

Notes on Partially Unimodular Structures (IV)

Dedicated to Professor Yoshizawa's 60th birthday

by

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This note is a continuation of Part I, II and III ([K-5~8]), and here we use the same notations.

§ 8. Foliations on Cylinders(II). Consider the product foliation \mathcal{F} on the product manifold $S' \times R^q = \{(x, y_1, \dots, y_q) \in S' \times R^q\}$, given by $\{y_j = \text{constant}, 1 \leq j \leq q\}$, and consider the strictly partially unimodular (s.p.u.) structure $\tau = dx$. In this section, we study derivations of the Lie algebra $\mathcal{L}_\tau(S' \times R^q)$ of all partially unimodular vector fields preserving the foliation \mathcal{F} . By Propositions 2 and 3 in [K-5], we have

$$\begin{aligned} C^\infty(S' \times R^q)^{\mathcal{F}} &= C^\infty(R^q) \quad \text{and} \\ \mathcal{L}_\tau(S' \times R^q) &= \mathcal{F}_\tau(S' \times R^q) \oplus \mathfrak{A}(R^q) \\ &= C^\infty(R^q) \partial_x \oplus \sum_{j=1}^q C^\infty(R^q) \partial_j, \end{aligned}$$

where we denote ∂_{y_j} by ∂_j ($1 \leq j \leq q$).

At first, we get the following proposition, similarly as Proposition 11 in [K-6].

Proposition 21. *Define the linear mappings A and C of $\mathcal{L}_\tau(S' \times R^q)$ to itself as*

$$\begin{aligned} A(\mathcal{F}_\tau(S' \times R^q)) &= 0, & A(\phi(y) \partial_j) &= (\partial_j \phi)(y) \partial_x; \\ C(\phi(y) \partial_x) &= \phi(y) \partial_x, & C(\mathfrak{A}(R^q)) &= 0 \end{aligned}$$

for any function ϕ in $C^\infty(R^q)$. Then A and C are derivations of $\mathcal{L}_\tau(S' \times R^q)$, and are properly outer, that is, they are not realizable by vector fields on $S' \times R^q$.

Moreover, the commutation relation $[C, A] = A$ holds.

Then we get

Theorem 22. (i) *Let D be a derivation of $\mathcal{L}_\tau(S' \times R^q)$. Then, there exists*

a vector field W on $S' \times R^q$ and constants a and $c \in R$ such that

$$D = \text{ad} W + aA + cC.$$

Moreover, the constants a and c are uniquely determined, and W is in $\mathcal{L}_\tau(S' \times R^q)$ and is unique modulo the center $\mathfrak{Z} = R\partial_x$ of $\mathcal{L}_\tau(S' \times R^q)$.

(ii) The first cohomology $H^1 = H^1(\mathcal{L}_\tau(S' \times R^q); \mathcal{L}_\tau(S' \times R^q))$ is of dimension 2, and as a Lie algebra H^1 has 2 generators A and C with the relation $[C, A] = A$.

Proof. Since the statement (i) implies (ii), we prove the first statement. Let D be a derivation of $\mathcal{L}_\tau(S' \times R^q)$.

Define the functions ψ_j and $\phi_{ij} \in C^\infty(R^q)$ ($1 \leq i, j \leq q$) as

$$D(\partial_j) = \psi_j(y) \partial_x + \sum_i \phi_{ij}(y) \partial_i.$$

Then we get that

$$\partial_j \phi_k = \partial_k \phi_j, \quad \partial_j \psi_{ki} = \partial_k \psi_{ji} \quad (1 \leq i, j, k \leq q).$$

In fact, apply D to the equality $[\partial_j, \partial_k] = 0$. Hence by Poincaré's lemma, we get the functions ϕ and $\psi_k \in C^\infty(R^q)$ ($1 \leq k \leq q$) such that

$$\partial_j \phi = \phi_j, \quad \partial_j \psi_k = \psi_{jk}, \quad \phi(0) = 0 \quad \text{and} \quad \psi_k(0) = 0.$$

Now we define the vector field W_1 as $W_1 = \phi \partial_x + \sum \psi_k \partial_k$ in $\mathcal{L}_\tau(S' \times R^q)$, and let $D_1 = D - \text{ad} W_1$. Then D_1 is a derivation of the Lie algebra $\mathcal{L}_\tau(S' \times R^q)$ such that $D_1(\partial_j) = 0$.

Put $J = \sum y_j \partial_j \in \mathcal{L}_\tau(S' \times R^q)$. Apply D_1 to the equality $[\partial_j, J] = \partial_j$, then we get the constants a and b_j ($1 \leq j \leq q$) such that $D_1(J) = qa \partial_x + \sum b_j \partial_j$.

