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Conformal Field Theory and the Braid Group

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Introduction

We discuss here conformal field theories (WZW-theory) on $\boldsymbol{P}^1$ associated with symmetry of affine Lie algebra of type $X_n^{(1)}$ and related monodromy representations of the braid group. Principally we describe on the same line as the one for $A_1^{(1)}$-symmetry in our previous work [TK] and give explicit formulae for monodromy representations in the case of $A_n^{(1)}$-symmetry. These representations factor through Iwahori's Hecke algebra $H_N(q)$ are thus obtained.

This expository work has an intermediate form between [TK] and [TK2], in the latter the notion of holonomic systems for N-point functions plays a main role rather than vertex operators (operator formalism), and some monodromy representations for the case of $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$-symmetry are determined. These representations factor through Birman-Wenzl-Murakami algebra, a q-analogue of Brauer's algebra.

Contents

Notations

$\mathfrak{g}$: simple Lie algebra of type $X_n=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$

$\bar{\Delta}$, $\bar{\Pi}=\{\alpha_1,\cdots,\alpha_n\}$: system of roots or simple roots of $(\mathfrak{g},\mathfrak{h})$

θ: maximum root

$(\ ,\ )$: nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ normalized as $(\theta,\theta)=2$.

$\mathfrak{k}=\boldsymbol{C}X_{-\theta}+\boldsymbol{C}\theta^\vee+\boldsymbol{C}X_\theta(\cong\mathfrak{sl}(2;\boldsymbol{C}))$; $X_{\pm\theta}\in\mathfrak{g}_{\pm\theta}$, $(X_\theta, X_{-\theta})=1$, $\theta^\vee=[X_\theta, X_{-\theta}]$.

$P=\sum_{i=1}^n\boldsymbol{Z}\bar{\Lambda}_i$: weight lattice of $(\mathfrak{g},\mathfrak{h})$; $<\bar{\Lambda}_i,\alpha_j^\vee>=\delta_{ij}$

$P_+=\sum_{i=1}^n\boldsymbol{Z}_{\geq0}\bar{\Lambda}_i$: the set of dominant integral weights of $(\mathfrak{g},\mathfrak{h})$

$\hat{\mathfrak{g}}=\mathfrak{g}\otimes\boldsymbol{C}[t,t^{-1}]\oplus\boldsymbol{C}c$: the affine Lie algebra of type $X_n^{(1)}\supset\mathfrak{g}=\mathfrak{g}\otimes1$

$X(n)=X\oplus t^n$ for $X\in\mathfrak{g}$ and $n\in\boldsymbol{Z}$

$[X(m),Y(n)]=[X,Y]\ (m+n)+m(X,Y)\delta_{m+n,0}c$

$\hat{\mathfrak{h}}=\mathfrak{h}\oplus\boldsymbol{C}_c$: the Cartan subalgebra of $\hat{\mathfrak{g}}$

$\mathfrak{m}_\pm=\mathfrak{g}\oplus t^\pm\boldsymbol{C}[t^\pm]$, $\hat{\mathfrak{n}}_+=\mathfrak{m}_+\oplus\mathfrak{n}_+$, $\hat{\mathfrak{n}}_-=\mathfrak{m}_-\oplus\mathfrak{n}_-$, $\mathfrak{p}_\pm=\mathfrak{m}_\pm\oplus\mathfrak{g}\oplus\boldsymbol{C}c$: subalgebras of $\hat{\mathfrak{g}}$

g: dual Coxeter number$=n+1$, $2n-1$, $n+1$, $2n-2$, 12, 18, 30, 9, 4 of $\hat{\mathfrak{g}}$, if it is of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$

$\mathscr{L}=\sum_{n\in\boldsymbol{Z}}\boldsymbol{C}e_n+\boldsymbol{C}e_0'$: the Virasoro algebra

$$[e_m,e_n]=(m-n)e_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}e_0';\ [e_m,e_0']=0$$

$\Omega=\sum_i X^iX_i\in U(\mathfrak{g})$: the Casimir element of $\mathfrak{g}$ ($\{X^i\}$, $\{X_i\}$ are dual bases of $\mathfrak{g}$)

$\circ X(m)Y(n)\circ$ the normal ordered product for $X(m)$, $Y(n)\in\mathfrak{g}\otimes\boldsymbol{C}[t,t^{-1}]$

$$=X(m)Y(n)\ (m<n);\ \frac{1}{2}(X(m)Y(n)+Y(n)X(m))\ (m=n);\ Y(n)X(m)\ (m>n)$$

$X(z)=\sum_{k\in\boldsymbol{Z}}X(n)z^{-n-1}$ $(z\in\boldsymbol{C}^*$, $X\in\mathfrak{g})$: a current

$T(z)=\dfrac{1}{2(g+c)}\sum_i\ \circ X^i(z)X_i(z)\circ\ (z\in\boldsymbol{C}^*)$

$\quad=\sum_{n\in\boldsymbol{Z}}L(m)z^{-m-2}$; the energy momentum tensor

$L(m)=\dfrac{1}{2(g+c)}\sum_{k\in\boldsymbol{Z}}\sum_i\circ\ X^i(-k)X_i(m+k)\circ$: the Sugawara form

l: the central charge (level of integrable representation) (we fix $l\in\boldsymbol{C}_{>0}$)

$\kappa=l+g$

$P_l=\{\lambda\in P_+;\ (\lambda,\theta)\leqslant l\}$

W_j: the irreducible left $\mathfrak{k}$-module of spin j for $j\in\frac{1}{2}\boldsymbol{Z}_{\geq0}$

V_λ: the irreducible left $\mathfrak{g}$-module of highest weight $\lambda \in P_l$, decomposed as a $\mathfrak{k}$-module:

$$V_\lambda = \sum_{2j \leqslant (\theta,\lambda)} m_j W_j$$

$W_j^\vee = \mathrm{Hom}(W_j, \boldsymbol{C})$, $V_\lambda^\vee = \mathrm{Hom}(V_\lambda, \boldsymbol{C})$: the dual (right) $\mathfrak{g}(\mathfrak{k})$-module of W_j, V_λ

$W_j^\dagger$: the irreducible right $\mathfrak{k}$-module of spin j for $j \in \frac{1}{2}\boldsymbol{Z}_{\geq 0}$

$V_\lambda^\dagger$: the irreducible right $\mathfrak{g}$-module of highest weight $\lambda \in P_l$

$\mathscr{M}_\lambda = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{p}_+)} |\lambda\rangle$: the Verma module as a left $\hat{\mathfrak{g}}$-module of highest weight $l\Lambda_0 + \lambda$

$(\lambda \in P_l)$ $(\mathfrak{m}_+|\lambda\rangle = \mathfrak{n}_+|\lambda\rangle = 0, c|\lambda\rangle = l|\lambda\rangle, h|\lambda\rangle = \langle\lambda, h\rangle|\lambda\rangle)$.

$\mathscr{M}_\lambda^\dagger = \langle\lambda| \otimes_{U(\mathfrak{p}_-)} U(\hat{\mathfrak{g}})$: the Verma module as a right $\hat{\mathfrak{g}}$-module of highest weight $l\Lambda_0 + \lambda$

$(\langle\lambda|\mathfrak{m}_- = \langle\lambda|\eta_- = 0,\ \langle\lambda|c = l\langle\lambda|,\ \langle\lambda|h = \langle\lambda, h\rangle\ \langle\lambda|)$.

$\mathscr{I}_\lambda = U(\mathfrak{p}_-)|J_\lambda\rangle$: the proper maximal $\hat{\mathfrak{g}}$-submodule of $\mathscr{M}_\lambda$

$\mathscr{I}_\lambda^\dagger = \langle J_\lambda|U(\mathfrak{p}_+)$: the proper maximal $\hat{\mathfrak{g}}$-submodule of $\mathscr{M}_\lambda^\dagger$

$$|J_\lambda\rangle = X_\theta(-1)^{l-(\lambda,\theta)+1}|\lambda\rangle, \quad \langle J_\lambda| = \langle\lambda|X_{-\theta}(1)^{l-(\lambda,\theta)+1}$$

π_λ: $\mathscr{M}_\lambda \to \mathscr{H}_\lambda$, $\pi_\lambda^\dagger$: $\mathscr{M}_\lambda^\dagger \to \mathscr{M}_\lambda^\dagger$: the canonical projection

$\mathscr{H}_\lambda = \mathscr{M}_\lambda/\mathscr{I}_\lambda$: the integrable highest weight left $\hat{\mathfrak{g}}$-module

$\mathscr{H}_\lambda^\dagger = \mathscr{I}_\lambda \backslash \mathscr{M}_\lambda^\dagger$: the integrable highest weight right $\hat{\mathfrak{g}}$-module

$\langle|\rangle$: $V_\lambda^\dagger \times V_\lambda \to \boldsymbol{C}$, $\mathscr{H}_\lambda^\dagger \times \mathscr{H}_\lambda \to \boldsymbol{C}$: the vacuum expectation values

$$\langle\lambda|\lambda\rangle = 1, \quad \langle ua|v\rangle = \langle u|av\rangle \quad \text{for any } a \in \mathfrak{g} \text{ or } \hat{\mathfrak{g}}$$

$\mathscr{H}_{\lambda,d}$ and $\mathscr{H}_{\lambda,d}^\dagger$: the eigenspaces of $\mathscr{H}_\lambda$ and $\mathscr{H}_\lambda^\dagger$ for the operator $L(0)$ belonging to the eigenvalue $\Delta_\lambda + d$ respectively

$\hat{\mathscr{H}}_\lambda = \prod_{d \geqslant 0} \mathscr{H}_{\lambda,d}$ and $\hat{\mathscr{H}}_\lambda^\dagger = \prod_{d \geqslant 0} \mathscr{H}_{\lambda,d}^\dagger$: completions of $\mathscr{H}_\lambda$ and $\mathscr{H}_\lambda^\dagger$ respectively

$$\mathscr{H} = \sum_{\lambda \in P_l} \mathscr{H}_\lambda \subset \hat{\mathscr{H}} = \sum_{\lambda \in P_l} \hat{\mathscr{H}}_\lambda;\ \mathscr{H}^\dagger = \sum_{\lambda \in P_l} \mathscr{H}_\lambda^\dagger \subset \hat{\mathscr{H}} = \sum_{\lambda \in P_l} \hat{\mathscr{H}}_\lambda^\dagger.$$

Π_λ: $\mathscr{H} \to \mathscr{H}_\lambda$, $\hat{\mathscr{H}} \to \hat{\mathscr{H}}_\lambda$, $\mathscr{H}^\dagger \to \mathscr{H}_\lambda^\dagger$, $\hat{\mathscr{H}}^\dagger \to \hat{\mathscr{H}}_\lambda^\dagger$: the projections to the λ-component

$\mathbb{V} = \{\mathbb{v} = \begin{pmatrix} & \lambda & \\ \lambda_2 & & \lambda_1 \end{pmatrix};\ \lambda, \lambda_1, \lambda_2 \in \bar{P}_+\}$: the set of vertices

$\cup$

$\mathbb{V}_l = \{\mathbb{v} \in \mathbb{V};\ \lambda_1, \lambda_2 \in P_l\}$

$\mathscr{V}er(\mathbb{v})$ = the set of all vertex operators of type $\mathbb{v}$

$\wr\wr$

$\mathscr{V}(\mathbb{v}) = \{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\lambda_1};\ V_{\lambda_2});\ \mathrm{proj}_{W_{j_2}} \cdot \varphi(W_j \otimes W_{j1}) = 0$ for any $\mathfrak{k}$-simple submodules W_j, W_{j_1}, W_{j_2} of V_λ, V_{λ_1}, V_{λ_2} resp. with $j + j_1 + j_2 > l\}$

$(lCG)=\{\mathbb{v}\in\mathbb{V}_l;\ \mathscr{V}(\mathbb{v})\neq 0\}$: the set of all lCG-vertices

$\Delta_\lambda=\dfrac{(\lambda,\ \lambda+2\rho)}{2k}=\dfrac{1}{2k}\Omega|_{V_\lambda}$: the conformal dimension of vertex operators of weight λ

$\Delta(\mathbb{v})=\Delta_\lambda$: the conformal dimension of a vertex $\mathbb{v}$

$\hat{\Delta}(\mathbb{v})=\Delta_\lambda+\Delta_{\lambda_1}-\Delta_{\lambda_2}$ for a vertex $\mathbb{v}$

$\Phi(u;\ z)=\Phi(z)\ (u\otimes\cdot)=\sum_{n\in\mathbf{Z}}\Phi_u(n)z^{-n-\hat{\Delta}(\mathbb{v})}$: the homogeneous decomposition of a vertex operator $\Phi(z)$ of type $\mathbb{v}$

$\Phi_\varphi(z)$: the vertex operator of type $\mathbb{v}$ whose initial term $\Phi.(0)$ is $\varphi\in\mathscr{V}(\mathbb{v})$ for each $\mathbb{v}\in(lCG)$ (considered as $V_\lambda\otimes\mathscr{H}_{\lambda_1}\to\hat{\mathscr{H}}_{\lambda_2}$)

λ^+: the anti-weight of λ, defined as $-\lambda^+$ is the lowest weight of V_λ, i.e. $\lambda^+=-w_0(\lambda)$, where w_0 is the longest element of the Weyl group of $(\mathfrak{g},\ \mathfrak{h})$

ν: $V_{\lambda^+}\to V_\lambda^\dagger$: the isomorphism defined by $\nu(|\lambda^+\rangle)=\langle\lambda^+|$ and $\nu(X|v\rangle)=-\nu(|v\rangle)X$ ($|v\rangle\in V_{\lambda^+}$, $X\in\mathfrak{g}$).

