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Fock Space Representations of $A_1^{(1)}$ and Kac−Kazhdan Conjecture.†

by
Yukihiro KANIE*
Department of Mathematics, Mie University

Introduction.

Let $\mathfrak{g}$ be an affine Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. Let Δ and Δ^{re} be the root system and real root system of $(\mathfrak{g},\mathfrak{h})$ respectively. The Lie algebra $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm = \sum_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}$ and Δ_+ is the set of positive roots.

For each $\Lambda \in \mathfrak{h}^*$, the Verma module $M(\Lambda)$ of $\mathfrak{g}$ is defined as the left $U(\mathfrak{g})$- and free $U(\mathfrak{n}_-)$- module generated by a nonzero vector $|\Lambda>$ satisfying

$$\mathrm{h}|\Lambda> = \Lambda(\mathrm{h})|\Lambda> \quad (\mathrm{h}\in\mathfrak{h}) \quad \text{and} \quad \mathfrak{n}_+|\Lambda> = 0$$

Then the $\mathfrak{g}$-module $M(\Lambda)$ has the weight space decomposition $M(\Lambda) = \sum_{\lambda\in\mathfrak{h}^*} M_\lambda(\Lambda)$. The formal character $\mathrm{ch}\, M(\Lambda)$ of $M(\Lambda)$ is defined by

$\mathrm{ch}\, M(\Lambda) = \sum_{\lambda\in\mathfrak{h}^*} (\dim M_\lambda(\Lambda))\, e^\lambda$ and is given as $\mathrm{ch}\, M(\Lambda) = e^\Lambda \prod_{\gamma\in\Delta_+} (1 - e^{-\gamma})^{-1}$.

The $\mathfrak{g}$-module $M(\Lambda)$ has a unique irreducible quotient $L(\Lambda)$. Then the Kac-Kazhdan conjecture ([3]) says that

$$\mathrm{ch}\, L(-\rho) = e^{-\rho} \prod_{\gamma\in\Delta_+^{re}} (1 - e^{-\gamma})^{-1},$$

where Δ_+^{re} is the set of positive real roots and ρ is the normalized half sum of positive roots.

In this paper, we show that this conjecture is valid for the affine Lie algebra of type $A_1^{(1)}$.

It is essential for the proof to introduce a polarization, which is inspired by the works of H.P. Jacobsen-V.G.Kac [1] and M. Wakimoto [6].

In § 1 , we construct a Fock space representation $\mathcal{H}$ of Bose commutation relations (1.1). In § 2 , for each $(\mu,\nu) \in \mathbb{C}^2$ we construct a representation of $\pi_{\mu,\nu}$ of $A_1^{(1)}$ on the Fock space $\mathcal{H}$. This $A_1^{(1)}$-module $(\mathcal{H}, \pi_{\mu,\nu})$ is a highest weight module

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* Department of Mathematics, Faculty of Education, Mie University.

with highest weight $\Lambda_{\mu,\nu} = (\frac{\nu^2}{2} - 2 - \mu)\,\Lambda_0 + \mu\Lambda_1$, where Λ_0 and Λ_1 are the fundamental weights. Its character $\mathrm{ch}\,(\mathcal{H}, \pi_{\mu,\nu})$ as an $A_1^{(1)}$-module is given as

$$\mathrm{ch}\,(\mathcal{H},\pi_{\mu,\nu}) = e^{\mu,\nu} \prod_{\gamma\in\Delta_+} (1-e^{-\gamma})^{-1}.$$

If $\nu = 0$, we can factorize the module $(\mathcal{H}, \pi_{\mu,0})$ to get a highest weight $A_1^{(1)}$-module $(\mathcal{K},\pi_{\mu,0})$ with highest weight $\Lambda_{\mu,0}$, whose character is given as

$$\mathrm{ch}\,(\mathcal{K},\pi_{\mu,0}) = e^{\Lambda_{\mu,0}} \prod_{\lambda\in\Delta_+^{re}} (1-e^{-\gamma})^{-1}.$$

In §3, we show that if $\mu \in \mathbb{C} \setminus \mathbb{Z}$ or $\mu = -1$, then the $A_1^{(1)}$-module $(\mathcal{K},\pi_{\mu,0})$ is irreducible. In particular, $(\mathcal{K},\pi_{-1,0})$ is $A_1^{(1)}$-isomorphic to $L(-\rho)$. Thus we get the Kac-Kazhdan conjecture for the affine Lie algebra of type $A_1^{(1)}$.

We dared to give accounts more in detail than mathematical standards for the character of this journal.

§ 1. Fock representations

Let W be the vector space over $\mathbb{C}$ spanned by $\{a_i, a_i^\dagger\ (i\in\mathbb{Z}),\ b_j\ (j\in\mathbb{Z}\setminus\{0\})\}$. Introduce the associative algebra $\mathcal{B}$ with unit, generated by W and the defining commutation relations:

$$(1.1)\qquad \begin{aligned} [a_i^\dagger, a_j] &= \delta_{i+j,0} && (i,j \in \mathbb{Z}),\\ [b_i, b_j] &= i\delta_{i+j,0} && (i,j \in \mathbb{Z}\setminus\{0\}), \end{aligned}$$

and all others vanish. We use the convention that $b_0 = 0$.

Consider the subspaces W_+ and W_- of W, defined by

$$W_+ = \sum_{j\geq 0} \mathbb{C}a_j + \sum_{j>0} (\mathbb{C}a_j^\dagger + \mathbb{C}b_j) \quad\text{and}\quad W_- = \sum_{j<0} (\mathbb{C}a_j + \mathbb{C}b_j) + \sum_{j\leq 0} \mathbb{C}a_j^\dagger.$$

Then any two elements in W_+ (or W_-) are commutative respectively, hence $\mathcal{B} = S(W_-) \otimes S(W_+)$ as a left $S(W_-)$- and right $S(W_+)$-module. We can define the *normal product*

$$:\ :\ : S(W) \longrightarrow \mathcal{B},$$

as the uniquely defined $\mathbb{C}$-linear isomorphism under the conditions that (i) : 1 : = 1 and (ii) : : is a left $S(W_-)$- and right $S(W_+)$- module mapping.

