

Notes on Partially Unimodular Structures (II)

by

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This note is a continuation of Part I ([K-6]), and here we use the same notations.

§ 5. Foliations on Torus (II). In this section, we study derivations of $\mathcal{L}_\tau(T^2)$ for the linear foliation \mathcal{F}_λ on the 2-dimensional torus T^2 with a rational slope λ and the corresponding s. p. u. structure τ_λ . Due to the remarks in § 4, we may consider the product foliation \mathcal{F} on $T^2 = S_x^1 \times S_y^1 = \{(x, y) \in \mathbb{R}^2 / (2\pi\mathbb{Z})^2\}$ given by $\{y = \text{constant}\}$, and the s. p. u. structure $\tau = dx$. Then, $Z = \partial_x$, $C^\infty(T^2) \not\cong C^\infty(S_y^1)$, $\mathcal{T}_\tau(T^2) = \mathcal{T}_{cr}(T^2) = C^\infty(S_y^1) \partial_x$ and $\mathcal{L}_\tau(T^2) = \mathcal{L}_{cr}(T^2) = \mathcal{T}_\tau(T^2) \oplus \mathfrak{A}(S_y^1) = C^\infty(S_y^1) \partial_x \oplus C^\infty(S_y^1) \partial_y$.

At first, we get easily the following

Lemma 10. (i) If $X \in \mathcal{L}_\tau(T^2)$ satisfies the relations $[X, \partial_y] = [X, \text{sing } \partial_y] = 0$, then X is expressed as $X = a\partial_x$ for some constant a .

(ii) The center \mathfrak{z} of $\mathcal{L}_\tau(T^2)$ coincides with $\mathbb{R}\partial_x$.

Proposition 11. Define the linear mappings A, B and C of $\mathcal{L}_\tau(T^2)$ to itself as

$$\begin{aligned} A(\mathcal{T}_\tau(T^2)) &= 0, & A(\phi(y)\partial_y) &= (\partial_y\phi)(y)\partial_x; \\ B(\mathcal{T}_\tau(T^2)) &= 0, & B(\phi(y)\partial_y) &= \phi(y)\partial_x; \\ C(\phi(y)\partial_x) &= \phi(y)\partial_x, & C(\mathfrak{A}(S_y^1)) &= 0 \end{aligned}$$

for any $\phi \in C^\infty(S_y^1)$. Then, A, B and C are derivations of $\mathcal{L}_\tau(T^2)$, and they are properly outer.

Moreover, $[A, B] = 0$, $[C, A] = A$ and $[C, B] = B$.

Proof. Since $[\phi Z, \psi Z] = 0$, $[\phi\partial_y, \psi Z] = \phi(\partial_y\psi)Z$ and $[\phi\partial_y, \psi\partial_y] = \{\phi(\partial_y\psi) - \psi(\partial_y\phi)\}\partial_y$ for any $\phi, \psi \in C^\infty(S_y^1)$, we can easily check the derivation property of A, B and C . And also by simple calculations, we can show their commutation relations.

Now, we show that A, B and C are properly outer.

At first, assume that a vector field X on T^2 gives the derivation A as $A = \text{ad}X$. Since $[X, \partial_x] = [X, \partial_y] = [X, \sin y \partial_x] = 0$, we get by the similar arguments as Lemma 10 that X is written as $X = \alpha \partial_x$ ($\alpha \in \mathbf{R}$).

Apply A to the vector field $\sin y \partial_y$, then

$$\cos y \partial_y = A(\sin y \partial_y) = [\alpha \partial_x, \sin y \partial_y] = 0.$$

This is impossible, that is, A is properly outer.

Secondly, assume that $X \in \mathfrak{N}(T^2)$ gives the derivation B as $B = \text{ad}X$. Since $[X, \partial_x] = 0$, the coefficient functions of X are independent of x , so X is written as $X = \phi(y) \partial_x + \psi(y) \partial_y$ ($\phi, \psi \in C^\infty(S_y^1)$).