ν: $\mathscr{H}_{\lambda^+}\to\mathscr{H}_\lambda^\dagger$: the isomorphism extended by the above ν and $\nu(X(m)|v\rangle)=-\nu(|v\rangle)\,X(m)$ ($|v\rangle\in\mathscr{H}_{\lambda^+}$, $X\in\mathfrak{g}$, $m\in\mathbf{Z}$)

Let $M=M_1\otimes\cdots\otimes M_N$ the tensor product of $\mathfrak{g}$-modules M_k, then

ρ_j: the $\mathfrak{g}$-action on the j-th component of M

$\Delta_{jk}=\rho_j+\rho_k$: the diagonal action on the j-th and k-th components of M

$$\Omega_{jk}=\sum\rho_j(X_i)\rho_k(X_i)=\frac{1}{2}(\Delta_{jk}(\Omega)-\Omega_{jj}-\Omega_{kk})$$

$$\Omega_m^\vee=-\sum_{1\leq i\neq k\leq m}\Omega_{ik}$$

$\Lambda=(\lambda_N,\cdots,\ \lambda_1)$: an N-ple of weights $\lambda_i\in P_l$

$V_\Lambda=V_{\lambda_N}\otimes\cdots\otimes V_{\lambda_1}$, $V_\Lambda^\vee=V_{\lambda_N}^\vee\otimes\cdots V_{\lambda_1}^\vee$

$V_{\mathfrak{g}}^\vee(\Lambda)=\mathrm{Hom}_{\mathfrak{g}}(V_\Lambda\ ;\ \mathbf{C})$ the space of all $\mathfrak{g}$-invariant forms of V_Λ

$\cup$

$\mathscr{V}(\Lambda)=\sum\mathscr{V}(\Lambda)$; $\mu=(\mu_{N-1},\cdots\mu_1)\in(P_l)^{N+1}$,

$$\mathscr{V}(\Lambda)_\mu=\mathscr{V}(\mathbb{v}_N(\mu))\otimes\cdots\otimes\mathscr{V}(\mathbb{v}_i(\mu))\otimes\cdots\otimes\mathscr{V}(\mathbb{v}_1(\mu))$$

$$\mathbb{v}_N(\mu)=\begin{pmatrix}\lambda_N\\ 0\ \ \mu_{N-1}\end{pmatrix},\cdots,\ \mathbb{v}_i(\mu)=\begin{pmatrix}\lambda_i\\ \mu_i\ \ \mu_{i-1}\end{pmatrix},\cdots\ \mathbb{v}_1(\mu)=\begin{pmatrix}\lambda_1\\ \mu_1\ \ 0\end{pmatrix}$$

$(\varphi_N\otimes\cdots\otimes\varphi_1)\ (u_N\otimes\cdots\otimes u_1)=\langle 0|\varphi_N(u_N)\circ\cdots\circ\varphi_1(u_1)(|0\rangle)$ ($\varphi_i\in\mathscr{V}(\mathbb{v}_i(\mu))$, $u_i\in V_{\lambda_i}$)

$\mathscr{V}er(\Lambda)$: the space of N-point functions of weight Λ;

$$\Phi_\varphi(z)=\langle\Phi_{\varphi_N}(z_N)\cdots\Phi_{\varphi_1}(z_1)\rangle\in\mathscr{V}er(\Lambda)\ \ (\varphi=\varphi_N\otimes\cdots\otimes\varphi_1\in\mathscr{V}(\Lambda))$$

$$\Delta_m^\vee(\Lambda)=-\sum_{i=1}^m\hat{\Delta}(\mathrm{v}_i(\Lambda))$$

$\Lambda=(\lambda_4,\ \lambda_3,\ \lambda_2,\ \lambda_1)$: a quadruple of $\lambda_i\in P_l$

$$V_{\mathfrak{g}}^\vee(\Lambda)_\mu^{12}=\mathrm{Hom}_{\mathfrak{g}}(V_\mu\otimes V_{\lambda_3},\ V_{\lambda_4^+})\otimes\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_2}\otimes V_{\lambda_1},\ V_\mu)\ \ (\mu\in P_+)$$

$$\mathscr{V}(\Lambda)_\mu^{12}=\mathscr{V}\begin{pmatrix} & \lambda_3 & \\ \lambda_4^+ & & \mu\end{pmatrix}\otimes\mathscr{V}\begin{pmatrix} & \lambda_2 & \\ \mu & & \lambda_1\end{pmatrix}\quad(\mu\in P_l)$$

$$C_\Lambda^{12}:\ \sum_{\mu\in P_+}V_{\mathfrak{g}}^\vee(\Lambda)_\mu^{12}\rightarrow V_{\mathfrak{g}}^\vee(\Lambda)$$

$$C_\Lambda^{12}(\varphi_2\otimes\varphi_1)\ (u_3\otimes u_2\otimes u_1)=\varphi_2(u_3\otimes\varphi_1(u_2\otimes u_1))\quad(\varphi_2\otimes\varphi_1\in V_{\mathfrak{g}}^\vee(\Lambda)_\mu^{12},\ u_i\in V_{\lambda_i})$$

$$I(\Lambda)=\{\mu\in P_+;\ V_{\mathfrak{g}}^\vee(\Lambda)_\mu^{12}\neq\{0\}\}\supset I_l(\Lambda)=\{\mu\in P_l;\mathscr{V}(\Lambda)_\mu^{12}\neq\{0\}\}$$

$$\Delta_4(\Lambda)=\hat{\Delta}(\mathrm{v}_2)+\hat{\Delta}(\mathrm{v}_1)=\Delta_{\lambda_1}+\Delta_{\lambda_2}+\Delta_{\lambda_3}-\Delta_{\lambda_4}$$

For each $\tau\in P_l$

$$\mathscr{V}(N;\ \tau)=\sum_{\mu}\mathscr{V}(N;\ \tau)_\mu,\qquad \mu=(\mu_{N-1},\cdots,\ \mu_1)\in(P_l)^{N-1}$$

$$\mathscr{V}(N;\ \tau)_\mu=\mathscr{V}\left(\begin{pmatrix} & \square & \\ \tau & & \mu_{N-1}\end{pmatrix}\right)\otimes\cdots\otimes\mathscr{V}\left(\begin{pmatrix} & \square & \\ \mu_i & & \mu_{i-1}\end{pmatrix}\right)\otimes\cdots\otimes\mathscr{V}\left(\begin{pmatrix} & \square & \\ \mu_1 & & 0\end{pmatrix}\right)$$

$X_N=\{(z_N,\cdots,\ z_1)\in\boldsymbol{C}^N;\ z_i\neq z_k\ \ (i=k)\}$
$\cup$
$M_N=\{(z_N,\cdots,\ z_1)\in(\boldsymbol{C}^*)^N;\ z_i\neq z_k\ \ (i\neq k)\}$

$\mathscr{R}_z=\{z=(z_N,\cdots,\ z_1)\in\boldsymbol{C}^N;\ |z_N|>\cdots>|z_1|\}\subset X_N$
$\cup$
$\mathscr{R}_{z,0}=\{(z_N,\cdots,\ z_1)\in\boldsymbol{C}^N;\ |z_N|>\cdots>|z_1|>0\}$
$\cup$
$I_N=\{(z_N,\cdots,\ z_1)\in\boldsymbol{R}^N;\ z_N>\cdots>z_1>0\}$

$\mathscr{R}_w=\{\mathrm{w}\in\boldsymbol{C}^N;1>|w_i|\ \ (1\leqslant i\leqslant N-1)\}$
$\cup$
$\mathscr{R}_{w,0}=\{\mathrm{w}=(w_N,\cdots,\ w_1)\in\boldsymbol{C}^N;\ w_N\neq0,\ 1>|w_i|>0\ \ (2\leqslant i\leqslant N-1),\ 1>|w_1|\}$

$(\tilde{X}_N,\ \tilde{\pi}_N,\ X_N)$: the universal covering space of X_N

$\mathfrak{S}_N$: the N-th symmetric group

$\{\sigma_1,\cdots,\ \sigma_{N-1}\}$: the canonical generators $(\sigma_i=(i,\ i+1))$ of $\mathfrak{S}_N$

p_N: $X_N\rightarrow\bar{X}_N=X_N/\mathfrak{S}_N$: the canonical projection

B_N: the braid group with N-strings of $\boldsymbol{C}$ $(\cong\pi_1\ (\bar{X}_N,*))$

B_N is generated by $\{b_1,\cdots,\ b_{N-1}\}$ and fundamental relations:

$$b_ib_{i+1}b_i=b_{i+1}b_ib_{i+1}\ \ (1\leqslant i\leqslant N-2);\ b_ib_j=b_jb_i\ \ (|i-j|\geqslant2)$$

π_N: $B_N \to \mathfrak{S}_N$: the homomorphism defined by $\pi_N(b_i)=\sigma_i$

$P_N=\ker\pi_N$: the pure braid group with N-strings of $\boldsymbol{C}$ ($\cong\pi_1(X_N, *)$)

$H_N(q)$: the Hecke algebra of type A_{N-1}

$\{T_1,\cdots, T_{N-1}\}$: the canonical generators of $H_N(q)$

$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$ $(1\leqslant i\leqslant N-2)$; $T_iT_j=T_jT_i$ $(|i-j|\geqslant 2)$;

$(T_i-q)\ (T_i+1)=0$

$\{e_1,\cdots, e_{N-1}\}$: the system of idempotent generators of $H_N(q)$; $e_j=(q-T_i)/[2]_q$

$\mathcal{Y}_N$: the set of all Young diagrams on N-nodes

$Y=[f_1,\cdots,f_k]$: the Young diagram such that the number of nodes of the i-th row is f_i

$(f_1\geqslant\cdots\geqslant f_k)$

$\mathcal{Y}_N^g$: the set of all Young diagrams on N-nodes with depth$\leqslant$g

$\mathcal{Y}_N^{(g,\kappa)}=\{Y=[f_1, f_2,\cdots, f_g]\in\mathcal{Y}_N^g;\ f_1-f_g\leqslant\kappa-g=l\}$ for type $A_n^{(1)}$

$\Gamma(z)$: the gamma function

$F(\alpha, \beta, \gamma,; z)$: the Gauss' hypergeometric function

$[\mathrm{i}]_q=\dfrac{q^i-1}{q-1}$ $(q\neq 1)$, i $(q=1)$: a q-integer $(i\in\boldsymbol{Z})$

$\dbinom{L}{\mathfrak{m}}=\dfrac{L!}{m_N!\cdots m_1!}$: the multinomial coefficent for $\mathfrak{m}=(m_N,\cdots, m_1)$ with $L=\sum m_k$

§1. Preliminaries

In this section, we summarize the facts about affine Lie algebras of type $X_n^{(1)}$, their representations and relations with the Virasoro algebra mainly after V. G. Kac [Ka]. Conformal field theory concerns with Virasoro algebra symmetries, and we treat here its Wess-Zumino theory, *i.e.* also with affine Lie algebra symmetries.

1.1) Simple Lie Algebras of type X_n

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\boldsymbol{C}$ of type X_n, and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denote by Δ the root system of $(\mathfrak{g}, \mathfrak{h})$. Then the Lie algebra $\mathfrak{g}$ has the root space decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus\sum_{\gamma\in\Delta}\mathfrak{g}_\gamma,$$

where

$$\mathfrak{g}_\gamma=\{X\in\mathfrak{g};\ [H, X]=\langle\gamma, H\rangle X \quad\text{for any } H\in\mathfrak{h}\}.$$

Choose a fundamental system $\Pi=\{\alpha_1,\cdots, \alpha_n\}$ of roots of $(\mathfrak{g}, \mathfrak{h})$ and denote by Δ_+

the set of the corresponding positive roots. Let $\{E_i \in \mathfrak{g}_{\alpha_i}, F_i \in \mathfrak{g}_{-\alpha_i} \ (1 \leqslant i \leqslant n)\}$ be the Chevalley generators and $\{H_i \in \mathfrak{h} \ (1 \leqslant i \leqslant n)\}$ be the coroot basis, that is,

$$[E_i, F_j] = \delta_{ij} H_j, \ [H_i, E_j] = \langle \alpha_j, H_i \rangle E_j, \ [H_i, F_j] = -\langle \alpha_j, H_i \rangle F_j$$

$(1 \leqslant i \leqslant n)$, and the matrix $[\langle \alpha_j, \mathrm{H}_i \rangle]_{1 \le i,j \le n}$ is the Cartan matrix of type X_n.

Let (,) be the non-degenerate, symmetric and invariant bilinear form on $\mathfrak{g}$ with the normalized condition $(\theta, \theta) = 2$, where θ is the maximum root.

Let $\{X_1, \cdots, X_n\}$ and $\{X^1, \cdots, X^n\}$ be dual bases of $\mathfrak{h}$ w.r.t. the form (,). For any $\gamma \in \Delta$, choose elements $X_\gamma \in \mathfrak{g}_\gamma$ such that $(X_\gamma, X_{-\gamma}) = 1$ and put $X^\gamma = X_{-\gamma}$. The *Casimir operator* of $\mathfrak{g}$ is defined by

$$\Omega = \sum_{j=1}^{n} X^j X_j + \sum_{\gamma \in \Delta} X^\gamma X_\gamma \in U(\mathfrak{g}).$$

Then Ω is a central element in $U(\mathfrak{g})$ and is independent of the choice of dual bases.

Denote by $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ generated by X_θ, $X_{-\theta}$ and $\theta^\vee = [X_\theta, X_{-\theta}]$, then $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$.

Introduce the *fundamental weights* $\Lambda_i \in \mathfrak{h}^* \ (1 \leqslant i \leqslant n)$ of $(\mathfrak{g}, \mathfrak{h})$ defined by

$$\langle \Lambda_i, H_j \rangle = \delta_{ij} \ (1 \leqslant i, j \leqslant n),$$

and the *weight lattice* P and the set P_+ of *dominant integral weights* of $(\mathfrak{g}, \mathfrak{h})$:

$$P = \{\lambda \in \mathfrak{h}^*; \ \langle \lambda, H_i \rangle \in \boldsymbol{Z} \ (1 \leqslant i \leqslant n)\} = \sum_{i=1}^{n} \boldsymbol{Z} \Lambda_i \supset P_+ = \sum_{i=1}^{n} \boldsymbol{Z}_{\ge 0} \Lambda_i.$$

Now we summarize the facts on finite dimensional $\mathfrak{sl}(2; \boldsymbol{C})$- and $\mathfrak{g}$-modules:

Proposition 1.1. *Fix any nonnegative half integer j. Then there exists a unique irreducible left and right $\mathfrak{k}$-module W_j and $W_j^\dagger$ of dimension $2j+1$ respectively. The modules W_j and $W_j^\dagger$ are generated by highest weight vectors $|j\rangle$ and $\langle j|$ with the fundamental relations:*

$$X_\theta |j\rangle = 0, \quad \theta^\vee |j\rangle = 2j|j\rangle, \quad X_{-\theta}^{2j+1} |j\rangle = 0;$$
$$\langle j| X_{-\theta} = 0, \quad \langle j| \theta^\vee = 2j \langle j|, \quad \langle j| X_\theta^{2j+1} = 0.$$

The spaces W_j and $W_j^\dagger$ have the weight space decompositions

$$W_j = \sum_{\substack{k=-j \\ k \equiv j(2)}}^{j} W_{j,k} \text{ and } W_j^\dagger = \sum_{\substack{k=-j \\ k \equiv j(2)}}^{j} W_{j,k}^\dagger,$$

where $W_{j,k} = \{|v\rangle \in W_j; \ \theta^\vee |v\rangle = 2\mathrm{k}|v\}$ and $W_{j,k}^\dagger = \{\langle u| \in W_j^\dagger; \ \langle u| \theta^\vee = 2k \langle u|\}$ are 1-dimensional.

Proposition 1.2. *Fix a dominant integral weight $\lambda \in P_+$.*

I) There exists a unique irreducible (finite-dimensional) left $\mathfrak{g}$-module V_λ with highest weight λ. The module V_λ is generated by a highest weight vector $|\lambda\rangle$ with the fundamental relations:

$$E_i |\lambda\rangle = 0, \quad H_i |\lambda\rangle = \langle \lambda, H_i \rangle |\lambda\rangle, \quad F_i^{\langle \lambda, H_i \rangle + 1} |\lambda\rangle = 0 \quad (1 \leqslant i \leqslant n).$$

The space V_λ *has the weight space decomposition*

$$V_\lambda = \sum_{\mu \in P(\lambda)} V_{\lambda,\mu},$$

where $P(\lambda)$ *is the set of weights in* V_λ *and* $V_{\lambda,\mu} = \{|v\rangle \in V_\lambda;\ H|v\rangle = \langle \mu, H\rangle |v\rangle\ \ (H \in \mathfrak{h})\}$.

II) There exists a unique irreducible (finite-dimensional) right $\mathfrak{g}$*-module* V_λ^+ *with highest weight* λ. *The module* V_λ^+ *is generated by a highest weight vector* $\langle \lambda|$ *with the fundamental relations:*

$$\langle \lambda | F_i = 0, \quad \langle \lambda | H_i = \langle \lambda, H_i\rangle \langle \lambda|, \quad \langle \lambda | E_i^{\langle \lambda, H_i\rangle + 1} = 0 \quad (1 \leqslant i \leqslant n).$$

The space V_λ^+ *has the weight space decomposition*

$$V_\lambda^+ = \sum_{\mu \in P(\lambda)} V_{\lambda,\mu}^+,$$

where $V_{\lambda,\mu}^+ = \{\langle u| \in V_\lambda;\ \langle \mathrm{u}|\mathrm{H} = \langle \mu, \mathrm{H}\rangle \langle \mathrm{u}|\ \ (\mathrm{H} \in \mathfrak{h})\}$.