Consider the left $\mathcal{B}$-module $\mathcal{H}$ with a cyclic vector $|0\rangle$ satisfying

$$(1.2)\qquad W_+\,|0\rangle = 0,$$

and also the right $\mathcal{B}$- module $\mathcal{H}^\dagger$ with a cyclic vector $\langle 0|$ satisfying

$$(1.2')\qquad \langle 0\,|\,W_- = 0.$$

Then the module $\mathcal{H}$ is a free $S(W_-)$-module and $\mathcal{H}^\dagger$ is a free $S(W_+)$-module, that is, the following two mappings are $\mathbb{C}$-linear isomorphisms:

(1.3) $$S(W_-) \ni a \longmapsto a \mid 0> \in \mathcal{H} = \mathcal{B} / \mathcal{B}W_+$$

and

(1.3′) $$S(W_+) \ni b \longmapsto <0 \mid b \in \mathcal{H}^\dagger = W_- \mathcal{B} \setminus \mathcal{B}$$

The *vacuum expectation value*

$$< \mid > : \mathcal{H}^\dagger \times \mathcal{H} \longrightarrow \mathbb{C}$$

is uniquely defined by the following conditions:

(i) $< \mid >$ is $\mathbb{C}$-bilinear ; (ii) $<0 \mid 0> = 1$;

(iii) $<va \mid u> = <v \mid au>$ for any $<v \mid \in \mathcal{H}^\dagger$, $\mid u> \in \mathcal{H}$ and $a \in \mathcal{B}$.

For an element $a \in \mathcal{B}$, the complex number $<0 \mid a \mid 0>$, denoted by $<a>$, is called the vacuum expectation value of a. Then

(1.4) $$<a_i> = <a_i^\dagger> = <b_j> = <a_i a_j> = <a_i^\dagger a_j^\dagger> = 0,$$
$$<b_i b_j> = -jY_-(j)\,\delta_{i+j,0}, \quad <a_i a_j^\dagger> = -Y_+(i)\,\delta_{i+j,0}$$

and

$$<a_j^\dagger a_i> = Y_-(i)\,\delta_{i+j,0}\ (i,j \in \mathcal{H}),$$

where

(1.5) $$Y_+(i) + Y_-(i) = 1 \text{ and } Y_+(i) = \begin{cases} 1 & j \geq 0 \\ 0 & j < 0. \end{cases}$$

Hence

(1.6) $$<a_i^\dagger a_j> - <a_j a_i^\dagger> = \delta_{i+j,0} \quad \text{and} \quad <b_i b_j> - <b_j b_i> = i\delta_{i+j,0}$$

and

(1.6′) $$<a_i a_\ell^\dagger><a_j^\dagger a_k> - <a_k a_j^\dagger><a_\ell^\dagger a_i> = \delta_{i+\ell,0}\,\delta_{j+k,0}\,(Y_-(\ell) - Y_-(k)),$$

since

(1.7) $$Y_+(k)\,Y_-(i) - Y_+(i)\,Y_-(k) = Y_-(i) - Y_-(k).$$

Hence, by the well-known Wick's theorem, we get

Lemma 1.1. *For any* $i, j, k, \ell, m, n \in \mathbb{Z}$,

$(1.8)_0$ $$[a_i^\dagger, a_j] = \delta_{i+j,0} \quad \textit{and} \quad [b_i, b_j] = i\delta_{i+j,0}.$$

$(1.8)_1$ $$[:a_i a_j^\dagger:, a_k^\dagger] = -\delta_{i+k,0}\,a_j^\dagger, \quad [:a_i a_j^\dagger:, a_k] = \delta_{j+k,0}\,a_i \quad \textit{and}$$
$$[:b_i b_j:, b_k] = -k\{\delta_{i+k,0}\,b_j + \delta_{j+k,0}\,b_i\}.$$

$(1.8)_2$ $$[:a_i a_j^\dagger:, :a_k a_\ell^\dagger:] =$$
$$= -\delta_{i+\ell,0} :a_k a_j^\dagger: + \delta_{j+k,0} :a_i a_\ell^\dagger: + \delta_{i+\ell,0}\,\delta_{j+k,0}\{Y_-(i) - Y_-(k)\},$$
$$[:b_i b_j:, :b_k b_\ell:] =$$

$$= i\{\delta_{i+k,0} : b_j b : + \delta_{i+\ell,0} : b_j b_k :\} + j\{\delta_{j+k,0} : b_i b_\ell : + \delta_{j+\ell,0} : b_i b_k :\}$$
$$+ ij\{\delta_{i+k,0}\,\delta_{j+\ell,0} + \delta_{i+\ell,0}\,\delta_{j+k,0}\}\{Y_-(-i)\,Y_-(-j) - Y_-(i)\,Y_-(j)\}.$$

(1.8)$_3$ $\quad [\,:a_i a_j^\dagger a_k^\dagger:\,,\ a_\ell^\dagger\,] = -\,\delta_{i+\ell,0}\, a_j^\dagger a_k^\dagger.$

(1.8′)$_3$ $\quad [\,:a_i a_j^\dagger a_k^\dagger:\,,\ a_\ell] = \delta_{j+\ell,0} : a_i a_k^\dagger : + \delta_{k+\ell,0} : a_i a_k^\dagger : .$

(1.8)$_4$ $\quad [\,: a_i a_j^\dagger :\,,\ \ : a_k a_\ell^\dagger a_m^\dagger :\,] =$

$$= -\,\delta_{i+\ell,0} : a_k a_j^\dagger a_m^\dagger : - \delta_{i+m,0} : a_k a_j^\dagger a_\ell^\dagger : + \delta_{j+k,0} : a_i a_\ell^\dagger a_m^\dagger :$$
$$+ \{Y_-(i) - Y_-(k)\}(\delta_{i+\ell,0}\,\delta_{j+k,0}\, a_m^\dagger + \delta_{i+m,0}\,\delta_{j+k,0}\, a_\ell^\dagger)$$

(1.8)$_5$ $\quad [\,: a_i a_j^\dagger a_k^\dagger :\,,\ \ : a_\ell a_m^\dagger a_n^\dagger :\,] = -\,\delta_{i+m,0} : a_\ell a_j^\dagger a_k^\dagger a_n^\dagger - \delta_{i+n,0} : a_\ell a_j^\dagger a_k^\dagger a_m^\dagger :$

$$+ \delta_{j+\ell,0} : a_\ell a_k^\dagger a_m^\dagger a_n^\dagger : + \delta_{k+\ell,0} : a_i a_j^\dagger a_m^\dagger a_n^\dagger :$$
$$+ \{Y_-(i) - Y_-(\ell)\}(\delta_{j+\ell,0}\, a_k^\dagger + \delta_{k+\ell,0}\, a_j^\dagger)(\delta_{i+m,0}\, a_n^\dagger + \delta_{i+n,0}\, a_m^\dagger)$$

For any $n \in \mathbb{Z}$, take the elements H_n, I_n and $J_n \in W \otimes W$ defined by

$$H_n = \sum_{j\in\mathbb{Z}} : a_{n-j}\, a_j^\dagger : ,\quad I_n = \sum_{j\in\mathbb{Z}} j : a_{n-j}\, a_j^\dagger : \text{ and } J_n = \sum_{j\in\mathbb{Z}} : b_{n-j}\, b_j : ,$$

then by (1.8)$_1$,

(1.9) $\quad [H_n, a_i] = a_{i+n}\,,\quad [I_n, a_i] = -\,i a_{i+n}\,,\quad [J_n, b_i] = -\,2ib_{i+n}\,,$