Now we define the vector field W_2 as $W_2 = \sum b_j \partial_j \in \mathcal{L}_\tau(S' \times R^q)$, and let $D_2 = D_1 - \text{ad} W_2 - aA$. Then D_2 is a derivation of $\mathcal{L}_\tau(S' \times R^q)$ such that $D_2(\partial_j) = D_2(J) = 0$.

Apply D_2 to the equalities $[\partial_x, \partial_j] = [\partial_x, J] = 0$, then we get the constant c such that $D_2(\partial_x) = c \partial_x$.

Let $D_3 = D_2 - cC$. Then D_3 is a derivation of $\mathcal{L}_\tau(S' \times R^q)$ such that $D_3(\partial_x) = D_3(\partial_j) = D_3(J) = 0$.

Now by the similar arguments as in the proof of Lemma 6.2 in [K-4], we can show that the derivation D_3 vanishes on the subalgebras of $\mathcal{L}_\tau(S' \times R^q)$ consisting of vector fields with polynomial coefficients.

On the other hand, we can prove the following, similarly as Proposition 1.4 in [K-2] and Proposition 6.5 in [K-4]. For any vector field X in $\mathcal{L}_\tau(S' \times R^q)$ whose 3-jet vanishes at a point p of the manifold ($j^3(X)(p) = 0$), there exist a

finite number of vector fields $Y_1, \dots, Y_{2r} \in \mathcal{L}_\tau(S' \times R^q)$ such that

$$X = \sum_{j=1}^r [Y_j, Y_{j+r}] \quad \text{and} \quad j'(Y_j)(\phi) = 0 \quad (1 \leq j \leq 2r).$$

Hence by Propositions 1.3 and 1.4 in [K-1], we can show that the derivation D_β vanishes on the whole algebra $\mathcal{L}_\tau(S' \times R^q)$, that is

$$D = \text{ad}(W_1 + W_2) + aA + cC.$$

For the uniqueness of the expression of D , it is sufficient to show that $a = c = 0$ and W is in \mathfrak{Z} , if the derivation $\text{ad}W + aA + cC$ ($W \in \mathcal{L}_\tau(S' \times R^q)$) is zero on $\mathcal{L}_\tau(S' \times R^q)$. In fact, apply this derivation to the vector fields ∂_x , ∂_j and J . *qed.*

§ 9. Foliations on Cylinders (III). Consider the product manifold $M = G \times V$, where $G = S'$ or R' , and $V = T^q \times R^r$. Here we introduce the global coordinates x on G (if $G = S'$, then $x \in R/(2\pi Z)$), y_1, \dots, y_q on $T^q = R^q/(2\pi Z)^q$ and z_1, \dots, z_r on R^r . Consider the product foliation \mathcal{F} on the manifold M , given by $\pi^{-1}(v)$ ($v \in V$), where π is the projection of M to the second factor, and consider the s.p.u. structure $\tau = dx$. In this section, we study derivations of $\mathcal{L}_\tau(M)$ for the foliation \mathcal{F} . By Propositions 2 and 3 in [K-5], we have

$$\begin{aligned} C^\infty(M)^{\mathcal{F}} &= C^\infty(V) \quad \text{and} \\ \mathcal{L}_\tau(M) &= \mathcal{F}_\tau(M) \oplus \mathfrak{A}(V) \\ &= C^\infty(V) \partial_x \oplus \sum_{j=1}^q C^\infty(V) \partial_j \oplus \sum_{\alpha=1}^r C^\infty(V) \partial_\alpha, \end{aligned}$$

where we denote ∂_{y_j} by ∂_j ($1 \leq j \leq q$) and ∂_{z_α} by ∂_α ($1 \leq \alpha \leq r$). We use Latin indices i, j, k, \dots for variables in T^q , and Greek indices α, β, \dots for variables in R^r , unless otherwise stated.

Put $J = \sum z_\alpha \partial_\alpha \in \mathcal{L}_\tau(M)$. Then we get the similar propositions as Lemma 10 and Proposition 11 in [K-6].

Lemma 22. (i) If $X \in \mathcal{L}_\tau(M)$ satisfies the relation $[X, \partial_j] = [X, \partial_\alpha] = [X, \text{sin} y_j \partial_k] = [X, J] = 0$ for any j and α and some k , then X is expressed as $X = c \partial_x$ for some constant c .

(ii) The center \mathfrak{Z} of $\mathcal{L}_\tau(M)$ coincides with $R \partial_x$.