III (i) The Casimir operator Ω *acts on* V_λ^+ *and* V_λ *as* $\Omega = (\lambda, \lambda + 2\rho)id$, *where* $2\rho = \sum_{\gamma \in \Delta_+} \gamma$.

(ii) There exists a unique nondegenerate bilinear form

$$\langle | \rangle:\ V_\lambda^+ \times V_\lambda \to \boldsymbol{C}$$

such that 1) $\langle va|v\rangle = \langle u|av\rangle$ *for any* $a \in \mathfrak{g}, \langle u| \in V_\lambda^+$ *and* $|v\rangle \in V_\lambda$, 2) $\langle \lambda|\lambda\rangle = 1$. *The restriction* $\langle | \rangle:\ V_{\lambda,\mu}^+ \times V_{\lambda,\mu'} \to \boldsymbol{C}$ *vanishes unless* $\mu = \mu'$, *and* $\langle | \rangle:\ V_{\lambda,\mu}^+ \times V_{\lambda,\mu} \to \boldsymbol{C}$ *is nondegenerate for any* $\mu \in P(\lambda)$.

IV) As $\mathfrak{k}$*-modules,* V_λ *and* V_λ^+ *are decomposed as follows:*

$$V_\lambda = \sum_j m_{\lambda,j} W_j \ \text{ and } \ V_\lambda^+ = \sum_j m_{\lambda,j} W_j^+,$$

where j *runs through the set* $\left\{ j \in \frac{1}{2}\boldsymbol{Z}_{\geq 0};\ 2j \leqslant (\theta, \lambda)\right\}$ *and* $m_{\lambda,j} \in \boldsymbol{Z}_{\geq 0}$ *are multiplicities.*

1.2 Affine Lie Algebrasof type $X_n^{(1)}$.

The affine Lie algebra $\hat{\mathfrak{g}}$ of type $X_n^{(1)}$ is defined by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \boldsymbol{C}[t, t^{-1}] \oplus \boldsymbol{C}c,$$

with the relations

$$\begin{cases} [X(m),\ Y(k)] = [X,\ Y]\ (m+k) + m\delta_{m+k,0}(X,\ Y)c \\ c \in \text{the center of } \hat{\mathfrak{g}}, \qquad (X,\ Y \in \mathfrak{g},\ m,\ k \in \boldsymbol{Z}) \end{cases}$$

where

$$X(m) = X \otimes t^m \in \hat{\mathfrak{g}} \qquad (X \in \mathfrak{g},\ m \in \boldsymbol{Z}).$$

The Lie algebra $\mathfrak{g}$ is included in $\hat{\mathfrak{g}}$ by identifying X with $X(0)$. Introduce the subspace $\mathfrak{g}(m) = \mathfrak{g} \otimes t^m$ of $\hat{\mathfrak{g}}$ for each $m \in \boldsymbol{Z}$, and the subalgebras $\mathfrak{m}_\pm$ and $\mathfrak{p}_\pm$ defined by

$$\mathfrak{p}_\pm = \mathfrak{m}_\pm \oplus \mathfrak{g} \oplus \boldsymbol{C}c \supset \mathfrak{m}_\pm = \sum_{m \geqslant 1} \mathfrak{g}(\pm m) = \mathfrak{g} \otimes t^\pm \boldsymbol{C}[t^\pm].$$

Then $\hat{\mathfrak{g}}$ is decomposed into

$$\hat{\mathfrak{g}}=\mathfrak{m}_+\oplus(\mathfrak{g}\oplus Cc)\oplus\mathfrak{m}_-=\mathfrak{p}_+\oplus\mathfrak{m}_-=\mathfrak{m}_+\oplus\mathfrak{p}_-.$$

The Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ is $\hat{\mathfrak{h}}=\mathfrak{h}\oplus Cc$, where we identify $\mathfrak{h}=\mathfrak{h}\otimes 1$ and $\mathfrak{g}=\mathfrak{h}\otimes 1$. The dual $\mathfrak{h}^*$ of $\mathfrak{h}$ is considered as the subspace of $\hat{\mathfrak{h}}^*$ by setting $\alpha|_{\mathfrak{h}}=\alpha$ and $\langle\alpha, c\rangle=0$ for $\alpha\in\mathfrak{h}^*$.

The Lie algebra $\hat{\mathfrak{g}}$ has the root space decomposition

$$\hat{\mathfrak{g}}=\hat{\mathfrak{h}}\oplus\sum_{k\neq 0}\mathfrak{h}\otimes t^k\oplus\sum_{k\in Z}\sum_{\gamma\in\Delta}\mathfrak{g}_\gamma\otimes t^k=\hat{\mathfrak{n}}_-\oplus\hat{\mathfrak{h}}\oplus\hat{\mathfrak{n}}_+,$$

where $\hat{\mathfrak{n}}_\pm=\mathfrak{m}_\pm\oplus\mathfrak{n}_\pm$ and $\mathfrak{n}_\pm=\sum_{\gamma\in\Delta_+}\mathfrak{g}_{\pm\gamma}$ are subalgebras of $\hat{\mathfrak{g}}$ and $\mathfrak{g}$ respectively.

Put $\alpha_0^\vee=c-\theta^\vee(0)$, $e_0=X_{-\theta}(1)$, $f_0=X_\theta(-1)$, $\alpha_j^\vee=H_j(0)$, $e_i=E_i(0)$ and $f_i=F_i(0)$ $(1\leqslant i\leqslant n)$. Then $\{\alpha_i^\vee\ (0\leqslant i\leqslant n)\}$ forms a basis of $\mathfrak{h}$ and

$$[e_i, f_j]=\delta_{ij}\alpha_i^\vee,\quad [\alpha_i^\vee, e_j]=a_{ij}e_j,\quad [\alpha_i^\vee, f_j]=-a_{ij}f_j\quad (0\leqslant i\leqslant n),$$

where $a_{ij}=\langle\alpha_j, \alpha_i^\vee\rangle$ and the matrix $A=(a_{ij})_{0\leq i,j\leq n}$ is the generalized Cartan matrix of type $X_n^{(1)}$.

Moreover $c=\alpha_0^\vee+\theta^\vee=\sum_{j=0}^n a_j^\vee\alpha_j^\vee$, where $a_0^\vee$, $a_1^\vee,\cdots$, $a_n^\vee$ are mutually prime positive integers whose sum $g=\sum_{j=0}^n a_j^\vee=\frac{1}{2}\{(\theta, \theta)+2(\theta, \rho)\}$ is called *dual Coxeter number.* It is known by the classification of the affine Lie algebras as

$$g=n+1,\ 2n-1,\ 2n-2,\ 12,\ 18,\ 30,\ 9,\ 4,$$

if $\hat{\mathfrak{g}}$ is of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ respectively.

Introduce the *fundamental weights* $\Lambda_i\in\hat{\mathfrak{h}}^*$ $(0\leqslant i\leqslant n)$ of $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ defined by

$$\langle\Lambda_i, \alpha_j^\vee\rangle=\delta_{i,j}\quad (0\leqslant i, j\leqslant n),$$

and the *weight lattice* $\hat{P}$ of $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ by

$$\hat{P}=\{\Lambda\in\mathfrak{h}^*;\ \langle\Lambda, \alpha_i^\vee\rangle\in\boldsymbol{Z}\ (0\leqslant i\leqslant n)\}=\sum_{i=0}^n\boldsymbol{Z}\Lambda_i\supset\hat{P}_+=\sum_{i=0}^n\boldsymbol{Z}_{\geq 0}\Lambda_i.$$

Note

$$\Lambda_i=a_i^\vee\Lambda_0+\bar{\Lambda}_i\qquad (1\leqslant i\leqslant n).$$

The number $\langle\Lambda, c\rangle$ is called of the *level* of $\Lambda\in\hat{P}$. For a fixed integer $l\geqslant 1$, the set $\hat{P}_l=\{\Lambda\in\hat{P}_+;\langle\Lambda, c\rangle=l\}$ is bijective with the set $P_l=\{\lambda\in P_+;\ (\lambda, \theta)\leqslant l\}$ by assigning $\Lambda_\lambda=l\Lambda_0+\lambda\in\hat{P}_l$ to $\lambda\in P_l$.

Proposition 1.3. *Irreducible integrable highest weight $\hat{\mathfrak{g}}$-modules are parametrized by the set $\{(l, \lambda);\ l\in\boldsymbol{Z}_{>0},\ \lambda\in P_l\}$. Fix such (l, λ).*

I. There exists a unique irreducible (integrable and highest weight) left $\hat{\mathfrak{g}}$-module $\mathscr{H}_\lambda$ with a nonzero vector $|\lambda\rangle$ (called vacuum) such that

$$\hat{\mathfrak{n}}_+|\lambda\rangle=0\ \textit{and}\quad h|\lambda\rangle=\langle\Lambda_\lambda, h\rangle|\lambda\rangle\qquad (h\in\hat{\mathfrak{h}}).$$

The $\hat{\mathfrak{g}}$-module $\mathscr{H}_\lambda$ is obtained from the $\mathfrak{g}$-module V_λ. Consider V_λ as a $\mathfrak{p}_+$-module by $\mathfrak{m}_+V_\lambda=0$ and $c|_{V_\lambda}=l\mathrm{id}_{V_\lambda}$, then the induced $\hat{\mathfrak{g}}$-module $\mathscr{M}_\lambda=U(\hat{\mathfrak{g}})\otimes_{\mathfrak{p}_+}V_\lambda$ (Verma *module) has the unique maximal proper* $\hat{\mathfrak{g}}$*-submodule* $\mathscr{I}_\lambda$ *which is generated by a single vector* $|J_\lambda\rangle = f_0^{l-(\lambda,\theta)+1}|\lambda\rangle$. *Thus* $\mathscr{H}_\lambda = \mathscr{M}_\lambda / \mathscr{I}_\lambda$.

The vacuum of $\mathscr{H}_\lambda$ *and the* $\mathfrak{g}$*-module* $\{|v\rangle \in \mathscr{H}_\lambda;\ \mathfrak{m}_+|v\rangle = 0\}$ *are identified with the class of* $1 \otimes |\lambda\rangle$ *and* $V_\lambda = 1 \otimes V_\lambda$ *respectively.*

Moreover $\hat{\mathfrak{n}}_+|J_\lambda\rangle = 0$ *and* $\alpha_0^\vee|J_\lambda\rangle = -(l-(\lambda,\theta)+2)|J_\lambda\rangle$.

II. There exists a unique irreducible (integrable and highest weight) right $\hat{\mathfrak{g}}$*-module* $\mathscr{H}_\lambda^+$ *with a nonzero vacuum vector* $\langle\lambda|$ *such that*

$$\langle\lambda|\hat{\mathfrak{n}}_- = 0 \text{ and } \langle\lambda|h = \langle\Lambda_\lambda, h\rangle\ \langle\lambda| \quad (h \in \hat{\mathfrak{h}}).$$

$\mathscr{H}_\lambda^+$ *is obtained as* $\mathscr{H}_\lambda^+ = \mathscr{I}_\lambda^+ \backslash \mathscr{M}_\lambda^+$, *where* $\mathscr{M}_\lambda^+$ *is the right* $\hat{\mathfrak{g}}$*-module* $\mathscr{M}_\lambda^+ = V_\lambda^+ \otimes_{\mathfrak{p}_-} U(\hat{\mathfrak{g}})$ *(the right* $\mathfrak{g}$*-module* V_λ^+ *is considered as a* $\mathfrak{p}_-$*-module, by setting* $V_\lambda^+\mathfrak{m}_- = 0$ *and* $c|v_\lambda^+ = l\,\mathrm{id}_{V_\lambda^+}$*). The* $\mathfrak{g}$*-module* $\{\langle u| \in \mathscr{H}_\lambda^+;\ \langle u|\mathfrak{m}_- = 0\}$ *is identified with the class of* $V_\lambda^+ = V_\lambda^+ \otimes 1$.

The unique maximal proper $\hat{\mathfrak{g}}$*-submodule* $\mathscr{I}_\lambda^+$ *of* $\mathscr{M}_\lambda^+$ *is generated by a single vector* $\langle J_\lambda| = \langle\lambda|e_0^{l-(\lambda,\theta)+1}$. *Moreover* $\langle J_\lambda|\hat{\mathfrak{n}}_- = 0$ *and* $\langle J_\lambda|\alpha_0^\vee = -(l-(\lambda,\theta)+2)\langle J_\lambda|$.

1.3. Segal-Sugawara Form.

In this paragraph, we give the actions on $\mathscr{H}_\lambda$ and $\mathscr{H}_\lambda^+$ of the *Virasoro Algebra* $\mathscr{L}$, where $\mathscr{L} = \sum_{n\in\boldsymbol{Z}} Ce_n + Ce_0'$ is the Lie algebra defined by the relations:

$$[e_m, e_n] = (m-n)e_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}e_0' \quad (m, n \in \boldsymbol{Z});$$

$$[e_0', e_m] = 0.$$

Definition 1.4.

i) **(current)** *For each* $X \in \mathfrak{g}$, *we define the formal Laurent series*

$$X(z) = \sum_{m\in\boldsymbol{Z}} X(n)z^{-n-1} \qquad (z \in \boldsymbol{C}^*).$$

ii) **(Energy Momentum Tensor; Segal-Sugawara Form** ([Se] *and* [*Su*])**)** *For* $z \in \boldsymbol{C}^*$, *define*

$$T(z) = \frac{1}{2(g+c)}\Big\{\sum_{j=1}^{n} {}^\circ_\circ X^j(z)X_j(z){}^\circ_\circ + \sum_{\gamma\in\Delta} {}^\circ_\circ X^\gamma(z)X_\gamma(z){}^\circ_\circ\Big\}$$
$$= \sum_{m\in\boldsymbol{Z}} L(m)z^{-m-2},$$

that is,

$$L(m) = \frac{1}{2(g+c)}\sum_{k\in\boldsymbol{Z}}\Big\{\sum_{j=1}^{n} {}^\circ_\circ X^j(-k)X_j(m+k){}^\circ_\circ + \sum_{\gamma\in\Delta} {}^\circ_\circ X^\gamma(-k)X_\gamma(m+k){}^\circ_\circ\Big\},$$

where $\{X^j, X^\gamma\}$ *and* $\{X_j, X_\gamma\}$ *are dual bases of* $\mathfrak{g}$ *taken as in* §1.1 *and the* **normal ordered products** *of elements of* $\mathfrak{g}\otimes\boldsymbol{C}[t, t^{-1}]$ *are defined by*

$${}^\circ_\circ X(m)Y(n){}^\circ_\circ = \begin{cases} X(m)Y(n) & (m<n) \\ \frac{1}{2}[X(m)Y(n) + Y(n)X(m)] & (m=n) \\ Y(n)X(m) & (m>n). \end{cases}$$

Then we get

Proposition 1.6.

i) For any $l \in \boldsymbol{Z}_{>0}$ *and* $\lambda \in \boldsymbol{P}_l$, *the operators* $L(m)$ $(m \in \boldsymbol{Z})$ *and* $L'(0) = \frac{\dim \mathfrak{g}}{g+l} l \mathrm{id}$ *act on* $\mathscr{H}_\lambda$ *and* $\mathscr{H}_\lambda^\dagger$.