$\qquad [H_n, a_i^\dagger] = -\,a_{i+n}^\dagger$ and $[I_n, a_i^\dagger] = -\,(i+n)\, a_{i+n}^\dagger\,,$

and by (1.8)$_2$,

(1.10) $\quad [H_n, : a_k a_\ell^\dagger :] = : a_{k+n} a_\ell^\dagger : - : a_k a_{\ell+n}^\dagger : + \delta_{n+k+\ell,0}(Y_-(-\ell) - Y_-(k)),$

$[I_n, : a_k a_\ell^\dagger :] = -(n+\ell) : a_{k+n} a_\ell^\dagger : - k : a_k a_{\ell+n}^\dagger : - k\delta_{n+k+\ell,0}(Y_-(-\ell) - Y_-(k)),$

$[J_n, : b_k b_\ell :] = -\,2(k : b_{k+n} b_\ell : - \ell : b_k b_{\ell+n} :),$

hence

(1.11) $\quad [H_m, H_n] = n\delta_{m+n,0}\,,\qquad [J_m, J_n] = 2\,(m-n)\,J_{m+n}\,,$

$$[H_m, I_n] = mH_{m+n} - \frac{m\,(m-1)}{2}\,\delta_{m+n,0}\,,$$

and

$$[I_m, I_n] = (m-n)\,I_{m+n} + \frac{m\,(m^2-1)}{6}\,\delta_{m+n,0}\,,$$

since

(1.12) $\quad \sum_{j\in\mathbb{Z}} \{Y_-(j) - Y_-(j+n)\} = n.$

By (1.8)$_3$ and (1.8′)$_3$,

(1.13) $\quad [\,: H_n a_k^\dagger :\,,\ a_\ell] = : a_{n+\ell}\, a_k^\dagger : + \delta_{k+\ell,0}\, H_n$ and $[\,: H_n a_k^\dagger :\,,\ a_\ell^\dagger] = -\,a_{n+\ell}^\dagger\, a_k^\dagger,$

and by (1.8)$_4$,

(1.14) $\quad [\,: a_i a_j^\dagger :\,,\ : H_k a_m^\dagger :\,] = : a_i a_{j+k}^\dagger a_m^\dagger : - : a_{k+i} a_j^\dagger a_m^\dagger : - \delta_{i+m,0} : H_k a_j^\dagger :$

$$+ (Y_-(i) - Y_-(-j))(\delta_{i+j+k,0}\, a_m^\dagger + \delta_{i+m,0}\, a_{j+k}^\dagger),$$

hence

(1.15) $$[H_n, :H_k a_m^\dagger:] = -:H_k a_{n+m}^\dagger: + k\delta_{n+k,0}\, a_m^\dagger + (Y_-(-m) - Y_-(-n-m))\, a_{m+n+k}^\dagger.$$

Finally by $(1.8)_5$,

(1.16) $$[:H_m a_k^\dagger:, :H_n a_p^\dagger:] = :H_m a_{k+n}^\dagger a_p^\dagger: - :H_n a_k^\dagger a_{m+p}^\dagger: + n\delta_{m+n,0}\, a_k^\dagger a_p^\dagger + (Y_-(-n-k) - Y_-(-k))a_p^\dagger a_{m+n+k}^\dagger + (Y_-(-p) - Y_-(-m-p))a_{m+n+p}^\dagger a_k^\dagger + (Y_-(-p) - Y_-(-k))a_{n+k}^\dagger a_{m+p}^\dagger.$$

Introduce the grading in $\mathcal{B}$ by setting $\deg(a_j) = \deg(a_j^\dagger) = \deg(b_j) = j$, then by (1.9), the *number operator* $\mathcal{N} = -I_0 - \frac{1}{2} J_0$ is the Euler operator corresponding to this grading, that is,

(1.17) $$[\mathcal{N}, a_j] = ja_j, \quad [\mathcal{N}, a_j^\dagger] = ja_j^\dagger \text{ and } [\mathcal{N}, b_j] = jb_j\ (j \in \mathbb{Z}),$$

hence $[\mathcal{N}, H_n] = nH_n$. The modules $\mathcal{H}$ and $\mathcal{H}^\dagger$ also admit the grading, by setting $\deg(\langle 0 | b) = n$ and $\deg(b | 0\rangle) = -n$ for a homogeneous element $b \in \mathcal{B}$ of degree n.

Denote by $\mathcal{J}$ the set of all sequences $N = (n_1, n_2, \cdots\cdots)$ of non-negative integers satisfying $\| N \| = \sum_{j\geq 1} jn_j < \infty$. For each $M,N \in \mathcal{J}$, we set $M + N = (m_1 + n_1, \cdots, m_j + n_j, \cdots) \in \mathcal{J}$. For any $N \in \mathcal{J}$, consider elements of $\mathcal{B}$ defined by

$$P_+^\dagger(N) = \cdots\cdot a_j^{\dagger n_j} \cdots a_2^{\dagger n_2} a_1^{\dagger n_1}, \qquad P_-(N) = a_{-1}^{n_1} a_{-2}^{n_2} \cdots\cdots a_{-j}^{n_j} \cdots,$$

$$P_+(N) = \cdots\cdot a_j^{n_j} \cdots a_2^{n_2} a_1^{n_1}, \qquad P_-^\dagger(N) = a_{-1}^{\dagger n_1} a_{-2}^{\dagger n_2} \cdots\cdots a_{-j}^{\dagger n_j} \cdots,$$

and

$$Q_+(N) = \cdots\cdot b_j^{n_j} \cdots b_2^{n_2} b_1^{n_1}, \qquad Q_-(N) = b_{-1}^{n_1} b_{-2}^{n_2} \cdots\cdots b_{-j}^{n_j} \cdots.$$

Bases of $\mathcal{H}$ and $\mathcal{H}^\dagger$ over $\mathbb{C}$ are given by

(1.18) $$|\ell, L, M, N\rangle = a_0^{\dagger \ell} P_-^\dagger(L) P_-(M) Q_-(N) | 0\rangle \quad (\ell \in \mathbb{Z}_{\geq 0},\ L, M, N \in \mathcal{J})$$

and

$$\langle N, M, L, \ell | = \langle 0 | Q_+(N) P_+^\dagger(M) P_+(L) a_0^\ell \quad (\ell \in \mathbb{Z}_{\geq 0},\ L, M, N \in \mathcal{J})$$

respectively. Then the vacuum expectation values are explicitly given by the formula