Proposition 23. Define the linear mappings A, B_j ($1 \leq j \leq q$) and C of $\mathcal{L}_\tau(M)$ to itself as

$$\begin{aligned} A(\mathcal{F}_\tau(M)) &= 0, & A(\phi \partial_k) &= \partial_k \phi \partial_x, & A(\psi \partial_\alpha) &= \partial_\alpha \psi \partial_x; \\ B_j(\mathcal{F}_\tau(M)) &= 0, & B_j(\phi \partial_k) &= \delta_{jk} \phi \partial_x, & B_j(\psi \partial_\alpha) &= 0; \\ C(\chi \partial_x) &= \chi \partial_x, & C(\mathfrak{A}(V)) &= 0 \end{aligned}$$

for any function ϕ , χ and ψ in $C^\infty(V)$. Then

- (i) A , B_j and C are derivations of $\mathcal{L}_\tau(M)$.
- (ii) A , B_j are properly outer.
- (iii) If G is S' , then C is also properly outer. If G is R' , then C is realized as $C = \text{ad}I$ by the vector field $I = x \partial_x \in \mathcal{I}_\tau(M)$.
- (iv) The following commutation relations hold :

$$[A, B_j] = 0, \quad [B_j, B_k] = 0, \quad [C, B_j] = B_j, \quad [C, A] = A.$$

Then we get

Theorem 24. (i) Let D be a derivation of $\mathcal{L}_\tau(M)$. Then, there exists a vector field W on M and constants a , b_j ($1 \leq j \leq q$) and $c \in R$ such that

$$D = \text{ad}W + aA + \sum_{j=1}^q b_j B_j + cC.$$

Moreover, the constants a and b_j are uniquely determined, and W is in $\mathcal{L}_\tau(M)$. If G is S' , then c is unique and W is in $\mathcal{L}_\tau(M)$ and is unique modulo the center $\mathfrak{Z} = R\partial_x$ of $\mathcal{L}_\tau(M)$. If G is R' , then W can be rewritten as $W = W_1 - c x \partial_x$, where W_1 is in $\mathcal{L}_\tau(M)$. Then c is also uniquely determined, and W_1 is unique modulo the center $\mathfrak{Z} = R\partial_x$ of $\mathcal{L}_\tau(M)$.

(ii) The first cohomology $H^1 = H^1(\mathcal{L}_\tau(M); \mathcal{L}_\tau(M))$ is of dimension $q+2$, and as a Lie algebra H^1 has $q+2$ generators A and C with the relation $[C, A] = A$.

Proof. Here we prove the first statement (i). Let D be a derivation of $\mathcal{L}_\tau(M)$.

Let D' be the linear mapping of $\mathfrak{U}(V)$ to itself which assigns to $X \in \mathfrak{U}(V)$ the $\mathfrak{U}(V)$ -component of $D(X)$. Then, similarly as in the proof of Theorem 12 in [K-6], we get a vector field $W_1 \in \mathfrak{U}(V) \subset \mathcal{L}_\tau(M)$ such that $D'(X) = [W_1, X]$ for $X \in \mathfrak{U}(V)$.

Let $D_1 = D - \text{ad}W_1$. Then D_1 is a derivation of the Lie algebra $\mathcal{L}_\tau(M)$ such that $D_1(\mathfrak{U}(V)) \subset \mathcal{I}_\tau(M)$.

Define the functions ϕ_j and $\psi_\alpha \in C^\infty(V)$ as $\phi_j = D_1(\partial_j)$ and $\psi_\alpha = D_1(\partial_\alpha)$ ($1 \leq j \leq q$, and $1 \leq \alpha \leq r$). Apply D_1 to the equalities $[\partial_j, \partial_k] = [\partial_j, \partial_\alpha] = [\partial_\alpha, \partial_\beta] = 0$, then we get that

$$\partial_j \phi_k = \partial_k \phi_j, \quad \partial_j \psi_\alpha = \partial_\alpha \phi_j \quad \text{and} \quad \partial_\alpha \psi_\beta = \partial_\beta \psi_\alpha$$

for any j, k, α and β . Put

$$b_j(y, z) = (2\pi)^{-1} \int_0^{2\pi} \phi_j(y_1, \dots, y_q, z_1, \dots, z_r) dy_j,$$

then we get that $\partial_k b_j = \partial_\alpha b_j = 0$, because the functions ϕ_j and ψ_α are 2π -periodic in the variables y_1, \dots, y_q . Hence these functions b_j are constants, so we may put

$$b_j = (2\pi)^{-1} \int_0^{2\pi} \phi_j(0, \dots, 0, y_j, 0, \dots, 0) dy_j.$$

Let

$$\begin{aligned} H(y, z) = & \sum_{j=1}^q \int_0^{y_j} \{ \phi_j(0, \dots, 0, t, y_{j+1}, \dots, y_q, z_1, \dots, z_r) - b_j \} dt \\ & + \sum_{\alpha=1}^r \int_0^{z_\alpha} \psi_\alpha(0, \dots, 0, t, z_{\alpha+1}, \dots, z_r) dt, \end{aligned}$$

then this C^∞ -function $H(y, z)$ on $V = T^q \times R^r$ is well-defined.