ii) For any $m, m' \in \boldsymbol{Z}$,

$$[L(m),\ L(m')] = (m-m')L(m+m') + \frac{m^3-m}{12}\delta_{m+m'\,0} L'(0).$$

iii) For each $m \in \boldsymbol{Z}$ *and* $X \in \mathfrak{g}$,

$$[L(m),\ X(z)] = z^m\left(z\frac{d}{dz} + m + 1\right)X(z);$$

$$[L(m),\ X(n)] = -nX(m+n) \qquad (m,\ n \in \boldsymbol{Z}).$$

Proposition 1.7.

i) There exists a uniquue bilinear form (called vacuum expectation values)

$$\langle\,|\,\rangle\colon\ \mathscr{H}_\lambda^\dagger \times \mathscr{H}_\lambda \to \boldsymbol{C}$$

such that 1) $\langle \lambda | \rangle = 1$ *and* 2) $\langle ua | v\rangle = \langle u | av\rangle$ *for any* $a \in \hat{\mathfrak{g}}$, $\langle u | \in \mathscr{H}_\lambda^\dagger$ *and* $|v\rangle \in \mathscr{H}_\lambda$. *This bilinear form* $\langle\,|\,\rangle$ *is nondegenerate and its restriction on* $V_\lambda^\dagger \times V_\lambda$ *coincides with the form given in Proposition* 1.2.

ii) $\mathscr{H}_\lambda$ *and* $\mathscr{H}_\lambda^\dagger$ *have the eigenspace decompositions w.r.t. the operator* $L(0)$:

$$\mathscr{H}_\lambda = \sum_{d=0}^{\infty} \mathscr{H}_{\lambda,d} \text{ and } \mathscr{H}_\lambda^\dagger = \sum_{d=0}^{\infty} \mathscr{H}_{\lambda,d}^\dagger,$$

where $\mathscr{H}_{\lambda,d}$ *and* $\mathscr{H}_{\lambda,d}^\dagger$ *are the eigenspaces in* $\mathscr{H}_\lambda$ *and* $\mathscr{H}_\lambda^\dagger$ *of the eigenvalue* $\Delta_\lambda + d$ *respectively*, $\Delta_\lambda = \frac{(\lambda, \lambda+2\rho)}{2(g+l)}$, $\dim \mathscr{H}_{\lambda,d} = \dim \mathscr{H}_{\lambda,d}^\dagger < \infty$ *and* $\mathscr{H}_{\lambda,0} = V_\lambda$, $\mathscr{H}_{\lambda,0}^\dagger = V_\lambda^\dagger$.

Moreover $\langle \mathscr{H}_{\lambda,d} | \mathscr{H}_{\lambda,d'}^\dagger \rangle = 0$ *unless* $d = d'$, *and the bilinear form* $\langle\,|\,\rangle$ *is nondegenerate on* $\mathscr{H}_{\lambda,d}^\dagger \times \mathscr{H}_{\lambda,d}$.

iii) For any $X \in \mathfrak{g}$, $m \in \boldsymbol{Z}$ *and* $d \geqslant 0$,

$$X(m)\mathscr{H}_{\lambda,d},\ L(m)\mathscr{H}_{\lambda,d} \subset \mathscr{H}_{\lambda,d-m} \quad \text{and} \quad \mathscr{H}_{\lambda,d}^\dagger X(m),\ \mathscr{H}_{\lambda,d}^\dagger L(m) \subset \mathscr{H}_{\lambda,d+m}^\dagger.$$

§2. Vertex Operators.

Throughout this paper we fix the value l (a positive integer) of the central element c on the spaces $\mathscr{H}$ and $\mathscr{H}^+$, and use the value $\kappa = l + g$ for convenience. We refer to our previous paper [TK] for details of notions and propositions in this section.

2.1) Field Operators.

For each $\lambda \in P_l$, introduce the completions $\hat{\mathscr{H}}_\lambda = \prod_{d \geqslant 0} \mathscr{H}_{\lambda,d}$ and $\mathscr{H}_\lambda^\dagger = \prod_{d \geqslant 0} \mathscr{H}_{\lambda,d}^\dagger$ of $\mathscr{H}_\lambda$ and

$\mathscr{H}_\lambda^\dagger$ respectively. Extend the $\hat{\mathfrak{g}}$-action on $\mathscr{H}_\lambda$ and $\mathscr{H}_\lambda^\dagger$ to their completions and the vacuum expectation $\langle | \rangle$: $\mathscr{H}_\lambda^\dagger \times \mathscr{H}_\lambda \to \boldsymbol{C}$ to continuous bilinear pairings $\langle | \rangle$: $\mathscr{H}_\lambda^\dagger \times \hat{\mathscr{H}}_\lambda \to \boldsymbol{C}$ and $\hat{\mathscr{H}}_\lambda^\dagger \times \mathscr{H}_\lambda \to \boldsymbol{C}$. Note $\mathscr{H}_\lambda^\dagger$ is naturally isomorphic to $\mathrm{Hom}_{\boldsymbol{C}}(\mathscr{H}_\lambda;\ \boldsymbol{C})$.

Consider the direct sums of these modules:

$$\mathscr{H} = \sum_{\lambda \in P_l} \mathscr{H}_\lambda \subset \hat{\mathscr{H}} = \sum_{\lambda \in P_l} \hat{\mathscr{H}}_\lambda; \qquad \mathscr{H}^+ = \sum_{\lambda \in P_l} \mathscr{H}_\lambda^\dagger \subset \hat{\mathscr{H}}^+ = \sum_{\lambda \in P_l} \hat{\mathscr{H}}_\lambda^\dagger,$$

where $P_l = \{\lambda \in P_+;\ (\lambda, \theta) \leqslant l\}$. The projection π_λ to the λ-th component: $\hat{\mathscr{H}} \to \hat{\mathscr{H}}_\lambda$, $\hat{\mathscr{H}}^+ \to \hat{\mathscr{H}}_\lambda^\dagger$ commutes with the action of $\hat{\mathfrak{g}}$.

An *operator* A on $\mathscr{H}$ means a linear mapping A: $\mathscr{H} \to \hat{\mathscr{H}}$, which is equivalent to give a bilinear map $\hat{A}$: $\mathscr{H}^+ \times \mathscr{H} \to \boldsymbol{C}$, and also to give a linear mapping A^+: $\mathscr{H}^+ \to \hat{\mathscr{H}}^+$ by the condition that for any $\langle v | \in \mathscr{H}^+$ and $|w\rangle \in \mathscr{H}$,

$$<v|Aw> = <v|\hat{A}|w> = <vA^+|w>.$$

The notions of the composability and the holomorphy of operators and operator valued functions are weakly defined (see [TK] for exact definitions).

Operator valued functions $X(z)$ $(X \in \mathfrak{g})$ and $T(z)$: $\mathscr{H} \to \hat{\mathscr{H}}$ are single-valued and holomorphic on $\boldsymbol{C}^* = \mathbb{P}^1 \backslash \{0, \infty\}$. By the same arguments of the proof of Proposition 2.6 of [TK], we get

Proposition 2.1.

i) Ordered pairs $\{X(\zeta), Y(z)\}$, $\{X(\zeta), T(z)\}$, $\{T(\zeta), X(z)\}$ *and* $\{T(\zeta), T(z)\}$ *of operators are composable for* $|\zeta| > |z| > 0$ $(X, Y \in \mathfrak{g})$, *and their composition* $X(\zeta)Y(z)$, $X(\zeta)T(z)$, $T(\zeta)X(z)$ *and* $T(\zeta)T(z)$ *are analytically continued to single-valued, operator-valued holomorphic functions on* $M_2 = \{(\zeta, z) \in (\boldsymbol{C}^*)^2;\ \zeta \neq z\}$. *As operators on* $\mathscr{H}$, *the following identities hold:*

$$\text{(I)} \quad X(\zeta)Y(z) = \frac{l(X, Y)}{(\zeta - z)^2}\mathrm{id} + \frac{1}{\zeta - z}[X, Y](z) + R_I \qquad (X, Y \in \mathfrak{g}).$$

$$\text{(II)} \quad T(\zeta)X(z) = \frac{1}{(\zeta - z)^2}X(z) + \frac{1}{\zeta - z}\frac{\partial}{\partial z}X(z) + R_{II} \qquad (X \in \mathfrak{g}).$$

$$\text{(III)} \quad T(\zeta)T(z) = \frac{l \dim \mathfrak{g}}{2\kappa(\zeta - z)^4}\mathrm{id} + \frac{2T(z)}{(\zeta - z)^2} + \frac{1}{\zeta - z}\frac{\partial}{\partial z}T(z) + R_{III}.$$

Here R_I, R_{II} *and* R_{III} *are regular at* $\zeta = z \in \boldsymbol{C}^*$. *Moreover*

$$T(\zeta)T(z) = T(z)T(\zeta),\ T(\zeta)X(z) = X(z)T(\zeta) \ \textit{and}\ X(\zeta)Y(z) = Y(z)X(\zeta).$$

ii) The normal product $\mathopen{\circ\atop\circ} X(\xi)Y(\zeta) \mathclose{\circ\atop\circ}$ is also regular at $\xi = \zeta$ and

$$\mathopen{\circ\atop\circ} X(\xi)Y(\zeta) \mathclose{\circ\atop\circ} = X(\xi)Y(\zeta) - \frac{l(X, Y)}{(\xi - \zeta)^2}\, id$$
$$- \frac{1}{\xi - \zeta} \sum_{m \in \boldsymbol{Z}} \{[X, Y]\ (2m) \frac{\xi + \zeta}{2(\xi\zeta)^{1+m}} + [X, Y]\ (2m+1) \frac{1}{(\xi\zeta)^{1+m}}\}.$$

As a corollary,

$$⸰X(\zeta)Y(\zeta)⸰=\frac{1}{2\pi\sqrt{-1}}\int_{C_{\zeta;0}}\frac{d\xi}{\xi-\zeta}X(\xi)Y(\zeta),$$

where $C_{\zeta;0}$ is a contour around ζ such that 0 is outside $C_{\zeta;0}$.

2.2 Vertex operators.

Vertex operators (or primary fields) are introduced by V. G. Knizhnik and A. B. Zamolodchikov [KZ].

A multi-valued, holomorphic, operator-valued function $\Phi(u;z)$ on $M_1=\boldsymbol{C}^*$ parametrized by $u\in V_\lambda$ is called a *vertex operator of weight* $\lambda\in P_+$, if for any $u\in V_\lambda$, an operator

$$\Phi(u;z)\colon \mathscr{H} \longrightarrow \hat{\mathscr{H}}$$

satisfies the conditions:

(V1) $\Phi(u;z)$ is linear in $u\in V_\lambda$;

(V2) $[X(m),\ \Phi(u;z)]=z^m\Phi(Xu;z) \qquad (X\in\mathfrak{g},\ m\in\boldsymbol{Z})$;

(V3) $[L(m),\ \Phi(u;z)]=z^m\left(z\frac{d}{dz}+(m+1)\Delta_\lambda\right)\Phi(u;z) \quad (m\in\boldsymbol{Z})$,

where the number $\Delta_\lambda=\dfrac{(\lambda,\ \lambda+2\rho)}{2\kappa}$ is called the *conformal dimension* of the vertex operator $\Phi(u;z)$. Denote by $\mathscr{V}\!er(\lambda)$ the space of all vertex operators of weight λ.

Remark 2.2. *i*) Vertex operators are sometimes considered as

$$\Phi(z)\colon V_\lambda\otimes\mathscr{H}\to\hat{\mathscr{H}} \quad \text{by } \Phi(z)(u,\ v)=\Phi(u;\ z)(v).$$

The conditions (V2) and (V3) are the gauge condition and the equations of motion for the field $\Phi(z)$ respectively.

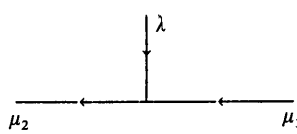
ii) The principal branch of $\Phi(z)$ is taken such as the value of $z^{-\hat{\Delta}(\mathbb{v})}$ is positive for $z\in\boldsymbol{R}_+=\{z\in\boldsymbol{R};\ z>0\}$ and uniquely continued to the region $\boldsymbol{C}_+=\{z\in\boldsymbol{C};\ Re z>0\}$, and we refer this for the value of $\Phi(z)$ on $\boldsymbol{C}_+$.

Introduce the sets $\mathbb{V}$ and $\mathbb{V}_l$ defined by

$$\mathbb{V}=\left\{\mathbb{v}=\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1\end{pmatrix};\ \lambda,\ \mu_1,\ \mu_2\in P_+\right\}\supset\mathbb{V}_l=\left\{\mathbb{v}=\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1\end{pmatrix}\in\mathbb{V};\ \mu_1,\ \mu_2\in P_l\right\}.$$

An element $\mathbb{v}$ of $\mathbb{V}$ is called a *vertex*. For a vertex $\mathbb{v}=\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1\end{pmatrix}\in\mathbb{V}$, we call that μ_1 is an *incoming weight*, μ_2 is an *outgoing weight* and λ is an *outer weight*, and denote $\Delta(\mathbb{v})=\Delta_\lambda$ and $\hat{\Delta}(\mathbb{v})=\Delta_\lambda+\Delta_{\mu_1}-\Delta_{\mu_2}$.

$\mathbb{V}$:



For a vertex $\mathbb{v}=\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1\end{pmatrix}\in\mathbb{V}_l$, a vertex opertor $\Phi(z)$ of weight λ is called *of type* $\mathbb{v}$, if $\Phi(u; z)=\Pi_{\mu_2}\Phi(u; z)\Pi_{\mu_1}$ for any $u\in V_\lambda$. Denote by $\mathscr{V}er(\mathbb{v})$ the space of all vertex operators of type $\mathbb{v}$, then

Proposition 2.3.

$$\mathscr{V}er(\lambda)=\sum_{\mu_1,\mu_2\in P_l}\mathscr{V}er\left(\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1\end{pmatrix}\right).$$

Similarly as Proposition 2.1 in [TK], we get

Proposition 2.4.

i) *Any vertex operator* Φ *of type* $\mathbb{v}(\in\mathbb{V}_l)$ *has a Laurent series expansion*

$$\Phi(u; z)=\sum_{m\in\boldsymbol{Z}}\Phi_u(m)z^{-m-\hat{\Delta}(\mathbb{v})}\quad(u\in V_\lambda)$$

and $\Phi_u(m)$ *satisfies*

$$[L(0),\ \Phi_u(m)]=(\Delta_{\mu_2}-\Delta_{\mu_1}-m)\Phi_u(m)\quad(m\in\boldsymbol{Z}),$$

that is,

$$\Phi_u(m):\ \mathscr{H}_{\mu_1,d}\to\mathscr{H}_{\mu_2,d-m},\ \mathscr{H}^+_{\mu_2,d}\to\mathscr{H}^+_{\mu_1,d+m}\qquad(m\in\boldsymbol{Z}).$$

ii) *For each* $u\in V_\lambda$,

$$[X(m),\ \Phi_u(m')]=[X(0),\ \Phi_u(m+m')]=\Phi_{Xu}(m+m')\qquad(X\in\mathfrak{g},\ m,\ m'\in\boldsymbol{Z})$$

and

$$[L(m),\ \Phi_u(m')]=\{(m+1)\Delta_\lambda-m-m'-\hat{\Delta}(\mathbb{v})\}\Phi_u(m+m')\quad(m,\ m'\in\boldsymbol{Z}).$$

Proposition 2.5.

i) *Introduce a trilinear form* φ: $V^+_{\mu_2}\otimes V_\lambda\otimes V_{\mu_1}\to\boldsymbol{C}$ *defined by*

$$\varphi(v,\ u,\ w)=\langle v|\Phi_u(0)|w\rangle z^{\hat{\Delta}(\mathbb{v})}|_{z=0}$$

for $u\in V_\lambda$, $v\in V^+_{\mu_2}$ *and* $w\in V_{\mu_1}$, *then* φ *is* $\mathfrak{g}$*-invariant*:

$$\varphi(vX,\ u,\ w)=\varphi(v,\ Xu,\ w)+\varphi(v,\ u,\ Xw)\quad(X\in\mathfrak{g}).$$

ii) *A vertex operator* Φ *of type* $\mathbb{v}$ *is uniquely determined by the form* $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(V^+_{\lambda_2}\otimes V_\lambda\otimes V_{\lambda_1},\ \boldsymbol{C})\cong\mathrm{Hom}_{\mathfrak{g}}(V_\lambda\otimes V_{\lambda_1},\ V_{\lambda_2})$ *defined in i). We call* φ *is the initial term of the vertex operator* Φ *and denote* $\Phi=\Phi_\varphi$.