(1.19) $$\langle N', M', L', \ell' | \ell, L, M, N\rangle = \delta_{\ell,\ell'}\, \delta_{L,L'}\, \delta_{M,M'}\, \delta_{N,N'} (-1)^{\ell+|L|} \ell!\, L!\, M!\, N!\, n^N,$$

where

(1.20) $$\delta_{N,N'} = \delta_{n_1,n'_1}\, \delta_{n_2,n'_2} \cdots\cdots \delta_{n_j,n'_j} \cdots,$$

$$|N| = n_1 + n_2 + \cdots\cdots + n_j + \cdots\cdots,$$

$$N! = n_1!\, n_2! \cdots\cdots n_j! \cdots\cdot,$$

and

$$n^N = 1^{n_1}\, 2^{n_2} \cdots\cdots j^{n_j} \cdots\cdot.$$

Note that $\deg |\ell, L, M, N> = \deg <N, M, L, \ell\ | = \| L + M + N \|$ for any $\ell \in \mathbb{Z}_{\geq 0}$ and $L, M, N \in \mathcal{J}$. The modules $\mathcal{H}$ and $\mathcal{H}^\dagger$ are decomposed into the sums of homogeneous components, that is,

$$(1.21) \qquad \mathcal{H} = \sum_{\ell \geq 0} \sum_{d \geq 0} \mathcal{H}_{\ell,d}, \qquad \mathcal{H}_{\ell,d} = \sum_{\|L+M+N\|=d} \mathbb{C}\,|\ell, L, M, N>,$$

and

$$\mathcal{H}^\dagger = \sum_{\ell \geq 0} \sum_{d \geq 0} \mathcal{H}^\dagger_{\ell,d}, \qquad \mathcal{H}^\dagger_{\ell,d} = \sum_{\|L+M+N\|=d} \mathbb{C} < N, M, L, \ell\ |$$

By enumerating the number of the basis vectors, we get

$$\dim \mathcal{H}_{\ell,d} = \dim \mathcal{H}^\dagger_{\ell,d} = q(d),$$

where $q(d)$ is expressed by the partition numbers $p(j)$ of j as

$$q(d) = \sum_{k=0}^{d} \sum_{j=0}^{k} p(j)\, p(k-j)\, p(d-k), \qquad \text{i. e.} \quad \sum_{d \geq 0} q(d)\, t^d = \prod_{k \geq 1} (1 - t^k)^{-3}.$$

By the formula (1.19), we get

Proposition 1. 2. *The vacuum expectation value value* $< | >$ *vanishes on* $\mathcal{H}^\dagger_{\ell',d'} \times \mathcal{H}_{\ell,d}$ *unless* $\ell = \ell'$ *and* $d = d'$, *and is non-degenerate if* $\ell = \ell'$ *and* $d = d'$.

Let σ be the $\mathbb{C}$-linear anti-automorphism of the associative algebra $\mathcal{B}$ defined by

$$\sigma(a_j) = a^\dagger_{-j}, \quad \sigma(a^\dagger_j) = a_{-j} \quad \text{and} \quad \sigma(b_j) = b_{-j},$$

then we get

$$(1.22) \qquad \sigma(H)_n = H_{-n} \qquad (n \in \mathbb{Z}) \quad \text{and} \quad \sigma(\mathcal{N}) = \mathcal{N},$$

and the $\mathbb{C}$-linear map $\tilde{\sigma} : \mathcal{H}^\dagger \longrightarrow \mathcal{H}$ defined by

$$\tilde{\sigma}(<N, M, L, \ell\ |) = |\ell, L, M, N> \qquad (\ell \in \mathbb{Z}_{\geq 0}, L, M, N \in \mathcal{J})$$

is an isomorphism over $(\mathcal{B}, \sigma)$, that is, an isomorphism satisfying

$$\tilde{\sigma}(<v|b\ ) = \sigma(b)\,(\tilde{\sigma}(<v\,|)) \qquad (b \in \mathcal{B},\ <v\,| \in \mathcal{H}^\dagger).$$

§2. Highest weight modules of $A_1^{(1)}$

Recall facts on the affine Lie algebra $\mathfrak{g}$ of type $A_1^{(1)}$ (see e.g. Kac's book [2]).

$$\mathfrak{g} = \hat{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d \supset \hat{\mathfrak{g}} = \sum_{n\in\mathbb{Z}} (\mathbb{C}e_n \oplus \mathbb{C}h_n \oplus \mathbb{C}f_n),$$

with the following commutation relations:

$$(2.1) \qquad \begin{aligned} &[f_m, f_n] = [e_m, e_n] = [c, \hat{\mathfrak{g}}] = [c, d] = 0, \\ &[e_m, f_n] = h_{m+n} + m\delta_{m+n,0}\, c, \quad [h_m, h_n] = 2m\delta_{m+n,0}\, c, \\ &[h_m, e_n] = 2e_{m+n}, \quad [h_m, f_n] = -2f_{m+n}, \\ &[d, h_n] = nh_n, \quad [d, e_n] = ne_n, \quad [d, f_n] = nf_n \end{aligned}$$

for all $m, n \in \mathbb{Z}$.

The subalgebra $\mathring{\mathfrak{g}} = \mathbb{C}e_0 \oplus \mathbb{C}h_0 \oplus \mathbb{C}f_0$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. The subspaces $\mathfrak{h} = \mathbb{C}h_0 \oplus \mathbb{C}c \oplus \mathbb{C}d \supset \mathring{\mathfrak{h}} = \mathbb{C}h_0$ are abelian (called Cartan subalgebra of $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ respectively). Denote by $\mathring{\Delta} = \{\pm\alpha\}$ the root system of $(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$. The dual $\mathfrak{h}^*$ of $\mathfrak{h}$ has the basis $\{\alpha/2, \Lambda_0, \delta\}$ dual to the basis $\{h_0, c, d\}$.