Now, we define the vector field W_2 as $W_2 = -H(y, z) \partial_x \in \mathcal{F}_\tau(M)$, and let $D_2 = D_1 - \text{ad} W_2 - \sum b_j B_j$. Then D_2 is a derivation of $\mathcal{L}_\tau(M)$ such that $D_2(\mathfrak{A}(V)) \subset \mathcal{F}_\tau(M)$. Here we can show that $D_2(\partial_j) = D_2(\partial_\alpha) = 0$ ($1 \leq j \leq q$, and $1 \leq \alpha \leq r$). In fact, by easy calculations, we get that

$$\partial_j H(y, z) = \phi_j(y, z) - b_j \quad \text{and} \quad \partial_\alpha H(y, z) = \psi_\alpha(y, z).$$

Apply D_2 to the equalities $[\partial_j, J] = 0$ and $[J, \partial_\alpha] = \partial_\alpha$, then we get the constant a such that $D_2(J) = ra \partial_x$.

Let $D_3 = D_2 - aA$. Then D_3 is a derivation of $\mathcal{L}_\tau(M)$ such that $D_3(\mathfrak{A}(V)) \subset \mathcal{F}_\tau(M)$ and $D_3(\partial_j) = D_3(\partial_\alpha) = D_3(J) = 0$.

Apply D_3 to the equalities $[\partial_x, \partial_j] = [\partial_x, \partial_\alpha] = [\partial_x, J] = [\partial_x, \sin y_k \partial_j] = 0$, then by Lemma 22 we get that $D_3(\partial_x) = c \partial_x$ for some constant $c \in R$.

Let $D_4 = D_3 - cC$. Then D_4 is a derivation of $\mathcal{L}_\tau(M)$ such that $D_4(\mathfrak{A}(V)) \subset \mathcal{F}_\tau(M)$, and $D_4(\partial_x) = D_4(\partial_j) = D_4(\partial_\alpha) = D_4(J) = 0$.

Since $[\sin y_k \partial_x, \partial_\alpha] = [\cos y_k \partial_x, \partial_\alpha] = [\sin y_k \partial_\alpha, \partial_\beta] = [\cos y_k \partial_\alpha, \partial_\beta] = 0$ for any integer k , we can use the similar arguments as in the proof of Lemma 17 in [K-8] and Theorem 12 in [K-6], to show that the derivation D_4 vanishes on the ideal $\mathcal{F}_\tau(M)$ and $D_4(\phi \partial_\alpha) = 0$ for any α and any function $\phi(y, z)$ in $C^\infty(V)$.

Moreover, we can show that $D_4(z_\alpha \partial_j) = 0$ for any j and α . In fact, apply D_4 to the equalities

$$[z_\alpha \partial_j, \partial_k] = [z_\alpha \partial_j, \partial_\beta] = 0 \quad \text{and} \quad [J, z_\alpha \partial_j] = z_\alpha \partial_j.$$

Finally, we can show that $D_4(\phi \partial_j) = 0$ for any function $\phi(y, z)$ in $C^\infty(V)$, because $\phi \partial_j$ can be written as

$$\phi \partial_j = [\phi \partial_\alpha, z_\alpha \partial_j] - z_\alpha (\partial_j \phi) \partial_\alpha.$$

Hence, D_1 vanishes on the whole algebra $\mathcal{L}_\tau(M)$, that is,

$$D = \text{ad}(W_1 + W_2) + aA + \sum_{j=1}^q b_j B_j + cC.$$

For the uniqueness of the expression of D , it is sufficient to show that $a = b_j = c = 0$ and W is in \mathfrak{g} , if the derivation $\text{ad}W + aA + \sum b_j B_j + cC$ ($W \in \mathcal{L}_\tau(M)$) vanishes on $\mathcal{L}_\tau(M)$. In fact, apply this derivation to the vector fields ∂_x , ∂_j , ∂_α , $\sin y_k \partial_j$ and J .

qed.

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