2.3) Existence of Vertex Operators.

Let $\mathbb{v}=\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1\end{pmatrix}$ be a vertex. The dimension $m(\mathbb{v})$ of the space $\mathrm{Hom}_{\mathfrak{g}}(V^+_{\mu_2}\otimes V_\lambda\otimes V_{\mu_1},\ \boldsymbol{C})$ is equal to the multiplicity of V_{μ_2} in the tensor product $V_\lambda\otimes V_{\mu_1}$. It is known as the Steinberg's formula, which is explicit but is not so easy to apply when the Weyl group is large.

Definition 2.6. *Introduce the space* $\mathscr{V}(\mathbb{v})$ *consisting all forms* $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(V_\lambda\otimes V_{\mu_1},$

V_{μ_2}) *satisfying the condition*:

$$\mathrm{proj}_{W_{j_2}} \circ \varphi(W_j \otimes W_{j_1}) = 0,$$

for any simple $\mathfrak{k}$*-submodules* W_j, W_{j_1} *and* W_{j_2} *of* V_λ, V_{μ_1} *and* V_{μ_2} *with* $j+j_1+j_2>l$, *where* $\mathrm{proj}_{W_{j_2}}$ *is the projection of* V_{μ_2} *onto the* $\mathfrak{k}$*-simple summand* W_{j_2}.

For any vertex operator $\Phi \in \mathscr{V}(\mathbb{v})$, its initial term φ is in $\mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\mu_1}, V_{\mu_2})$ and $\varphi = \Phi(0)|_{V_\lambda \otimes \mathscr{H}_{\mu_1}(0)} = \mathrm{proj}_{V_{\mu_2}} \circ z^{\Delta(\mathbb{v})}\Phi(z)|\, V_\lambda \otimes V_{\mu_1}$. Under this correspondence,

Theorem 2.7. *The space* $\mathscr{V}\!\mathit{er}(\mathbb{v})$ *of N-point functions of type* $\mathbb{v}$ *is isomorphic with the space* $\mathscr{V}(\mathbb{v})$ *of initial terms of type* $\mathbb{v}$.

Proof. The initial term φ of $\Phi(z) \in \mathscr{V}\!\mathit{er}(\mathbb{v})$ must satisfy

$$\varphi(v,\ X_\theta^{l-(\mu_1,\theta)+1}u,\ |\mu_1\rangle) = 0 \quad \text{for any } v \in V_{\mu_2}^+,\ u \in V_\lambda$$

and

$$\varphi(\langle\mu_2|,\ X_{-\theta}^{l-(\mu_2,\theta)+1}u,\ w) = 0 \quad \text{for any } u \in V_\lambda,\ w \in V_{\mu_1}.$$

Decompose V, V_{μ_1} and V_{μ_2} as $\mathfrak{k}$-modules, and apply Lemma 2.2 in [TK]. Then $\varphi \in \mathscr{V}(\mathbb{v})$. The surjectivity is also due to the same lemma, and Proposition 2.5 implies the injectivity. *q.e.d.*

Remark 2.8. i) A vertex $\mathbb{v}$ with $\mathscr{V}(\mathbb{v}) \neq 0$ is said to satisfy the *l-constrained generalized Clebsh-Gordan condition and write* $\mathbb{v} \in (lCG)$.

ii) Let $\Phi(u; z)$ be a vertex operator of weight $\lambda (u \in V_\lambda)$, then as a formal Laurent series,

$$\Phi(u; z) = z^{L(0) - \Delta_\lambda} \Phi(u; 1) z^{-L(0)} \qquad (u \in V_\lambda).$$

Proposition 2.9.

There exist no nonzero vertex operators of weight λ, *unless* $\lambda \in P_l$.

Proof. For a weight $\lambda \in P_+$ with $(\theta, \lambda) > l$, take $\varphi \in \mathscr{V}\left(\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{pmatrix}\right)$ and let $\Phi(z) = \Phi_\varphi(z)$. *We must show that* $\Phi = 0$. By Proposition 2.5 ii), it is sufficient to prove that $\varphi(v, |\lambda\rangle w) = 0$ for any $v \in V_{\mu_2}^+$ and $w \in V_{\mu_1}$.

The subalgebra $\hat{\mathfrak{k}} = \mathfrak{k} \otimes \boldsymbol{C}[t, t^{-1}] + \boldsymbol{C}c$ of $\hat{\mathfrak{g}}$ is isomorphic to the affine Lie algebra of type $A_1^{(1)}$. $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{k} \cap \mathfrak{h} = \boldsymbol{C}\theta^\vee$ and $\hat{\mathfrak{h}}_{\mathfrak{k}} = \hat{\mathfrak{k}} \cap \hat{\mathfrak{h}} = \boldsymbol{C}\alpha_0^\vee \oplus \boldsymbol{C}\theta^\vee(0)$ are Cartan subalgebras of $\mathfrak{k}$ and $\hat{\mathfrak{k}}$ respectively. The sets $\left\{\frac{\theta}{2}\right\} \subset \mathfrak{h}_{\mathfrak{k}}^* \subset \mathfrak{h}^*$ and $\left\{\Lambda_0,\ \Lambda_0 + \frac{\theta}{2}\right\} \subset \hat{\mathfrak{h}}_{\mathfrak{k}}^* \subset \hat{\mathfrak{h}}^*$ give fundamental weights of $(\hat{\mathfrak{k}}, \mathfrak{h}_{\mathfrak{k}})$ and $(\hat{\mathfrak{k}}, \hat{\mathfrak{h}}_{\mathfrak{k}})$ respectively. Let W_j be the irreducible left $\mathfrak{k}$-module with highest weight $j\theta\left(j \in \frac{1}{2}\boldsymbol{Z}_{\geq 0}\right)$ and denote by W_j^+ the corresponding right $\mathfrak{k}$-module.

Decompose V_{μ_1} and $V_{\mu_2}^+$ as $\mathfrak{k}$-modules:

$$V_{\mu_1} = \bigoplus_{i=1}^{m_1} W_{j_i} \text{ and } V_{\mu_2}^+ = \bigoplus_{i=1}^{m_2} W_{j_i}^+,$$

then j_i, $k_i \leqslant \frac{l}{2}$, since $(\mu, \theta) \leqslant (\lambda, \theta)$ for any $\mu \in P(\lambda)$. By restricting initial terms of the vertex operator $\Phi(z)$ for $\hat{\mathfrak{g}}$ to $V_{\mu_2}^{+} \otimes U(\mathfrak{k})|\lambda\rangle \otimes V_{\mu_1}$, we get a sum of forms in $\mathrm{Hom}_{\mathfrak{k}}(W_{k_i}^{+} \otimes W_{(\lambda,\theta)/2} \otimes W_{j_i})$ satisfying similar conditions of Definition 2.6. However such forms must vanish by Lemma 2.2 and Remark 2.2′ of [TK]. *q.e.d.*

Proposition 2.10. *Let $\hat{\mathfrak{g}}$ be an affine Lie algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ or $D_n^{(1)}$. If a vertex $\mathbb{v}$ has a form $\mathbb{v}=\begin{pmatrix} & \bar{\Lambda}_1 & \\ \mu_2 & & \mu_1 \end{pmatrix}$, then*

$$\mathscr{V}(\mathbb{v})=\mathrm{Hom}_{\mathfrak{g}}(V_{\bar{\Lambda}_1} \otimes V_{\mu_1}, V_{\mu_2}).$$

Proof. Consider the weight structure of the vector representation $V_{\bar{\Lambda}_1}$ of $\mathfrak{g}$. *q.e.d.*

2.4) Operator Product Expansions and Actions of $\hat{\mathfrak{g}}$ and $\mathscr{L}$ on Vertex Operators.

Operator product expansions are obtained similarly as Propositions 2.6 and 2.7 of [TK]:

Proposition 2.11.

i) Let $\Phi(u;z)$ be a vertex operator of weight $\lambda\,(u \in V_\lambda)$. Ordered pairs $\{X(\zeta), \Phi(u;z)\}$, $\{\Phi(u;\zeta), X(z)\}$, $\{T(\zeta), \Phi(u;z)\}$ and $\{\Phi(u;\zeta), T(z)\}$ of operators are composable for $|\zeta|>|z|>0$ $(X \in \mathfrak{g})$, and their compositions $X(\zeta)\Phi(u;z)$, $\Phi(u;\zeta)X(z)$, $T(\zeta)\Phi(u;z)$ and $\Phi(u;\zeta)T(z)$ are analytically continued to multi-valued, operator-valued holomorphic functions on M_2. As operators on $\mathscr{H}$, the following identities hold:

(IV) $$X(\zeta)\Phi(u;z)=\frac{1}{\zeta-z}\Phi(Xu;z)+R_{IV} \quad (X \in \mathfrak{g}).$$

(V) $$T(\zeta)\Phi(u;z)=\frac{\Delta_\lambda}{(\zeta-z)^2}\Phi(u;z)+\frac{1}{\zeta-z}\frac{\partial}{\partial z}\Phi(u;\ z)+R_V.$$

Here R_{IV} and R_V are regular at $\zeta=z \in \boldsymbol{C}^*$.

Moreover $X(\zeta)\Phi(u;z)$ and $T(\zeta)\Phi(u;z)$ $(X \in \mathfrak{g})$ are single-valued and holomorphic function on $\zeta \in \boldsymbol{P}^1 \backslash \{0, z, \infty\}$ for any fixed $z \in \boldsymbol{C}^*$, and

$$X(\zeta)\Phi(u;\ z)=\Phi(u;\ z)X(\zeta) \text{ and } T(\zeta)\Phi(u;\ z)=\Phi(u;\ z)T(\zeta).$$

ii) Let $u \in V_\lambda$ and $\Phi(z)$ be a vertex operator of type $\mathbb{v}=\begin{pmatrix} & \lambda & \\ \lambda_2 & & \lambda_1 \end{pmatrix} \in (lCG)$, Let $A_N(z_N), \cdots, A_1(z_1)$ be operators of the form $T(z)$, $X(z)$ $(X \in \mathfrak{g})$ or $\Phi(u;\ z)$, and assume that there is at most one number i_0 such that $A_{i_0}(z_{i_0})=\Phi(u;\ z_{i_0})$ and $A_i(z_i)$ is not a vertex operator for $i \neq i_0$.

Then $\{A_N(z_N), \cdots, A_1(z_1)\}$ is composable in the range $|z_N|>\cdots>|z_1|$, and the composed operator $A_N(z_N)\cdots A_1(z_1)$ is analytically continued to a multivalued and holomorphic function on $M_N=\{(z_N, \cdots, z_1) \in (\boldsymbol{C}^*)^N;\ z_i \neq z_j\ (i \neq j)\}$. If we fix $(z_N, \cdots, \hat{z}_j, \cdots, z_1)$ $(j \neq i_0)$, then this function is single-valued in $z_j \in \boldsymbol{P}^1 \backslash \{\infty, z_N, \cdots, \hat{z}_j, \cdots, z_1, 0\}$.

Notation. For any points $\zeta_1,\cdots,\zeta_a$ and $\xi_1,\cdots,\xi_b \in \boldsymbol{C}$, denote by $C=C_{\zeta_1,\dots,\zeta_a;\xi_1,\dots,\xi_b}$ a positively oriented contour such that $\zeta_1,\cdots,\zeta_a$ are inside C and $\xi_1,\cdots,\xi_b$ are outside C.

For each $\varphi\in\mathscr{V}(\mathbb{v})$, introduce the $\mathfrak{g}$-module $\mathscr{P}(\varphi)$ defined by

$$\mathscr{P}(\varphi)=\{\Phi(u;z);\ u\in V_\lambda\};\qquad X\Phi(u;z)=\Phi(Xu;z)\ (X\in\mathfrak{g}).$$

Now introduce the space $\mathcal{O}(\varphi)$ of operators on $\mathscr{H}$ as the $\boldsymbol{C}$-vector space spanned by the set

$$\left\{\frac{1}{(2\pi\sqrt{-1})^N}\int_{C_N}\cdots\int_{C_1} d\zeta_N\cdots d\zeta_1(\zeta_N-z)^{m_N}\cdots(\zeta_1-z)^{m_1}X_N(\zeta_N)\cdots X_1(\zeta_1)\Phi(u;z);\right.$$
$$\left. N\in\boldsymbol{Z}_{\geq 0},\ X_i\in\mathfrak{g},\ m_i\in\boldsymbol{Z}\ (1\leqslant i\leqslant N),\ u\in V_\lambda\right\},$$

where the contours C_i $(1\leqslant i\leqslant N)$ are taken as $C_i=C_{z,\zeta_1,\dots,\zeta_{i-1};0}$

Let $A(z)\in\mathcal{O}(\varphi)$, $X\in\mathfrak{g}$ and $m\in\boldsymbol{Z}$, then define

$$\hat{X}(m)A(z)=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^m X(\zeta)A(z)\in\mathcal{O}(\varphi)$$

and

$$\hat{L}(m)A(z)=\frac{1}{2\pi\sqrt{-1}}\int_C d\zeta(\zeta-z)^{m+1}T(\zeta)A(z)\in\mathcal{O}(\varphi)$$

for some contour $C=C_{z;0}$. Then by Proposition 2.11,

Proposition 2.12.

i) *The assignation $X(m)\mapsto\hat{X}(m)$ and $c\mapsto l\mathrm{id}$ defines the $\hat{\mathfrak{g}}$-module structure on $\mathcal{O}(\varphi)$.*

ii) *Let $u\in V_\lambda$, then*

$$\hat{X}(m)\Phi(u;z)=0,\quad \hat{L}(m)\Phi(u;z)=0\qquad (m\geqslant 1,\ X\in\mathfrak{g}),$$
$$\hat{X}(0)\Phi(u;z)=[X(0),\ \Phi(u;z)]=\Phi(Xu;\ z)\qquad (X\in\mathfrak{g}),$$
$$\hat{L}(0)\Phi(u;z)=\Delta_\lambda\Phi(u;z)\ \text{and}\ \hat{L}(-1)\Phi(u;z)=\frac{\partial}{\partial z}\Phi(u;z).$$

iii) *The $\mathscr{L}$-action on $\mathcal{O}(\varphi)$ defined by assigning $L(m)\mapsto\hat{L}(m)$ and $L'(0)\mapsto\frac{\dim\mathfrak{g}}{\kappa}l\mathrm{id}$ is compatible with the $\hat{\mathfrak{g}}$-action.*

By Theorem 2.7 and Proposition 2.12,

Proposition 2.13.

