(2-\alpha(f(x))) & \text{if } a < x \leqq 1 \end{cases} \tag{5}$$

Then we can easily show that $T(M) \subseteqq M$. T is a contraction, because

$$\| T(\alpha) - T(\beta) \| = 2^{-1} \| \alpha - \beta \|, \quad \alpha \in M, \quad \beta \in M$$

holds in view of (5). Hence there is the unique fixed point φ of T in M. It remains to show that φ is a one to one mapping of $[0, 1]$. Assume $x \neq y$ and $\varphi(x) = \varphi(y)$ then $T(\varphi)(x) = T(\varphi)(y)$ and hence $T^n(\varphi)(x) = T^n(\varphi)(y)$ for all integers $n \geqq 1$. Since $x \neq y$ we can pick the smallest k for which $f^k(x) \leqq a < f^k(y)$ or $f^k(y) \leqq a < f^k(x)$ holds. In the former case, we can derive

$$N^k(T^{k+1}(\varphi))(x) = 2^{-1}\varphi(f^{k+1}(x))$$

and

$$N^k(T^{k+1}(\varphi))(y) = 2^{-1}(2 - \varphi(f^{k+1}(y))).$$

Therefore, noting that $2^{-1}w \leqq 2^{-1} < 2^{-1}(2-z)$ for any $0 \leqq w \leqq 1$ and $0 \leqq z < 1$, we get $T^{k+1}(\varphi)(x) \neq T^{k+1}(\varphi)(y)$ which gives a contradiction. The latter case can be similarly treated. Therefore, if $x < y$ then $\varphi(x) < \varphi(y)$.

We define the closed subset H^A_σ of M by

$$H^A_\sigma = \{ \alpha \in M \mid \sup_{0 \leqq x < y \leqq 1} | \alpha(x) - \alpha(y) | / | x - y |^\sigma \leqq A \}$$

for σ defined in (4) and $A > 0$. We get

$$| T(\alpha)(x) - T(\alpha)(y) | / | x - y |^\sigma \leqq 2^{-1} A \mid f(x) - f(y) |^\sigma / | x - y |^\sigma$$

for $\alpha \in M$ and $0 \leqq x < y \leqq 1$. Hence, Since $| f(x) - f(y) | \leqq 2^{-1/\sigma} | x - y |$ we find $T(H^A_\sigma) \subseteqq H^A_\sigma$, so we can find the fixed point φ in H^A_σ. The proof is therefore complete.

Example 1. Let $f(x) = 4x(1-x)$, which appears in the biological science. Then it satisfies the assumptions of our theorem. Hence there exists a Hölder continuous homeomorphism φ of $[0, 1]$ for which $N(\varphi(x)) = \varphi(f(x))$. As is well known,

$\varphi(x)=2\pi^{-1}\sin^{-1}(x^{1/2})$ which is absolutely continuous, see [2] .

Example 2. Let us put

$$f(x)=\begin{cases} px & \text{if } 0\leqq x\leqq 1/p \\ (p/(p-1))(1-x) & \text{if } 1/p<x\leqq 1. \end{cases}$$

Then it also satisfies the assumptions. However we can easily see that φ is Hölder continuous but not absolutely continuous.

References

[1] H. Rüssmann and E. Zehnder: *On a normal form of symmetric maps of* [*0, 1*] , Commun. Math. Phys. , 72(1980), 49-53.

[2] R. M. May: *Simple mathematical models with very complicated dynamics,* Nature, 261(1976), 459-467.