On the other hand, the function ϕ has the constant derivative -1 . In fact,

$$\partial_x = B(\partial_y) = [X, \partial_y] = -(\partial_y \phi) \partial_x - (\partial_y \psi) \partial_y.$$

This contradicts that periodic functions have not non-zero constant derivatives.

Finally, assume that $X \in \mathfrak{N}(T^2)$ gives derivation C as $C = \text{ad}X$.

Since $[X, \mathfrak{N}(S_y^1)] = 0$, it follows easily that X is written as $X = f(x) \partial_x$ for some $f \in C^\infty(S_x^1)$.

Apply C to the vector field ∂_x , then

$$\partial_x = C(\partial_x) = [X, \partial_x] = -(\partial_x f)(x) \partial_x.$$

hence $\partial_x f = -1$. This contradicts the periodicity of f .

Q. E. D.

Theorem 12. (i) Let D be a derivation of $\mathcal{L}_\tau(T^2)$. Then, there exists a vector field W on T^2 and constants a, b and $c \in \mathbf{R}$ such that

$$D = \text{ad}W + aA + bB + cC.$$

Moreover, the constants a, b and c are uniquely determined, and W is in $\mathcal{L}_\tau(T^2)$ and is unique modulo center $\mathfrak{z} = \mathbf{R}\partial_x$ of $\mathcal{L}_\tau(T^2)$.

(ii) There are no natural outer derivations of $\mathcal{L}_\tau(T^2)$. The first cohomology $H^1 = H^1(\mathcal{L}_\tau(T^2); \mathcal{L}_\tau(T^2))$ is of dimension 3, and as a Lie algebra H^1 has three generators A, B and C with relations $[A, B] = 0$, $[C, A] = A$ and $[C, B] = B$.

Proof. Let D be a derivation of $\mathcal{L}_\tau(T^2)$.

Let D' be the linear mapping of $\mathfrak{N}(S_y^1)$ to itself which assigns to $X \in \mathfrak{N}(S_y^1)$ the $\mathfrak{N}(S_y^1)$ -part of $D(X)$. Then D' gives a derivation of $\mathfrak{N}(S_y^1)$, because $\mathcal{I}_\tau(T^2)$ is an ideal. Hence by F. Takens[T-1], there exists a unique vector field $W_1 \in \mathfrak{N}(S_y^1)$ such that $D'(X) = [W_1, X]$ for any $X \in \mathfrak{N}(S_y^1)$.

Let $D^1 = D - \text{ad}W_1$, then D^1 is a derivation of $\mathcal{L}_\tau(T^2)$ such that $D^1(\mathfrak{N}(S_y^1)) \subset \mathcal{I}_\tau(T^2)$.

Define the function $\phi_0(y) \in C^\infty(S_y^1)$ as $D_1(\partial_y) = \phi_0(y) \partial_x$. Put

$$b = (2\pi)^{-1} \oint \phi_0(y) dy \quad \text{and} \quad \Phi(y) = \int_0^y \{\phi_0(y) - b\} dy,$$

then the function $\Phi(y)$ is well-defined on S_y^1 .

Let $W_2 = -\Phi(y) \partial_x \in \mathcal{T}_\tau(T^2)$ and $D_2 = D_1 - \text{ad} w_2 - bB$, then D_2 is a derivation of $\mathcal{L}_\tau(T^2)$ such that $D_2(\mathfrak{N}(S_y^1)) \subset \mathcal{T}_\tau(T^2)$ and $D_2(\partial_y) = 0$. In fact,

$$D_2(\partial_y) D_1(\partial_y) + [W_2, \partial_y] - dB(\partial_y) = \{\phi_0(y) - (\partial_y \Phi)(y) - b\} \partial_x = 0.$$

Apply D_2 to the equalities $[\partial_x, \partial_y] = [\partial_x, \sin y \partial_y] = 0$, then by Lemma 10 we get that $D_2(\partial_x) = c \partial_x$ for some constant $c \in \mathbb{R}$.