For each $\varphi\in\mathscr{V}(\mathbb{v})$, the assignation $V_\lambda\in u\mapsto\Phi(u;z)$ defines the $\mathfrak{g}$-isomorphism of V_λ onto the space $\mathscr{P}(\mathbb{v})$, and it is extended to a surjective $\hat{\mathfrak{g}}$-module mapping Φ: $\mathscr{M}_\lambda\to\mathcal{O}(\varphi)$.

Theorem 2.14 (Nuclear Democracy).

For any $\varphi\in\mathscr{V}(\mathbb{v})$ with $\mathbb{v}\in(lCG)$ the $\hat{\mathfrak{g}}$-mapping Φ factors to $\mathscr{H}_\lambda$ and gives the $\hat{\mathfrak{g}}$-

isomorphism of $\mathscr{H}_\lambda$ *onto* $\mathcal{O}(\varphi)$.

Proof. First note that the following fact is important: The only one additonal relation of $\mathscr{H}_\lambda$ to the Verma module $\mathscr{M}_\lambda$ is the equality $f_0^L|\lambda\rangle=0$, where $L=l-(\theta, \lambda)+1\geqslant 1$.

Let $\varphi\in\mathscr{V}(\mathbb{v})$ with $\mathbb{v}\in(lCG)$. Since the kernel of the projection of $\mathscr{M}_\lambda$ onto $\mathscr{H}_\lambda$ is generated by a vector $|J_\lambda\rangle\in\mathscr{M}_\lambda$ over $U(\hat{\mathfrak{g}})$, it is sufficient to show that $\Phi(|J_\lambda\rangle;\ z)=0$.

First note that $|J_\lambda\rangle=f_0^L|\lambda\rangle\in\mathscr{M}_\lambda$ is a weight vector of weight $l\Lambda_0+\lambda-L\alpha_0\notin\hat{P}_+$ and $\mathscr{I}_\lambda\cong\mathscr{M}_{\lambda+L\theta}$.

Any $\Psi(z)\in U(\mathfrak{g})\Phi(|J_\lambda\rangle;z)$ satisfies

$$\hat{X}(m)\Psi(z)=0 \qquad \text{for any } m>0,\ X\in\mathfrak{g},$$

since $\mathfrak{m}_+U(\mathfrak{g})|J_\lambda\rangle=0$. By Proposition 2.11, we get

$$[X(m),\ \Psi(z)]=\frac{1}{2\pi\sqrt{-1}}\int_{C_{z;0}} d\zeta\ \zeta^m X(\zeta)\Psi(z),$$

so

$$[X(0),\ \Psi(z)]=\hat{X}(0)\Psi(z) \text{ and } [X(m),\ \Psi(z)]=z^m[X(0),\ \Psi(z)] \quad (m\in\boldsymbol{Z}).$$

Hence by Proposition 2.9, we get $\Phi(|J_\lambda>;\ z)=0$ since $(\lambda+L\theta,\ \theta)=L+l+1>l$.

q.e.d.

Here we summarize the relations satisfied by vertex opertors:

Fundamental Relations for Vertex Operators.

Let $\Phi(z)$ *be a vertex operator of weight* λ. *Then*

$$\begin{aligned}
&\hat{X}(m)\Phi(u;z) &&=0 &&(m\geqslant 1,\ X\in\mathfrak{g},\ u\in V_\lambda);\\
&\hat{X}(0)\Phi(u;z) &&=[X(0),\ \Phi(u;z)]=\Phi(Xu;z) &&(X\in\mathfrak{g},\ u\in V_\lambda);\\
&\hat{L}(m)\Phi(u;z) &&=0 &&(m\geqslant 1,\ u\in V_\lambda);\\
&\hat{L}(0)\Phi(u;z) &&=\Delta_\lambda\Phi(u;z) &&(u\in V_\lambda);\\
&\hat{L}(-1)\Phi(u;z) &&=\frac{\partial}{\partial z}\Phi(u;z) &&(u\in V_\lambda);
\end{aligned}$$

and

$$\hat{X}_\theta(-1)^{l-(\lambda,\theta)+1}\Phi(|\lambda\rangle;z)=0.$$

§3. Differential Equations of N-point Functions and Composabilty of Vertex Operators.

In this section, we will give the system of differential equations of N-point functions and show the composability of vertex operators.

3.1. N-point Functions and their Differential Equations.

The vacuums $|0\rangle$ and $\langle 0|$ of $\mathscr{H}_0$ and $\mathscr{H}_0^+$ are of special importance (and are called *Virasoro vacuums*):

$$\mathfrak{p}_+|0\rangle=0 \quad \text{and} \quad L(m)|0\rangle=0 \qquad (m\geqslant -1);$$
$$\langle 0|\mathfrak{p}_-=0 \quad \text{and} \quad \langle 0|L(m)=0 \qquad (m\leqslant 1).$$

For an operator A on $\mathscr{H}$, define its *vacuum expectation value* as $\langle A\rangle=\langle 0|A|0\rangle$.

Denote by ρ_i the $\mathfrak{g}$-action on the i-th component of the tensor product $M=M_N\otimes\cdots\otimes M_1$ of $\mathfrak{g}$-modules M_i. Denote by Δ_{ik} $(1\leqslant i,\ k\leqslant N)$ the $\mathfrak{g}$-diagonal action on the i-th and k-th component of M, that is, $\Delta_{ik}=\rho_i+\rho_k$, and introduce the operator Ω_{ik} on M defined by

$$\Omega_{ik}=\sum_{j=1}^{n}\rho_i(X^j)\rho_k(X_j)+\sum_{\gamma\in\Delta}\rho_i(X^\gamma)\rho_k(X_\gamma),$$

where $\{X^j\ (1\leqslant j\leqslant n),\ X^\gamma(\gamma\in\Delta)\}$ and $\{X_j\ (1\leqslant j\leqslant n),\ X_\gamma(\gamma\in\Delta)\}$ are dual bases of $\mathfrak{g}$ taken in §1.1. Denote $\Omega_i=\Omega_{ii}=\pi_i(\Omega)$, then

$$\Omega_{ik}=\frac{1}{2}\{\Delta_{ik}(\Omega)-\Omega_i-\Omega_k\},$$

and

$$[\Omega_{ik},\ \Delta_{ik}(X)]=[\Omega_{ik},\ \rho_j(X)]=0 \quad (i\neq k,\ X\in\mathfrak{g},\ j\neq i,\ k).$$

For each $\Lambda=(\lambda_N,\cdots,\ \lambda_1)\in(P_l)^N$, denote

$$V_\Lambda=V_{\lambda_N}\otimes\cdots\otimes V_{\lambda_1},\ V_\Lambda^\vee=V_{\lambda_N}^\vee\otimes\cdots\otimes V_{\lambda_1}^\vee,\ V_{\mathfrak{g}}^\vee(\Lambda)=(V_\Lambda^\vee)^{\mathfrak{g}}\simeq\mathrm{Hom}_{\mathfrak{g}}(V_\Lambda,\ \boldsymbol{C}).$$

Then the operators Ω_{ik} act on V_Λ, $V_\Lambda^\vee$, $V_{\mathfrak{g}}^\vee(\Lambda)$ and $\mathscr{P}^{\otimes N}$ where $\mathscr{P}=\sum_{\mathrm{v}}\sum_{\varphi\in\mathscr{V}(\mathrm{v})}\mathscr{P}(\varphi)$.

Let $\Phi^i(z_i)$ be a vertex operator of weight λ_i $(1\leqslant i\leqslant N)$, then the vacuum expectation valuue of the composed operator

$$\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle$$

is considered as a $V_\Lambda^\vee$-valued, formal Laurent series on $(z_N,\cdots,\ z_1)$ and is called an *N-point function of weight Λ*. Denote by $\mathscr{V}er(\Lambda)$ the space of all N-point functions of weight Λ.

If $\Phi^i(z_i)$ is of type v_i $(1\leqslant i\leqslant N)$,

$$\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle=\prod_{i=1}^{N}z_i^{-\hat{\Delta}(\mathrm{v}_i)}\sum_{m_N\geq 0}\cdots\sum_{m_k\in\mathbb{Z}}\cdots\sum_{m'\leq 0}C_{m_N\cdots m_1}z_N^{-m_N}\cdots z_1^{-m_1},$$

where

$$C_{m_N\cdots m_1}=\langle 0|\Phi^N_{\cdot}(m_N)\Phi^{N-1}_{\cdot}(m_{N-1})\cdots\Phi^2_{\cdot}(m_2)\Phi^1_{\cdot}(m_1)|0\rangle\in V_\Lambda^\vee.$$

It is shown (Theorem 3.3) that N-point functions define multivalued holomorphic functions on M_N.

First we get a system of differential equations of N-point functions ((i)~(iii) are due to V. G. Knizhnik and A. B. Zamolodchikov [KZ]).

Theorem 3.1.

Let $\Phi^i(z_i)$ be a vertex operator of weight λ_i $(1\leqslant i\leqslant N)$, then the N-point function

$\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle$ *satisfies the following equations:*

(*I*) **(gauge invariance)** *For any* $X\in\mathfrak{g}$,

$$\sum_{i=1}^{N}\rho_i(X)\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle=0,$$

(*II*) *For each* $i=1,\cdots,N$,

$$\left(\kappa\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right)\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle=0,$$

where $\kappa=l+g$.

(*III*) *For each* i $(1\leqslant i\leqslant N)$ *and any* $u_k\in V_{\lambda_k}$ $(k\neq i)$, *put* $L_i=l-(\lambda_i,\theta)+1$.

$$\sum_{|\mathfrak{m}_i|=L_i}\binom{L_i}{\mathfrak{m}_i}\prod_{k\neq i}(z_k-z_i)^{-m_k}\langle\Phi^N(X_\theta^{m_N}u_N;z_N)\cdots\Phi^i(|\lambda_i\rangle;z_i)\cdots\Phi^1(X_\theta^{m_1}u_1;z_1)\rangle=0,$$

where $\mathfrak{m}_i=(m_N,\cdots,\hat{m}_i,\cdots,m_1)\in(\boldsymbol{Z}_{\geq0})^{N-1}$, $|\mathfrak{m}_i|=\sum_{k\neq i}m_k$ *and* $\binom{L_i}{\mathfrak{m}_i}$ *is the multinomial coefficient.*

Proof. (*I*)

$$\begin{aligned}&\sum_{i=1}^{N}\rho_i(X)\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle\\&=\frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{N}\int_{C_i}d\xi\langle X(\xi)\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)\rangle\\&=\frac{-1}{2\pi\sqrt{-1}}\int_{C_0}d\xi\langle X(\xi)\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)\rangle=0,\end{aligned}$$

where $C_i=C_{z_i;z_1,\dots,\hat{z}_i,\dots,z_N,0}$ $(1\leqslant j\leqslant N)$ and $C_0=C_{0;z_1,\dots,z_N}$

(II) By Proposition 2.1 (ii), for any X, $Y\in\mathfrak{g}$, $u\in V_\lambda$ and $\Phi\in\mathscr{V}\!\mathit{er}(\lambda)$,

$$\int_{C_{z;0}}d\zeta\,{}^{\circ}_{\circ}X(\zeta)Y(\zeta){}^{\circ}_{\circ}\,\Phi(u;z)=\int_{C_{z;0}}\frac{d\xi}{\xi-z}\{X(\xi)\Phi(Yu;z)+Y(\xi)\Phi(Xu;z)\},$$

so

$$2\kappa\hat{L}(-1)\Phi(u;z)=\frac{1}{\pi\sqrt{-1}}\int_{C_{z;0}}\frac{d\xi}{\xi-z}\sum_{k=1}^{\dim\mathfrak{g}}X^k(\xi)\Phi(X_ku;z),$$

where $\{X^k\}$ and $\{X_k\}$ are dual bases of $\mathfrak{g}$, Hence

$$\begin{aligned}&\kappa\frac{\partial}{\partial z_i}\langle\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)\rangle=\kappa\rho_i(\hat{L}(-1))\langle\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)\rangle\\&=\frac{1}{2\pi\sqrt{-1}}\int_{C_i}\frac{d\xi}{\xi-z_i}\sum_{k=1}^{\dim\mathfrak{g}}\langle X^k(\xi)\Phi_N(u_N;z_N)\cdots\Phi_i(X_ku_i;z_i)\cdots\Phi_1(u_1;z_1)\rangle\\&=\frac{-1}{2\pi\sqrt{-1}}\sum_{k=1}^{\dim\mathfrak{g}}\sum_{\substack{j=1\\j\neq i}}^{N}\int_{C_j}\frac{d\xi}{\xi-z_i}\rho_i(X_k)\langle X^k(\xi)\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)\rangle\end{aligned}$$

$$= -\sum_{k=1}^{\dim\mathfrak{g}} \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{z_j - z_i} \rho_i(X_k)\rho_j(X^k)\langle\Phi_N(u_N; z_N)\cdots\Phi_1(u_1; z_1)\rangle$$

$$= \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\Omega_{ij}}{z_j - z_i} \langle\Phi_N(u_N;\ z_N)\cdots\Phi_1(u_1; z_1)\rangle.$$

The equations (III) are nothing but the following:

$$\rho_i(\hat{X}_\theta(-1))^{L_i}\langle\Phi_N(u_N; z_N)\cdots\Phi_1(u_1; z_1)\rangle = 0. \qquad q.e.d.$$

Remark 3.2.

i) The equations (I) mean that $\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle \in V_{\mathfrak{g}}^{\vee}(\Lambda)$.

ii) The equations (II) and (III) imply the *projective invariance*: *For* $m = -1$, 0 and 1,

$$\sum_{i=1}^{N} z_i^m\left(z_i\frac{\partial}{\partial z_i} + (m+1)\Delta_{\lambda_i}\right)\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle = 0.$$

iii) As a corollary of the above remark ii), N-point functions are translation invariant:

$$\langle\Phi_N(z_N + z)\cdots\Phi_1(z_1 + z)\rangle = \langle\Phi_N(z_N)\cdots\Phi_1(z_1)\rangle.$$

3.2) Solutions of Fundamental Equation.

Consider the systems $KZ(\Lambda)$ of differential equations and $lC(\Lambda)$ of algebraic equations for $V_{\mathfrak{g}}^{\vee}(\Lambda)$-valued functions $\Phi(z_N, \cdots, z_1)$ on the manifold $X_N = \{(z_N, \cdots, z_1) \in \boldsymbol{C}^N;\ z_i \neq z_k\ (i \neq k)\} \supset M_N$;

$$KZ(\Lambda) \qquad \left(\kappa\frac{\partial}{\partial z_i} - \sum_{\substack{k=1\\ k\neq i}}^{N} \frac{\Omega_{ik}}{z_i - z_k}\right)\Phi(z_N, \cdots, z_1) = 0 \qquad (1 \leq i \leq N)$$

and for each i $(1 \leq i \leq N)$ and any $u_k \in V_{\lambda_k}$ $(k \neq i)$,

$$lC(\Lambda) \qquad \sum_{|\mathfrak{m}_i| = L_i} \binom{L_i}{\mathfrak{m}_i} \prod_{k\neq i} (z_k - z_i)^{-m_k}\Phi(z_N, \cdots, z_1)\ (X_\theta^{m_N}u_N, \cdots, |\lambda_i\rangle, \cdots, X_\theta^{m_1}u_1) = 0,$$

where $\mathfrak{m}_i = (m_N, \cdots, \hat{m}_i, \cdots, m_1) \in (\boldsymbol{Z}_{\geq 0})^{N-1}$, $|\mathfrak{m}_i| = \sum_{k\neq i} m_k$ and $L_i = l - (\lambda_i, \theta) + 1$.