We can introduce a nondegenerate symmetric and invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$ as

$$(e_0, f_0) = 1, \quad (h_0, h_0) = 2, \quad (e_0, e_0) = (f_0, f_0) = 0,$$

$$(x_m, y_n) = \delta_{m+n,0}(x_0, y_0) \qquad (x \text{ and } y \text{ are ones of } e, h \text{ and } f);$$

$$(c, d) = 1 \quad \text{and} \quad (c, c) = (d, d) = 0$$

for any $m, n \in \mathbb{Z}$, and a nondegenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ as

$$(\alpha, \alpha) = 2, \quad (\delta, \Lambda_0) = 1, \quad (\alpha, \Lambda_0) = (\alpha, \delta) = (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0.$$

The root system Δ of $(\mathfrak{g}, \mathfrak{h})$ and positive root system Δ_+ corresponding to the simple root system $\Pi = \{\alpha_0 = \delta - \alpha, \quad \alpha_1 = \alpha\}$ are given as

$$\begin{aligned} &\Delta = \{\pm\alpha + n\delta,\ n\delta (n \neq 0)\} \cup \mathring{\Delta} \\ &\cup \\ &\Delta_+ = \{\pm\alpha + n\delta,\quad n\delta (n > 0)\} \cup \{\alpha\}. \end{aligned}$$

The Lie algebra $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where

$$\mathfrak{n}_\pm = \sum_{\beta\in\Delta_+} \mathfrak{g}_{\pm\beta}, \quad \mathfrak{g}_{\alpha+n\delta} = \mathbb{C}e_n, \quad \mathfrak{g}_{-\alpha+n\delta} = \mathbb{C}f_n, \quad \text{and} \quad \mathfrak{g}_{n\delta} = \mathbb{C}h_n.$$

The root lattice Q is given as

$$Q = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 = \mathbb{Z}\delta + \mathbb{Z}\alpha \supset Q_+ = \mathbb{Z}_{\geq 0}\,\alpha_0 + \mathbb{Z}_{\geq 0}\,\alpha_1.$$

The real root system Δ^{re} of $(\mathfrak{g}, \mathfrak{h})$ is given as

$$\Delta^{re} = \{\pm\,\alpha + n\delta (n \in \mathbb{Z})\} \supset \Delta_+^{re} = \Delta_+ \cap \Delta^{re}$$

A $\mathfrak{g}$-module V is called a *highest weight module of highest weight* $\Lambda \in \mathfrak{h}^*$, if V admits a weight space decomposition $V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda$ by finite dimensional weight spaces $V_\lambda = \{v \in V ; hv = \langle \lambda, h \rangle v \quad (h \in \mathfrak{h})\}$ with $\dim V_\Lambda = 1$ and the set $P(\lambda) = \{\lambda \in \mathfrak{h}^* ; V_\lambda \neq \{0\}\}$ of weights is contained in $\Lambda - Q_+$. Fix a non-zero vector $v_\Lambda \in V_\Lambda$, which we call a *vacuum vector*. (Note that this definition is slightly different from the one in Kac's book [4]). Then we define a *formal character* ch$V$ of a highest weight $\mathfrak{g}$-module V by

$$\mathrm{ch} V = \sum_{\lambda \in P(V)} \dim V_\lambda\, e^\lambda.$$

Now fix $(\mu, \nu) \in \mathbb{C}^2$. Introduce the $\mathbb{C}$-linear mapping $\pi_{\mu,\nu} : \mathfrak{g} \longrightarrow \mathcal{B}$ defined by the following: for any $n \in \mathbb{Z}$,

$$(2.2) \quad \pi_{\mu,\nu}(e_n) = a_n,\ \pi_{\mu,\nu}(f_n) = \left(n\left(\frac{\nu^2}{2} - 2\right) - \mu\right) a_n^\dagger - \sum_{j \in \mathbb{Z}} :H_j\, a_{n-j}^\dagger: - \nu \sum_{j \in \mathbb{Z}} b_j\, a_{n-j}^\dagger,$$

$$\pi_{\mu,\nu}(h_n) = \mu\delta_{n,0} + 2H_n + \nu b_n,\quad \pi_{\mu,\nu}(d) = \mathcal{N} \ \text{ and } \ \pi_{\mu,\nu}(c) = \frac{\nu^2}{2} - 2.$$

Then $\deg(\pi_{\mu,\nu}(e_n)) = \deg(\pi_{\mu,\nu}(h_n)) = \deg(\pi_{\mu,\nu}(f_n)) = n$ and $\deg(\pi_{\mu,\nu}(c)) = \deg(\pi_{\mu,\nu}(d)) = 0$.

Proposition 2.1. *Fix* $(\mu, \nu) \in \mathbb{C}^2$.

(i) $\pi_{\mu,\nu}$ *gives a homomorphism as Lie algebras.*

(ii) $(\mathcal{H}, \pi_{\mu,\nu})$ *is a highest weight left* $\mathfrak{g}$*-module with highest weight* $\Lambda_{\mu,\nu}$, *where* $\Lambda_{\mu,\nu} = \mu\frac{\alpha}{2} + \left(\frac{\nu^2}{2} - 2\right)\Lambda_0 = \left(\frac{\nu^2}{2} - 2 - \mu\right)\Lambda_0 + \mu\Lambda_1.$

(iii) The weight space decomposition of the $\mathfrak{g}$*-module* $(\mathcal{H}, \pi_{\mu,\nu})$ *is given as* $\mathcal{H} = \sum_{\gamma \in Q_+} \mathcal{H}_\gamma$, *where* $\mathcal{H}_\gamma$ *denotes the weight space of* $\mathcal{H}$ *belonging to the weight* $\Lambda_{\mu,\nu} - \gamma$.

(iv) $\mathcal{H}_\gamma\,(\gamma \in Q_+)$ *is spanned by vectors* $|\ell, L, M, N\rangle$ *with* $\|L + M + N\| = m$ *and* $\ell + |L - M| = n - m$ *for* $\gamma = m\alpha_0 + n\alpha_1 = m\delta + (n - m)\,\alpha$, *hence* $\dim \mathcal{H}_\gamma = P(\gamma)$, *where* $P(\gamma)$ *is the Kostant's partition function, that is, the number of sets* $\{k_\beta\,(\in \mathbb{Z}_{\geq 0}) ; \beta \in \Delta_+\}$ *such that* $\gamma = \sum_{\beta \in \Delta_+} k_\beta \beta$.

Therefore

(2.3) $$\mathrm{ch}\,(\mathcal{H}, \pi_{\mu,\nu}) = e^{\Lambda_{\mu,\nu}} \prod_{\beta\in\Delta_+} (1 - e^{-\beta})^{-1}.$$

(v) $(\mathcal{H}^\dagger, \pi_{\mu,\nu})$ *is a highest weight right* $\mathfrak{g}$- *module with highest weight* $\Lambda_{\mu,\nu}$.

(vi) The weight space decomposition of the $\mathfrak{g}$-*module* $(\mathcal{H}^\dagger, \pi_{\mu,\nu})$ *is given as* $\mathcal{H}^\dagger = \sum_{\gamma\in Q_+} \mathcal{H}^\dagger_\gamma$, *where* $\mathcal{H}^\dagger_\gamma$ *denotes the weight space of* $\mathcal{H}^\dagger$ *belonging to the weight* $\Lambda_{\mu,\nu} - \gamma$.

(vii) $\mathcal{H}^\dagger_\gamma$ $(\gamma \in Q_+)$ *is spanned by vectors* $< N, M, L, \ell \mid$ *with* $\| L + M + N \| = m$ *and* $\ell + | L - M | = n - m$ *for* $\gamma = m\alpha_0 + n\alpha_1$, *hence* $\dim \mathcal{H}^\dagger_\gamma = P(\gamma)$.