Let $D_3 = D_2 - cC$, then D_3 is a derivation of $\mathcal{L}_\tau(T^2)$ such that $D_3(\mathfrak{N}(S_y^1)) \subset \mathcal{T}_\tau(T^2)$ and $D_3(\partial_x) = D_3(\partial_y) = 0$.

Define the functions ϕ_k, ϕ'_k, ψ_k and $\psi'_k \in C^\infty(S_y^1)$ as

$$D_3(\sin ky \partial_x) = \phi_k \partial_x + \phi'_k \partial_y \quad \text{and} \quad D_3(\cos ky \partial_x) = \psi_k \partial_x + \psi'_k \partial_y.$$

Apply D_3 to the equalities $k \sin ky \partial_x = [\cos ky \partial_x, \partial_y]$ and $k \cos ky \partial_x = [\partial_y, \sin ky \partial_x]$ ($k \in \mathbb{Z}$), then we get

$$\begin{aligned} k \phi_k \partial_x + k \phi'_k \partial_y &= [\psi_k \partial_x + \psi'_k \partial_y, \partial_y] = -(\partial_y \psi_k) \partial_x - (\partial_y \psi'_k) \partial_y, \\ k \psi_k \partial_x + k \psi'_k \partial_y &= [\partial_y, \phi_k \partial_x + \phi'_k \partial_y] = (\partial_y \phi_k) \partial_x + (\partial_y \phi'_k) \partial_y, \end{aligned}$$

hence $\{\phi_k, \psi_k\}$ and $\{\phi'_k, \psi'_k\}$ satisfy the conditions of the following lemma.

Lemma 13. *Let ϕ and ψ be functions satisfying the differential equations*

$$\partial_y \phi = k \psi \quad \text{and} \quad \partial_y \psi = -k \phi$$

for some integer k . Then, there are constants α and $\beta \in \mathbb{R}$ such that

$$\phi(y) = \alpha \sin ky + \beta \cos ky \quad \text{and} \quad \psi(y) = -\beta \sin ky + \alpha \cos ky.$$

Proof. The functions ϕ and ψ satisfy the equations $\partial_y^2 \phi = -k^2 \phi$ and $\partial_y^2 \psi = -k^2 \psi$. So, as is well known, ϕ and ψ are linear combinations of $\sin ky$ and $\cos ky$. Write ϕ and ψ as

$$\phi(y) = \alpha \sin ky + \beta \cos ky \quad \text{and} \quad \psi(y) = \gamma \sin ky + \delta \cos ky$$

for some constants α, β, γ and δ . Put them into the equation $\partial_y \phi = k \psi$, then

$$k \alpha \cos ky - k \beta \sin ky = k \gamma \sin ky + k \delta \cos ky,$$

hence $\alpha = \delta$ and $\beta = -\gamma$.

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Thus, there are constants $\alpha_k, \beta_k, \alpha'_k$ and β'_k such that

$$\phi_k(y) = \alpha_k \sin ky + \beta_k \cos ky, \quad \psi_k(y) = -\beta_k \sin ky + \alpha_k \cos ky,$$

$$\phi'_k(y) = \alpha'_k \sin ky + \beta'_k \cos ky \quad \text{and} \quad \psi'_k(y) = -\beta'_k \sin ky + \alpha'_k \cos ky.$$