Remark 3.3. The system $KZ(\Lambda)$ of differential equations is completely integrable. The integrability condition of $KZ(\Lambda)$ is nothing but the *infinitesimal pure braid relations* of Ω_{ik}:

$$[\Omega_{ik}, \Omega_{mn}] = 0 \quad (\text{if } i, k, m, n \text{ are mutually disjoint});$$

and

$$[\Omega_{im}, \Omega_{ik} + \Omega_{km}] = 0 \quad (\text{if } i, k, m \text{ are mutually disjoint}).$$

These relations were originally noted by K. Aomoto (see [A1] and [A2]). Moreover these pure braid relations are equivalent to the classical Yang-Baxter equations for $\mathfrak{sl}_2$ obtained by C. N. Yang [Y] and A. A. Belavin-V. G. Drinfel'd [BD].

The space $V_{\mathfrak{g}}^{\vee}(\Lambda)$ is decomposed as

$$V_{\mathfrak{g}}^{\vee}(\Lambda)=\sum V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu};\qquad \mu=(\mu_{N-1},\cdots,\ \mu_1)\in(P_+)^{N+1},$$

where

$$V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu}\overset{C_\Lambda}{\cong}\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_N}\otimes V_{\mu_{N-1}},\ V_0)\otimes\cdots\otimes$$

$$\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_i}\otimes V_{\mu_{i-1}},\ V_{\mu_i})\otimes\cdots\otimes\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1}\otimes V_0,\ V_{\mu_1}).$$

The identification C_Λ is given by

$$C_\Lambda(\varphi_N\otimes\cdots\otimes\varphi_1)\ (u_N\otimes\cdots\otimes u_1)=\langle 0|\varphi_N(u_N\otimes\varphi_{N-1}(\cdots\otimes\varphi_2(u_1\otimes|0\rangle)\cdots))\rangle$$
$$=\langle 0|\varphi_N(u_N)\cdots\varphi_1(u_1)\ (|0\rangle),$$

for $\varphi_i\in\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_i}\otimes V_{\mu_{i-1}},\ V_{\mu_i})\cong\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_i},\ \mathrm{Hom}(V_{\mu_{i-1}},\ V_{\mu_i}))$ $(1\leqslant i\leqslant N;\ \mu_N=\mu_0=0)$, and $u_N\otimes\cdots\otimes u_1\in V_\Lambda$.

Introduce the subspace $\mathscr{V}(\Lambda)$ of $V_{\mathfrak{g}}^{\vee}(\Lambda)$ defined through C_Λ by

$$\mathscr{V}(\Lambda)=\sum_{\mu}\mathscr{V}(\Lambda)_{\mu};\qquad \mu=(\mu_{N-1},\cdots,\ \mu_1)\in(P_e)^{N+1},$$

where

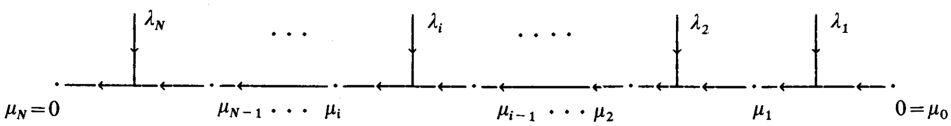
$$\mathscr{V}(\Lambda)_{\mu}=\mathscr{V}(\mathbb{v}_N(\mu))\otimes\cdots\otimes\mathscr{V}(\mathbb{v}_i(\mu))\otimes\cdots\otimes\mathscr{V}(\mathbb{v}_1(\mu))\subset V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu},$$

and

$$\mathbb{v}_N(\mu)=\begin{pmatrix} & \lambda_N & \\ 0 & & \mu_{N-1}\end{pmatrix},\cdots,\ \mathbb{v}_i(\mu)=\begin{pmatrix} & \lambda_i & \\ \mu_i & & \mu_{i-1}\end{pmatrix},\cdots,\ \mathbb{v}_1(\mu)=\begin{pmatrix} & \lambda_1 & \\ \mu_1 & & 0\end{pmatrix}.$$

Then the space $\mathscr{V}(\Lambda)$ is isomorphic to $\mathscr{V}e\imath(\Lambda)$ of N-point functions of weight Λ as follows: to each $\varphi=\varphi_N\otimes\cdots\otimes\varphi_1\in\mathscr{V}(\Lambda)_{\mu}$, assign the N-point function

$$\Phi_\varphi(z)=\langle\Phi_{\varphi_N}(z_N)\cdots\Phi_{\varphi_1}(z_1)\rangle\in\mathscr{V}e\imath(\Lambda).$$



Introduce the operators $\Omega_m^{\vee}=\sum_{1\leq i\neq j\leq m}\Omega_{ij}$ on $V_\Lambda^{\vee}$ for m $(2\leqslant m\leqslant N)$, then

$$\Omega_m^{\vee}=\hat{\Omega}_m-\sum_{i=1}^{m}\Omega_{ii},$$

where $\hat{\Omega}_m$ is the diagonal action of Ω on $V_{\lambda_m}^{\vee}\otimes\cdots V_{\lambda_1}^{\vee}$. By the pure braid relations (Remark 3.3), we get that $[\Omega_m^{\vee},\ \Omega_n^{\vee}]=0$.

These operators are scalar on each $V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu}$:

$$\Omega_m^{\vee}=2\kappa\Delta_m^{\vee}(\mu)\mathrm{id}\qquad \text{on } V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu}\ \left(\mu=(\mu_{N-1},\cdots,\ \mu_1);\ \mathbb{v}_i=\begin{pmatrix} & \lambda_i & \\ \mu_i & & \mu_{i-1}\end{pmatrix}\right)$$

where

$$\Delta_m^\vee(\mu)=\Delta_{\mu_m}-\sum_{i=1}^{m}\Delta_{\lambda_i}=-\sum_{i=1}^{m}\hat{\Delta}(\nu_i)\quad(2\leqslant m\leqslant N).$$

In fact, for each $i=2,\cdots,N$,

$$\hat{\Omega}_i=2\kappa\Delta_{\mu_i}\ \mathrm{id}\qquad\text{and}\qquad\Omega_{ii}=2\kappa\Delta_{\lambda_i}\ \mathrm{id}\qquad\text{on } V_{\mathfrak{g}}^\vee(\Lambda)_\mu.$$

For each $\mu\in(P_l)^{N-1}$ and $\varphi\in\mathscr{V}(\Lambda)_\mu$, the N-point function $\Phi_\varphi(z)$ is a formal Laurent series solution of the joint system $KZ(\Lambda)$ and $lC(\Lambda)$ by Theorem 3.1, where its Laurent series expansion is given as

$$\begin{aligned}\Phi_\varphi(z)&=\prod_{i=1}^{N}z_i^{-\hat{\Delta}(\nu_i)}\sum_{m_N\geq 0}\cdots\sum_{m_i\in\mathbb{Z}}\cdots\sum_{m_1\leq 0}C_{m_N\cdots m_1}z_N^{-m_N}\cdots z_1^{-m_1}\\&=\prod_{i=1}^{N}z_i^{-\Delta_{\lambda_i}}\langle z_N^{L(0)}\Phi^N(1)\left(\frac{z_{N-1}}{z_N}\right)^{L(0)}\Phi^{N-1}(1)\cdots\left(\frac{z_1}{z_2}\right)^{L(0)}\Phi^1(1)z_1^{-L(0)}\rangle\end{aligned}$$

where $\Phi^i=\Phi_{\varphi_i}$ and

$$C_{m_N\cdots m_1}=\langle 0|\Phi^N_\cdot(m_N)\Phi^{N-1}_\cdot(m_{N-1})\cdots\Phi^1_\cdot(m_2)\Phi^1_\cdot(m_1)|0\rangle.$$

Moreover

Theorem 3.4.

i) *For any $\varphi\in\mathscr{V}(\Lambda)_\mu$, the Laurent series $\Phi_\varphi(z)$ is absolutely convergent in the region $\mathscr{R}_z$, and is analytically continued to a multivalued holomorphic function on X_N, where $\mathscr{R}_z$ is defined by*

$$\mathscr{R}_z=\{z=(z_N,\cdots,z_1)\in\mathbf{C}^N;\ |z_N|>\cdots>|z_1|\}\subset X_N.$$

ii) *The solution space of the joint system $KZ(\Lambda)$ and $lC(\Lambda)$ is isomorphic with $\mathscr{V}_{e\ell}(\Lambda)$, hence with $\mathscr{V}(\Lambda)$.*

Proof. The statement i) is similarly proved as Theorem 3.3 i) of [TK]: Change coordinates z to w by $w_N=z_N$ and $w_i=z_i/z_{i+1}$ $(1\leqslant i\leqslant N-1)$, and apply the theory of partial differential equations with regular singular points.

ii) The equations $lC(\Lambda)$ are related only with the $\mathfrak{k}$-module structures of V_λ's, where $\mathfrak{k}=\mathbf{C}X_\theta+\mathbf{C}\theta^\vee+\mathbf{C}X_{-\theta}$. So similarly as Proposition 2.9, decompose V_λ into a sum of irreducible $\mathfrak{k}$-modules and apply the arguments of the proof Theorem 3.3 ii) of [TK]. *q.e.d.*

3.3) Composability of vertex operators.

The right $\mathfrak{g}$-module $V_\lambda^\dagger$ can be identified with the dual (right) $\mathfrak{g}$-module $V_\lambda^\vee=\mathrm{Hom}(V_\lambda,\mathbf{C})$ through the vacuum expectation values:

$$v(u)=\langle v|u\rangle\quad\text{for}\quad v\in V_\lambda^\dagger\text{ and }u\in V_\lambda.$$

Let w_0 be the longest element of the Weyl group W for $(\mathfrak{g},\mathfrak{h})$. It is well known that the group W acts on $\mathfrak{h}^*$, the weight lattice P, the set $P(\lambda)$ of weights in V_λ and the module V_λ for any dominant integral weight $\lambda\in P_+$. For each $\lambda\in P_+$, $w_0\lambda$ is the lowest weight of the $\mathfrak{g}$-module V_λ and $\lambda^+=-w_0\lambda$ is also dominant integral and is called *anti-weight* of λ. The invariance of (,) under the group W implies $\Delta_{\lambda^+}=\Delta_\lambda$ and $(\lambda^+,\theta)=(\lambda,\theta)$. There exists a $\mathfrak{g}$-isomorphism ν: $V_{\lambda^+}\to V_\lambda^\dagger$ over the anti-automorhism $\nu_{\mathfrak{g}}$ of $\mathfrak{g}$ defined by $\nu(|-\lambda\rangle)=\langle\lambda|$, that is,

$$\nu(X|u\rangle)=\nu(|u\rangle)\nu_{\mathfrak{g}}(X) \quad \text{for any } X\in\mathfrak{g} \text{ and } |u\rangle\in V_{\lambda^+}\ ,$$

where $\nu_{\mathfrak{g}}(X)=-X$ $(X\in\mathfrak{g})$ and $|-\lambda\rangle=w_0|\lambda^+\rangle$ is a lowest vector in V_{λ^+}. Note that ν is a generator of $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda^+}\otimes V_\lambda;\ \boldsymbol{C})\simeq\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda^+},\ V_\lambda^\dagger)$. By the classification, λ^+ is known as follows:

If $\mathfrak{g}$ is of type B_n, C_n, E_7, E_8, F_4 or G_2, $\lambda^+=\lambda$ for any $\lambda\in P_+$.

If $\mathfrak{g}$ is of type A_n, $(\sum_{i=1}^n a_i\Lambda_i)^+=\sum_{i=1}^n a_{n+1-i}\Lambda_i$.

If $\mathfrak{g}$ is of type D_n with even n, $\lambda^+=\lambda$ for any $\lambda\in P_+$.

If $\mathfrak{g}$ is of type D_n with odd n,

$$(\sum_{i=1}^n a_i\Lambda_i)^+=\sum_{i=1}^{n-2} a_i\Lambda_i+a_n\Lambda_{n-1}+a_{n-1}\Lambda_n.$$

If $\mathfrak{g}$ is of type E_6,

$$(\sum_{i=1}^6 a_i\Lambda_i)^+=\sum_{i=1}^5 a_{6-i}\Lambda_i+a_6\Lambda_6.$$

Proposition 3.5.

i) Let $\mathbb{v}=\begin{pmatrix} & \lambda \\ \lambda & 0\end{pmatrix}\in\mathbb{V}_l$. *Then* $\hat\Delta(\mathbb{v})=0$ *and* $\mathscr{V}(\mathbb{v})\cong\mathrm{Hom}_{\mathfrak{g}}(V_\lambda,\ V_\lambda)=\boldsymbol{C}\mathrm{id}_{V_\lambda}$, *hence vertex operators of type* $\mathbb{v}$ *exist uniquely up to a constant multiple. Let* $\Phi(z)$ *be the vertex operator with the initial term* $\varphi=\mathrm{id}_{V_\lambda}$. *Then*

$$\lim_{z\searrow 0}\ \Phi(w;z)|0\rangle=|w\rangle \qquad (w\in V_\lambda).$$

ii) Let $\mathbb{v}=\begin{pmatrix} & \lambda^+ \\ 0 & \lambda\end{pmatrix}\in\mathbb{V}_l$. *Then* $\hat\Delta(\mathbb{v})=2\Delta_\lambda$ *and* $\mathscr{V}(\mathbb{v})\cong\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda^+},\ V_\lambda^\dagger)=\boldsymbol{C}\nu$, *hence vertex operators of type* $\mathbb{v}$ *exist uniquely up to a constant multiple. Let* $\Phi(z)$ *be the vertex operator with the initial term* $\varphi=\nu$. *Then*

$$\lim_{z\nearrow\infty}\ z^{2\Delta_\lambda}\langle 0|\Phi(w;\ z)=\langle\nu(w)| \qquad (w\in V_{\lambda^+}).$$

By Theorem 3.4 and Proposition 3.5, we get the following similarly as Theorem 3.4 of [TK]:

Theorem 3.6.

Let $\Phi_i(z_i)$ *be a vertex operator of weight* λ_i *and* $u_i\in V_{\lambda_i}$ $(1\leqslant i\leqslant N)$. *Then the sequence* $\{\Phi_N(u_N;\ z_N),\cdots,\ \Phi_1(u_1;\ z_1)\}$ *is composable in the region* $\mathscr{R}_{z,0}=\{(z_N,\cdots,\ z_1)\in\boldsymbol{C}^N;\ |z_N|>\cdots>|z_1|>0\}$ *and the composed operator* $\Phi_N(u_N;z_N)\cdots\Phi_1(u_1;z_1)$ *is analytically* *continued to a multivalued holomorphic function on* M_N.