Therefore

$$\mathrm{ch}\,(\mathcal{H}^\dagger, \pi_{\mu,\nu}) = e^{\Lambda_{\mu,\nu}} \prod_{\beta\in\Delta_+} (1 - e^{-\beta})^{-1}.$$

(viii) The vacuum expectation value $< \mid >$ *vanishes on* $\mathcal{H}^\dagger_{\gamma'} \times \mathcal{H}_\gamma$ *unless* $\gamma = \gamma'$ *and is non-degenerate if* $\gamma = \gamma'$.

Proof. (i) In this proof, we abbreviate $\pi_{\mu,\nu}(x)$ to x, for $x \in \mathfrak{g}$. $[e_m, e_n] = 0$ is obvious. And we get that $[h_m, h_n] = 2m\delta_{m+n,0}\, c$ from (1.1,11) and $[h_m, e_n] = 2e_{m+n}$ from (1.9).

By (1.9, 15 and 12), we get $[h_m, f_n] = -2f_{m+n}$,.

Note the identity

$$2\sum_{j\in\mathbb{Z}} j\, a^\dagger_{m+j}\, a^\dagger_{n+j} + (m-n)\sum_{j\in\mathbb{Z}} a^\dagger_{m+j}\, a^\dagger_{n-j} = 0,$$

then we get that $[e_m, f_n] = h_{m+n} + m\delta_{m+n,0}\, c$ from (1.1, 13 and 12), and $[f_m, f_n] = 0$ from (1.13, 16 and 12).

For (ii)～(iv), note

$$e_n\, |0> = h_n\, |0> = f_n\, |0> = e_0 |0> = 0 \qquad (n > 0)$$

and

$$h_0\, |0> = \mu\, |0>, \quad c\, |0> = (\tfrac{\nu^2}{2} - 2)\, |0>.$$

It is easily seen that $c \mid \ell, L, M, N> = (\tfrac{\nu^2}{2} - 2) \mid \ell, L, M, N>$,

$$h_0 \mid \ell, L, M, N> = \{\mu - 2(\ell + |L|) + 2|M|\} \mid \ell, L, M, N>$$

and

$$d \mid \ell, L, M, N> = - \| L + M + N \| \mid \ell, L, M, N>.$$

(v)～(vii) are similarly obtained.

(vii) follows from the formula (1.19).

qed.

Introduce the $\mathcal{B}$- modules $\mathcal{K} = \mathcal{B}/\mathcal{B}(W_+ + \sum_{j<0} \mathbb{C}b_j)$ and $\mathcal{K}^\dagger = (W_- + \sum_{j>0} \mathbb{C}b_j)\mathcal{B} \setminus \mathcal{B}$. Then $\mathcal{K}$ and $\mathcal{K}^\dagger$ have the basis

$$|\ell, L, M> = a_0^{\dagger\ell} P_-^\dagger(L) P_-(M)|0> \qquad (\ell \in \mathbb{Z}_{\geq 0}, L, M \in \mathcal{G})$$

and

$$< M, L, \ell| = < 0|P_+^\dagger(M) P_+(L) a_0^\ell \qquad (\ell \in \mathbb{Z}_{\geq 0}, L, M \in \mathcal{G})$$

respectively. Then $\mathcal{K}$ and $\mathcal{K}^\dagger$ can be considered as subspaces of $\mathcal{H}$ and $\mathcal{H}^\dagger$ by the $\mathbb{C}$ -linear mapplings $\iota : \mathcal{K} \longrightarrow \mathcal{H}$ and $\iota^\dagger : \mathcal{K}^\dagger \longrightarrow \mathcal{H}^\dagger$ defined by

$$\iota(|\ell, L, M>) = |\ell, L, M, \mathbf{0}> \quad \text{and} \quad \iota^\dagger(< M, L, \ell|) = < \mathbf{0}, M, L, \ell|$$

for $\ell \in \mathbb{Z}_{\geq 0}$ and $L, M \in \mathcal{G}$ respectively, where $\mathbf{0} = (0, 0, \cdots) \in \mathcal{G}$. Note that $\tilde{\sigma}(\mathcal{K}^\dagger) = \mathcal{K}$.

Then by the formula (2.2), these subspaces $\mathcal{K}$ and $\mathcal{K}^\dagger$ are $\mathfrak{g}$-modules through $\pi_{\mu,0}$ for $\nu = 0$, and all above facts are valid for $(\mathcal{K}, \pi_{\mu,0})$ and $(\mathcal{K}^\dagger, \pi_{\mu,0})$. For $\mu \in \mathbb{C}$, set

$$\pi_\mu(x) = \pi_{\mu,0}(x) \quad (x \in \hat{\mathfrak{g}}), \qquad \pi_\mu(c) = -2 \quad \text{and} \quad \pi_\mu(d) = -I_0.$$

then we get

Proposition 2.2. *Fix* $\mu \in \mathbb{C}$.

(i) $(\mathcal{K}, \pi_\mu)$ *is a highest weight left* $\mathfrak{g}$- *module with highest weight* $\Lambda_{\mu,0}$, *where* $\Lambda_{\mu,0} = \mu\frac{\alpha}{2} - 2\Lambda_0$.

(ii) The weight space decomposition of the $\mathfrak{g}$- *module* $(\mathcal{K}, \pi_{\mu,0})$ *is given as* $\mathcal{K} = \sum_{\gamma \in Q_+} \mathcal{K}_\gamma$, *where* $\mathcal{K}_\gamma = \mathcal{K} \cap \mathcal{H}_\gamma$.

(iii) $\mathcal{K}_\gamma (\gamma \in Q_+)$ *is spanned by vectors* $|\ell, L, M>$ *with* $\| L + M \| = m$ *and* $\ell + |L - M| = n - m$ *for* $\gamma = m\alpha_0 + n\alpha_1 = m\delta + (n-m)\alpha$, *hence* $\dim \mathcal{K}_\gamma = P^{re}(\gamma)$, *where* $P^{re}(\gamma)$ *is the number of sets* $\{k_\beta (\in \mathbb{Z}_{\geq 0}); \beta \in \Delta_+^{re}\}$ *such that* $\gamma = \sum_{\beta \in \Delta_+^{re}} k_\beta \beta$.

(iv) $(\mathcal{K}^\dagger, \pi_\mu)$ *is a highest weight right* $\mathfrak{g}$- *module with highest weight* $\Lambda_{\mu,0}$.