Apply D_3 to the equality $k\partial_x = [\sin ky \partial_y, -\cos ky \partial_x] + [\cos ky \partial_y, \sin ky \partial_x]$, then

$$\begin{aligned} 0 &= [\sin ky \partial_y, (\beta_k \sin ky - \alpha_k \cos ky) \partial_x + (\beta'_k \sin ky - \alpha'_k \cos ky) \partial_y] \\ &\quad + [\cos ky \partial_y, (\alpha_k \sin ky + \beta_k \cos ky) \partial_x + (\alpha'_k \sin ky + \beta'_k \cos ky) \partial_y] \\ &= k \{ \beta_k \sin ky \cos ky + \alpha_k \sin^2 ky + \alpha_k \cos^2 ky - \beta_k \cos ky - \beta_k \cos ky \sin ky \} \partial_x \\ &\quad + k \{ \alpha'_k \sin^2 ky + \alpha'_k \cos^2 ky + \alpha'_k \cos^2 ky + \alpha'_k \sin^2 ky \} \partial_y \\ &= k(\alpha_k \partial_x + 2\alpha'_k \partial_y), \end{aligned}$$

hence $\alpha_k = \alpha'_k = 0$. Apply D_3 to the equality $\sin 2y \partial_x = 2[\sin y \partial_y, \sin y \partial_x]$, then

$$\begin{aligned} \beta_2 \cos 2y \partial_x + \beta'_2 \cos 2y \partial_y &= 2[\sin y \partial_y, \beta_1 \cos y \partial_x + \beta'_1 \cos y \partial_y] \\ &= -2\beta_1 \sin^2 y \partial_x - 2(\beta'_1 \sin^2 y + \beta_1 \cos^2 y) \partial_y = -2\beta_1 \sin^2 y \partial_x - 2\beta'_1 \partial_y, \end{aligned}$$

hence $\beta_2 \cos 2y = \beta_1 (\cos 2y - 1)$ and $\beta'_2 \cos 2y = -2\beta'_1$. These imply that $\beta_1 = \beta'_1 = 0$, that is, $D_3(\sin y \partial_x) = D_3(\cos y \partial_x) = 0$.

Let $\phi \in C^\infty(S_y^1)$. Then we get that $D_3(\phi \partial_x) = 0$, by the formula

$$\phi \partial_x = [\phi \sin y \partial_y, -\cos y \partial_x] + [\phi \cos y \partial_y, \sin y \partial_x].$$

Hence $D_3(\mathcal{I}_\tau(T^2)) = 0$.

Define the functions ϕ'' and $\psi'' \in C^\infty(S_y^1)$ as

$$D_3(\sin y \partial_y) = \phi''(y) \partial_x \quad \text{and} \quad D_3(\cos y \partial_y) = \psi''(y) \partial_x.$$

Apply D_3 to the equalities $\sin y \partial_y = [\cos y \partial_y, \partial_y]$ and $\cos y \partial_y = [\partial_y, \sin y \partial_y]$, then we get similarly as above that $\partial_y \phi'' = \psi''$ and $\partial_y \psi'' = -\phi''$. Then by Lemma 13, there are constants α'' and β'' such that

$$\phi''(y) = \alpha'' \sin y + \beta'' \cos y \quad \text{and} \quad \psi''(y) = -\beta'' \sin y + \alpha'' \cos y.$$

Apply D_3 to the equality $\partial_y = [\cos y \partial_y, \sin y \partial_y]$, then we get

$$\begin{aligned} 0 &= [(\alpha'' \cos y - \beta'' \sin y) \partial_x, \sin y \partial_y] + [\cos y \partial_y, (\alpha'' \sin y + \beta'' \cos y) \partial_x] \\ &= \alpha'' \partial_x, \end{aligned}$$

hence $\alpha'' = 0$.

Put $a = \beta''$ and let $D_4 = D_3 - aA$, then D_4 is a derivation of $\mathcal{L}_\tau(T^2)$ such that $D_4(\mathfrak{A}(S_y^1)) \subset \mathcal{I}_\tau(T^2)$ and $D_4(\partial_x) = D_4(\partial_y) = D_4(\sin y \partial_y) = D_4(\cos y \partial_y) = 0$.

Let $f \in C^\infty(S_y^1)$. Define the functions $f_i \in C^\infty(S_y^1)$ ($i = 0, 1, 2$) as

$$D_4(f \partial_y) = f_0 \partial_x, \quad D_4(f \sin y \partial_y) = f_1 \partial_x \quad \text{and} \quad D_4(f \cos y \partial_y) = f_2 \partial_x.$$

Apply D_4 to the equalities

$$2f \partial_y = [f \cos y \partial_y, \sin y \partial_y] - [f \sin y \partial_y, \cos y \partial_y],$$

$$\begin{aligned}
 2f \sin y \partial_y &= [f \cos y \partial_y, \partial_y] - [f \partial_y, \cos y \partial_y] \\
 2f \cos y \partial_y &= [f \partial_y, \sin y \partial_y] - [f \sin y \partial_y, \partial_y], \\
 [\partial_y, f \partial_y] + [f \sin y \partial_y, \sin y \partial_y] + [f \cos y \partial_y, \cos y \partial_y] &= 0,
 \end{aligned}$$

then we get

$$\begin{aligned}
 2f_0 \partial_x &= [f_2 \partial_x, \sin y \partial_y] - [f_1 \partial_x, \cos y \partial_y] = \{(\partial_y f_1) \cos y - (\partial_y f_2) \sin y\} \partial_x, \\
 2f_1 \partial_x &= [f_2 \partial_x, \partial_y] - [f_0 \partial_x, \cos y \partial_y] = \{(\partial_y f_0) \cos y - (\partial_y f_2)\} \partial_x, \\
 2f_2 \partial_x &= [f_0 \partial_x, \sin y \partial_y] - [f_1 \partial_x, \partial_y] = \{(\partial_y f_1) - (\partial_y f_0) \sin y\} \partial_x, \\
 [\partial_y, f_0 \partial_x] + [f_1 \partial_x, \sin y \partial_y] + [f_2 \partial_x, \cos y \partial_y] \\
 &= \{(\partial_y f_0) - (\partial_y f_1) \sin y - (\partial_y f_2) \cos y\} \partial_x = 0.
 \end{aligned}$$

hence,

$$\begin{aligned}
 2f_0 &= (\partial_y f_1) \cos y - (\partial_y f_2) \sin y, & 2f_1 &= (\partial_y f_0) \cos y - \partial_y f_2, \\
 2f_2 &= \partial_y f_1 - (\partial_y f_0) \sin y & \text{and} & & \partial_y f_0 &= (\partial_y f_1) \sin y + (\partial_y f_2) \cos y.
 \end{aligned}$$

In particular,

$$\begin{aligned}
 2f_2 &= (\partial_y f_1) (\sin^2 y + \cos^2 y) - (\partial_y f_0) \sin y \\
 &= (\partial_y f_0) \sin y - (\partial_y f_2) \sin y \cos y + 2f_0 \cos y + (\partial_y f_2) \sin y \cos y - (\partial_y f_0) \sin y \\
 &= 2f_0 \cos y,
 \end{aligned}$$

thus $f_2 = f_0 \cos y$, similarly $f_1 = f_0 \sin y$. Hence

$$\begin{aligned}
 &2f_0(\partial_y f_1) \cos y - (\partial_y f_2) \sin y \\
 &= \{(\partial_y f_0) \sin y + f_0 \cos y\} \cos y - \{(\partial_y f_0) \cos y - f_0 \sin y\} \sin y \\
 &= f_0(\cos^2 y + \sin^2 y) = f_0.
 \end{aligned}$$

Hence we get that $f_0 = 0$, that is, $D_4(f\partial_y) = 0$. Consequently we get that $D_4(\mathcal{L}_\tau(T^2)) = 0$, that is, $D = \text{ad}(W_1 + W_2) + aA + bB + cC$.

For the uniqueness of the expression of D , it is sufficient to show that $a = b = c = 0$ and $W \in \mathfrak{s}$, if the derivation $\text{ad}W + aA + bB + cC$ ($W \in \mathcal{L}_\tau(T^2)$) is zero on $\mathcal{L}_\tau(T^2)$. In fact, apply this derivation to the vector fields ∂_x, ∂_y and $\sin y \partial_y$.

Q. E. D.

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