(v) The weight space decomposition of the $\mathfrak{g}$- *module* $(\mathcal{K}^\dagger, \pi_\mu)$ *is given as* $\mathcal{K}^\dagger = \sum_{\gamma \in Q_+} \mathcal{K}_\gamma^\dagger$, *where* $\mathcal{K}_\gamma^\dagger = \mathcal{H}_\gamma^\dagger \cap \mathcal{K}$.

(vi) $\mathcal{K}_\gamma^\dagger (\gamma \in Q_+)$ *is spanned by vectors* $< M, L, \ell|$ *with* $\| L + M \| = m$

and $\ell + |L - M| = n - m$ *for* $\gamma = m\alpha_0 + n\alpha_1$, *hence* $\dim \mathcal{K}_\gamma^\dagger = P^{re}(\gamma)$.

(vii)

(2.5) $$\mathrm{ch}\,(\mathcal{K}, \pi_\mu) = \mathrm{ch}\,(\mathcal{K}^\dagger, \pi_\mu) = e^{\Lambda_{\mu,0}} \prod_{\beta \in \Delta_+^{re}} (1 - e^{-\beta})^{-1}.$$

(viii) The vacuum expectation value $<\,|\,>$ *vanishes on* $\mathcal{K}_{\gamma'}^\dagger \times \mathcal{K}_\gamma$ *unless* $\gamma = \gamma'$ *and is non-degenerate* if $\gamma = \gamma'$.

For any $\Lambda \in \mathfrak{h}^*$, the *Verma module* $M(\Lambda)$ is defined as the left $U(\mathfrak{g})$- and free $U(\mathfrak{n}_-)$ - module generated by $|\Lambda>$ satisfying

$$h|\Lambda> = <\Lambda, h>|\Lambda> \quad (h \in \mathfrak{h}) \quad \text{and} \quad \mathfrak{n}_+|\Lambda> = 0.$$

Then it is well-known (see e.g. Kac [2]) that $M(\Lambda) = U(\mathfrak{n}_-)|\Lambda>$ has the unique irreducible quotient $L(\Lambda)$ and $\mathrm{ch}\, M(\Lambda) = e^\Lambda \prod_{\gamma \in \Delta_+} (1 - e^{-\gamma})^{-1}$.

The dual Verma module $M^\dagger(\Lambda)$ is defined as the right $U(\mathfrak{g})$- and free $U(\mathfrak{n}_+)$-module generated by $<\Lambda|$ satisfying

$$<\Lambda|h = <\Lambda|<\Lambda, h> \quad (h \in \mathfrak{h}) \quad \text{and} \quad <\Lambda|\mathfrak{n}_- = 0.$$

V.G. Kac and D.A. Kazhdan [3] conjectured

(2.6) $$\mathrm{ch}\, L(-\rho) = e^{-\rho} \prod_{\gamma \in \Delta_+^{re}} (1 - e^{-\gamma})^{-1},$$

where ρ is the renormalized half sum of positive roots, and $\rho = \frac{\alpha}{2} + 2\Lambda_0$ in our case $\mathfrak{g} = A_1^{(1)}$. We will show it in the next section (Proposition 3.4).

§3. Singular vectors

A vector v in a highest weight $\mathfrak{g}$-module V is called *singular* if $\mathfrak{n}_+ v = 0$, and $\mathcal{S}(V)$ denotes the set of all singular vectors in the module V. Then it is obvious that $\mathcal{S}(V) \supset V_\Lambda = \mathbb{C} v_\Lambda$. For any $(\mu, \nu) \in \mathbb{C}^2$, set

(3.1) $$\mathcal{S}_{\mu,\nu} = \mathcal{S}(\mathcal{H}, \pi_{\mu,\nu}) \supset \mathcal{S}_\mu = \mathcal{S}_{\mu,0} \cap \mathcal{K} = \mathcal{S}(\mathcal{K}, \pi_\mu),$$

and

$$\mathcal{S}_{\mu,\nu}^\dagger = \mathcal{S}(\mathcal{H}^\dagger, \pi_{\mu,\nu}) \supset \mathcal{S}_\mu^\dagger = \mathcal{S}_{\mu,0}^\dagger \cap \mathcal{K}^\dagger = \mathcal{S}(\mathcal{K}^\dagger, \pi_\mu).$$

By the standard arguments (e.g. [4]), we get

Proposition 3.1. *Let V or $V^\dagger$ be a highest weight left or right $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$ respectively. Assume that $\dim V_{\Lambda-\gamma} = \dim V^\dagger_{\Lambda-\gamma}$ for any $\gamma \in Q_+$, and that the vacuum expectation value $< | >$ is defined on $V^\dagger \times V$ such that*

(i) $< | >$ is $\mathbb{C}$-bilinear ; (ii) $< v^\dagger_\Lambda | v_\Lambda > = 1$;

(iii) $< vx | u > = < v | xu >$ for any $v \in V^\dagger$, $u \in V$ and $x \in \mathfrak{g}$,

(iv) The vacuum expectation value $< | >$ vanishes on $V^\dagger_{\gamma'} \times V_\gamma$ unless $\gamma = \gamma'$, and is non-degenerate if $\gamma = \gamma'$.

Then

(i) The following conditions (a)~(c) are equivalent.

(a) $\mathcal{S}(V) = V_\Lambda \,(= \mathbb{C} v_\Lambda)$;

(b) $V^\dagger$ is generated by the vacuum vector $v^\dagger_\Lambda$ as a $\mathfrak{g}$-module;

(c) The canonical map $M^\dagger(\Lambda) \longrightarrow V^\dagger$ sending vacuum to vacuum is surjective.

(ii) The following conditions (a)~(c) are equivalent.

(a) $\mathcal{S}(V^\dagger) = V^\dagger_\Lambda \,(= \mathbb{C} v^\dagger_\Lambda)$;

(b) V is generated by the vacuum vector v_Λ as a $\mathfrak{g}$-module;

(c) The canonical map $M(\Lambda) \longrightarrow V$ sending vacuum to vacuum is surjective.

(iii) If $\mathcal{S}(V) = V_\Lambda$ and $\mathcal{S}(V^\dagger) = V^\dagger_\Lambda$, then the canonical map factors into the isomorphism $L(\Lambda) \longrightarrow V$.

The module $(\mathcal{B}, \mathcal{H})$ is realized as

(3.2) $$\mathcal{H} = \mathbb{C}[\, x_m\, (m \geq 0),\, y_n,\, z_n \;(n > 0)]$$
$$\supset \mathcal{K} = \mathbb{C}[x_m\,(m \geq 0),\; y_n\;(n > 0)] \ni 1 = |0>,$$
$$a_m = -\frac{\partial}{\partial x_m} \text{ and } a^\dagger_{-m} = \text{multiplication by } x_m \quad (m \geq 0),$$
$$a^\dagger_n = \frac{\partial}{\partial y_n} \text{ and } a_{-n} = \text{multiplication by } y_n \quad (n > 0),$$

and

$$b_n = n\frac{\partial}{\partial z_n} \text{ and } b_{-n} = \text{multiplication by } z_n \quad (n > 0).$$

In this realization, H_n's are expressed as

(3.3) $$H_n = \sum_{j>n} y_{j-n}\frac{\partial}{\partial y_1} - \sum_{j=1}^{n} \frac{\partial}{\partial x_{n-j}}\frac{\partial}{\partial y_j} - \sum_{j\geq 0} x_j \frac{\partial}{\partial x_{n+j}} \quad (n \geq 0),$$

and

$$H_{-n} = \sum_{j>0} y_{j+n}\frac{\partial}{\partial y_j} + \sum_{j=0}^{n-1} y_{n-j}\, x_j - \sum_{j\geq 0} x_{n+j}\frac{\partial}{\partial x_j} \quad (n > 0).$$

The vector $|\ell, L, M, N>$ is realized as

$$x_0^{\ell}\, \mathbf{x}^L\, \mathbf{y}^M\, \mathbf{z}^N = x_0^{\ell}\, x_1^{\ell_1} \cdots x_j^{\ell_j} \cdots y_1^{m_1} \cdots y_j^{m_j} \cdots z_1^{n_1} \cdots z_j^{n_j} \cdots$$

for all $\ell \in \mathbb{Z}_{\geq 0}$ and $L, M, N \in \mathcal{I}$.

Now determine singular vectors in $(\mathcal{K}, \pi_\mu)$. At first, the vacuum vector $1 = |\, 0 >$ is singular. Assume that v is a singular vector of degree m. In the realization (3.1), v is a polynomial of x_n's ($n \geq 0$) and y_n's ($n \geq 1$). By the equation $e_n v = - \dfrac{\partial}{\partial x_n} v = 0$ ($n \geq 0$), the polynomial v does not depend on x_n's ($n \geq 0$).

By the realization (3.2) of H_n, the equation $h_n v = 0$ changes into

$$\sum_{j \geq 1} y_j \frac{\partial}{\partial y_{j+n}} v(y_1, y_2, \cdots) = 0 \qquad (n \geq 1).$$

Hence $h_n v = 0$ is fulfilled automatically for all $n \geqq m$. From $h_{m-1} v = 0$, we know that v does not depend on y_m, and inductively that v depends only on y_1.

Since $[f_1, h_n] = 2f_{n+1}$, it is sufficient that v satisfies the equation $f_1 v = 0$. By the definition of f_1 and (3.2), we get

$$f_1 v = -(\mu + 2)\frac{\partial}{\partial y_1} v - y_1 \frac{\partial^2}{(\partial y_1)^2} v = 0,$$

hence $v = \gamma_0 + \gamma_1 y_1^{-(\mu+1)}$ for some γ_0 and $\gamma_1 \in \mathbb{C}$. Thus we get

Proposition 3. 2.

(i) $\mathcal{I}_\mu \neq \mathbb{C}|\, 0 >$, *if and only if* μ *is an integer with* $\mu \leq -2$.

(ii) In that case, $\mathcal{S}_\mu\, \mathbb{C}1 + \mathbb{C} y_1^{-(\mu+1)} = \mathbb{C}\,|0> + \mathbb{C} e_{-1}^{-(\mu+1)}\,|0>$.

The isomorphism $\tilde{\sigma}$ of $\mathcal{K}^\dagger$ to $\mathcal{K}$ over $(\mathcal{B}, \sigma)$ gives an isomophism of the right module $(\mathcal{K}^\dagger, \pi_\mu)$ to the left $\mathfrak{g}$-module $(\mathcal{K}, \sigma \circ \pi_\mu)$.

Now determine singular vectors in $(\mathcal{K}^\dagger, \pi_\mu)$. At first, the vacuum vector $< 0\,|$ is singular. Assume that $< u\,| \in \mathcal{K}^\dagger$ is a singular vector of degree m. As above, the image $|\, v> = \tilde{\sigma}(< u\,|)$ is a polynomial in x_n's ($n \geqq 0$) and y_n's ($n \geqq 1$) of degree m. The equation $< u\,| e_{-n} = 0$ ($n \geq 1$) implies the equation $a_n^\dagger |\, v> = \dfrac{\partial}{\partial y_n} v = 0$ ($n \geq 1$), hence the polynomial v does not depend on y_n's ($n \geqq 1$).

By the realization (3.2) of H_n, the equation $< u\,| h_{-n} = 0$ ($n \geq 1$) implies $H_n\,|\, v> = 0$, and further into

$$\sum_{j \geqq 0} x_j \frac{\partial}{\partial x_{j+n}} v(x_1, x_2, \cdots) = 0 \quad (n \geq 1).$$

Hence $< u\,| h_{-n} = 0$ is fulfilled automatically for all $n > m$. From the equation

$< u \mid h_{-m} = 0$, we know that v does not depend on x_m, and inductively that v depends only on x_0.

Since $[f_0, h_n] = 2f_n$, it is sufficient that $<u|$ satisfies the equation $< u \mid f_0 = 0$, that is, $\sigma(f_0) \mid v >= 0$. By the definition of f_0 and (3.2), we get

$$\sigma(f_0) \mid v > = \mu \frac{\partial}{\partial x_0} v - x_0 \frac{\partial^2}{(\partial x_0)^2} v = 0,$$

hence $\mid v > = \gamma_0 + \gamma_0 x_0^{\mu+1} = \gamma_0 |0> + \gamma_1 a_0^{\dagger \mu+1} |0>$ for some γ_0 amd $\gamma_1 \in \mathbb{C}$, that is, $<u| = <0| \gamma_0 + \gamma_1 <0| e_0^{\mu+1}$. Thus we get

Proposition 3.3. *The right* $\mathfrak{g}$*-module* $(\mathcal{H}^\dagger, \pi_\mu)$ *admits singular vectors of positive degree, if and only if* μ *is an integer with* $\mu \geqq 0$. *Moreover, in that case,* $\mathcal{S}_\mu^\dagger = <0|\mathbb{C} + <0| e_0^{\mu+1} \mathbb{C}$.

Then by Propositions 2.2 and 3.1~3, we get

Proposition 3.4. If $\mu \in \mathbb{C} \setminus \mathbb{Z}$ or $\mu = -1$, *then there exists a* $\mathfrak{g}$*-isomorphism*

$$L(\mu \frac{\alpha}{2} - 2\Lambda_0) \longrightarrow (\mathcal{H}, \pi_\mu).$$

In particular, $L(-\rho)$ *is isomorphic with* $(\mathcal{H}, \pi_{-1})$, *hence Kac-Kazhdan conjecture* (2.6) *holds.*

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