Remark 3.7. If we take the value l of the central element c of $\hat{\mathfrak{g}}$ as $l \notin \mathbf{Q}$, then we can construct an analogous theory without the l-constraint condition. In this case, the Verma module $\mathscr{M}_\lambda$ is irreducible for any dominant integral form $\lambda \in P_+$, and the space $\mathscr{H}$ is taken as $\mathscr{H} = \sum \mathscr{M}_\lambda$, where λ runs over P_+. Then the space $\mathscr{V}\!\mathit{er}(\mathbb{V})$ of vertex operators on $\mathscr{H}$ of type $\mathbb{V} = \begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{pmatrix} \in \mathbb{V}$ is isomorphic with $\mathscr{V}(\mathbb{\Lambda}) = \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\mu_1}, V_{\mu_2})$. In this case, $\mathcal{O}(\varphi) \cong \mathscr{M}_\lambda$ for any $\varphi \in \mathscr{V}(\mathbb{V})$, so the last equation $\hat{X}_\theta(-1)^{l-(\lambda,\theta)+1}\Phi(|\lambda\rangle; z) = 0$ is eliminated among the fundamental equations for vertex operators.

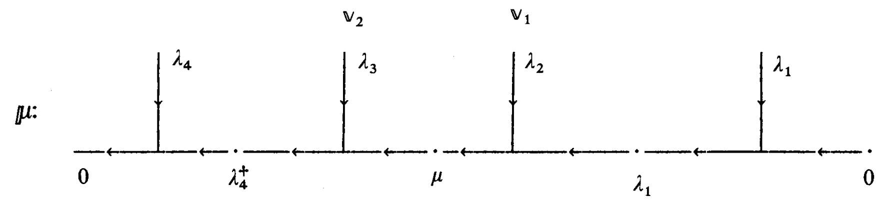
§4. Commutation Relations and Fusions of Vertex Operators.

4.1) Commutation Relations.

Fix a quadruple $\mathbb{\Lambda} = (\lambda_4, \lambda_3, \lambda_2, \lambda_1) \in (P_l)^4$. For each $\mu \in P_l$, denote

$$\boldsymbol{\mu} = (\lambda_4^+, \mu, \lambda_1), \quad \mathbb{V}_2(\mu) = \begin{pmatrix} & \lambda_3 & \\ \lambda_4^+ & & \mu \end{pmatrix} \quad \mathbb{V}_1(\mu) = \begin{pmatrix} & \lambda_2 & \\ \mu & & \lambda_1 \end{pmatrix}$$

and introduce the number $\Delta_4(\mathbb{\Lambda}) = \hat{\Delta}(\mathbb{V}_2) + \hat{\Delta}(\mathbb{V}_1) = \Delta_{\lambda_1} + \Delta_{\lambda_2} + \Delta_{\lambda_3} - \Delta_{\lambda_4}$ (independent of μ).



The space $V_{\mathfrak{g}}^{\mathbb{V}}(\mathbb{\Lambda})$ is identified with $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_3} \otimes V_{\lambda_2} \otimes V_{\lambda_1}, V_{\lambda_4^+})$ and is decomposed as

$$V_{\mathfrak{g}}^{\mathbb{V}}(\mathbb{\Lambda}) \xleftarrow[\cong]{C_{\Lambda}^{12}} \sum_{\mu \in P_+} V_{\mathfrak{g}}^{\mathbb{V}}(\mathbb{\Lambda})_\mu^{12}$$

where

$$V_{\mathfrak{g}}^{\mathbb{V}}(\mathbb{\Lambda})_\mu^{12} = \mathrm{Hom}_{\mathfrak{g}}(V_\mu \otimes V_{\lambda_3}, V_{\lambda_4^+}) \otimes \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_2} \otimes V_{\lambda_1}, V_\mu)$$

and

$$C_{\Lambda}^{12}(\varphi_2 \otimes \varphi_1)\ (u_3 \otimes u_2 \otimes u_1) = \varphi_2(u_3 \otimes \varphi_1(u_2 \otimes u_1)) \quad (u_i \in V_{\lambda_i}).$$

Thus the subspace $\mathscr{V}(\mathbb{\Lambda})$ is also identified, by this C_{Λ}^{12}, with

$$\mathscr{V}(\mathbb{\Lambda}) \xleftarrow[\cong]{C_{\Lambda}^{12}} \sum_{\mu \in P_l} \mathscr{V}(\mathbb{\Lambda})_\mu^{12}; \quad \mathscr{V}(\mathbb{\Lambda})_\mu^{12} = \mathscr{V}\begin{pmatrix} & \lambda_3 & \\ \lambda_4^+ & & \mu \end{pmatrix} \otimes \mathscr{V}\begin{pmatrix} & \lambda_2 & \\ \mu & & \lambda_1 \end{pmatrix}.$$

By Theorem 3.6, for each $\varphi_i \in \mathscr{V}(\mathrm{v}_i(\mu))$, the vertex operators $\Phi_{\varphi_2}(w)$ and $\Phi_{\varphi_1}(z)$ are composable in the region $\mathscr{R}_2=\{(w, z)\in \boldsymbol{C}^2;\ |w|>|z|>0\}$, and the composed operator $\Phi_{\varphi_2}(w)\Phi_{\varphi_1}(z)$ is analytically continued to a multi-valued holomorphic and $\mathrm{Hom}_c(\mathscr{H}_{\lambda_1}, \hat{\mathscr{H}}_{\lambda_4^+})$-valued function on $M_2=\{(w, z)\in(\boldsymbol{C}^*)^2;\ w\neq z\}$. The $V_{\mathfrak{g}}^{\vee}(\Lambda)$-valued holomorphic function

$$\Psi_{\varphi_2\otimes\varphi_1}(w, z)\ (\mathbb{u})=\langle v(u_4)|\Phi_{\varphi_2}(u_3; w)\Phi_{\varphi_1}(u_2; z)|u_1\rangle \quad (\mathbb{u}\in V_\Lambda)$$

$$=\lim_{\zeta\nearrow\infty}\lim_{\xi\searrow 0}\zeta^{2\Delta_4}\langle\Phi_v(u_4;\zeta)|\Phi_{\varphi_2}(u_3; w)\Phi_{\varphi_1}(u_2; z)\Phi(u_1;\xi)\rangle$$

on M_2 has a convergent Laurent series expansion in the region $\mathscr{R}_2$:

$$z^{-\Delta_4(\Lambda)}\sum_{n\geq 0}\left(\frac{z}{w}\right)^{n+\hat{\Delta}(\mathrm{v}_2)}\langle v(u_4)|\Phi_{\varphi_2,u_3}(n)\Phi_{\varphi_1,u_2}(-n)|u_1\rangle .$$

Its initial term is $C_\Lambda^{12}(\varphi_2\otimes\varphi_1)$.

For $V_{\mathfrak{g}}^{\vee}(\Lambda)$-valued functions $\Psi(w, z)$ on M_2, introduce the systems $KZ_2(\Lambda)$ and $lC_2(\Lambda)$ of equations:

$$KZ_2(\Lambda)\quad\left(\kappa\frac{\partial}{\partial w}-\frac{\Omega_{13}}{w}-\frac{\Omega_{23}}{w-z}\right)\Psi(w,z)=\left(\kappa\frac{\partial}{\partial z}-\frac{\Omega_{12}}{z}-\frac{\Omega_{23}}{z-w}\right)\Psi(w, z)=0.$$

and

$$lC_2(\Lambda)\quad \sum_{m=0}^{L_1}\binom{L_1}{m}w^{-m}z^{m-L_1}\Psi(w, z)\ (u_4,\ X_\theta^m u_3,\ X_\theta^{L_1-m}u_2,\ |\lambda_1\rangle)=0,$$

$$\sum_{m=0}^{L_2}\binom{L_2}{m}(w-z)^{-m}(-z)^{m-L_2}\Psi(w, z)\ (u_4,\ X_\theta^m u_3,\ |\lambda_2\rangle,\ X_\theta^{L_2-m}u_1)=0,$$

$$\sum_{m=0}^{L_3}\binom{L_3}{m}(w-z)^{-m}(-w)^{m-L_3}\Psi(w, z)\ (u_4,\ |\lambda_3\rangle,\ X_\theta^m u_2,\ X_\theta^{L_3-m}u_1)=0,$$

$$\sum_{|\mathbb{m}|=L_4}\binom{L_4}{\mathbb{m}}\Psi(w, z)\ (|\lambda_4\rangle,\ X_\theta^{m_3}u_3,\ X_\theta^{m_2}u_2,\ X_\theta^{m_1}u_1)=0,$$

where $L_i=l-(\lambda_i, \theta)+1$ $(1\leqslant i\leqslant 4)$ *and* $\mathbb{m}=(m_3, m_2, m_1)\in(\boldsymbol{Z}_{\geq 0})^3$.

By Proposition 2.5 and Theorems 2.7, 3.4, we get

Proposition 4.1.

i) The space $\{\sum_{\mu\in P_l}\sum_{\varphi_i\in\mathscr{V}(\mathrm{v}_i(\mu))}\Phi_{\varphi_2}(w)\Phi_{\varphi_1}(z)\}$ *of* $\mathrm{Hom}_C(\mathscr{H}_{\lambda_1}, \hat{\mathscr{H}}_{\lambda_4^+})$*-valued functions on* M_2 *is isomorphic with* $\mathscr{V}_{e\imath}(\Lambda)\simeq\mathscr{V}(\Lambda)$.

ii) The solution space $\mathscr{S}_2(\Lambda)$ *of the joint systems* $KZ_2(\Lambda)$ *and* $lC_2(\Lambda)$ *is* $\{\Psi_{\varphi_2\otimes\varphi_1}(w, z);\ \varphi_i\in\mathscr{V}(\mathrm{v}_i(\mu)),\ \mu\in P_l\}$.

Now introduce the $\mathfrak{g}$-isomorphism T: $V_\Lambda^\vee\to V_{T\Lambda}^\vee$ defined by

$$(T\varphi)\ (u_4\otimes u_2\otimes u_3\otimes u_1)=\varphi(u_4\otimes u_3\otimes u_2\otimes u_1)$$

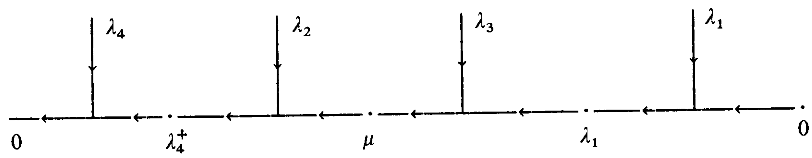
for $\varphi\in V_\Lambda^\vee$, $u_4\otimes u_2\otimes u_3\otimes u_1\in V_{T\Lambda}$ and $T\Lambda=(\lambda_4, \lambda_2, \lambda_3, \lambda_1)$. Then

$$T(V_{\mathfrak{g}}^{\vee}(\Lambda))=V_{\mathfrak{g}}^{\vee}(T\Lambda),\quad T(\mathscr{V}(\Lambda))=\mathscr{V}(T\Lambda)\quad\text{and}\quad \Delta_4(\Lambda)=\Delta_4(T\Lambda).$$

For each $\mu\in P_l$, let $\tilde{v}_2(\mu)=\begin{pmatrix} & \lambda_2 \\ \lambda_4^+ & \mu\end{pmatrix}$ and $\tilde{v}_1(\mu)=\begin{pmatrix} & \lambda_3 \\ \mu & \lambda_1\end{pmatrix}$. $V_{\mathfrak{g}}^{\vee}(T\Lambda)$-valued holomorphic functions $\Psi_{\varphi_2\otimes\varphi_1}(w, z)$ on M_2 with $\varphi_i\in\mathscr{V}(\tilde{v}_i)$ form the solution space of the joint system $KZ_2(T\Lambda)$ and $IC_2(T\Lambda)$, which is also isomorphic with $\mathscr{V}(T\Lambda)$. In the region $\mathscr{R}_2$, this function $\Psi_{\varphi_2\otimes\varphi_1}(w, z)$ also has a convergent Laurent series expansion:

$$\begin{aligned}&\Psi_{\varphi_2\otimes\varphi_1}(w, z)\ (u_4\otimes u_2\otimes u_3\otimes u_1)\\ &=z^{-\Delta_4(\Lambda)}\sum_{n\geq 0}\left(\frac{z}{w}\right)^{n+\hat{\Delta}(v_2)}\langle v(u_4)|\Phi_{\varphi_2,u_2}(n)\Phi_{\varphi_1,u_3}(-n)|u_1\rangle.\end{aligned}$$

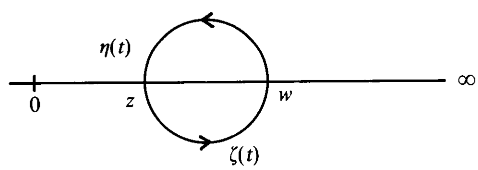
with the initial term $C_{T\Lambda}^{12}(\varphi_2\otimes\varphi_1)\in\mathscr{V}(T\Lambda)$.



Now introduce the subsets $I_2=\{(w, z)\in\mathbf{R}^2;\ w>z>0\}$ and $\bar{I}_2=\{(z, w)\in\mathbf{R}^2;\ w>z>0\}$ of the manifold M_2 and the functions

$$\eta(t)=\frac{w+z}{2}+e^{\pi\sqrt{-1}}\ \frac{w-z}{2},\quad \zeta(t)=\frac{w+z}{2}-e^{\pi\sqrt{-1}}\ \frac{w-z}{2}\quad (t\in[0,\ 1])$$

for $(w, z)\in I_2$. Then $\gamma(t)=(\eta(t), \zeta(t))$ is a path from a point (w, z) in the set I_2 to the point (z, w) in the set $\bar{I}_2$ on M_2.



Recall that $\Psi_{\varphi_2\otimes\varphi_1}(w,\ z)$ is a convergent Laurent series in the region $\mathscr{R}_2\supset I_2$. Denote by $\Psi_{\varphi_2\otimes\varphi_1}(z,\ w)$ its analytic continuation along the path $\gamma(t)$ and consider $\Psi_{\varphi_2\otimes\varphi_1}(z, w)$ near $\bar{I}_2$, then the $V_{\mathfrak{g}}^{\vee}(T\Lambda)$-valued function $T\Psi_{\varphi_2\otimes\varphi_1}(z, w)$ satisfies the equations $KZ_2(T\Lambda)$ and $IC_2(T\Lambda)$, so we get a linear mapping

$$C(\Lambda)=C_{\gamma}(\Lambda)\colon\ \mathscr{V}(\Lambda)\longrightarrow\mathscr{V}(T\Lambda)$$

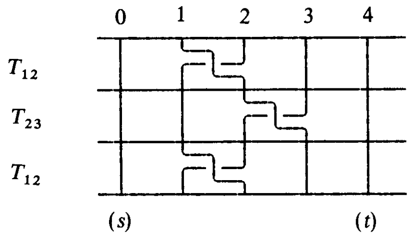
by Theorem 3.4 Sometimes we denote by $C(z,\ w)$ the endomorphism of $\sum_{\Lambda}\mathscr{V}(\Lambda)$ defined by $C_{\gamma}(\Lambda)$.

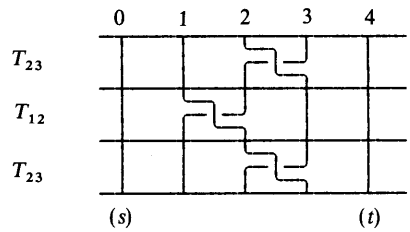
Hence by Proposition 4.1,

Proposition 4.2. *i) The mapping $C(\Lambda)$ gives an isomorphism.*
ii) Let $\Lambda=(\lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0)$, then the **braid relation** *holds:*

$$C_{12}C_{23}C_{12}=C_{23}C_{12}C_{23},$$

where $C_{ij}=C(z_i, z_j)$ $(1\leq i<j\leq 3)$.





Now our fundamental problem is:

Fundamental Problem.

Determine the isomorphism $C(\Lambda)$ for any quadruple Λ.

4.2) Reduced Equation.

Introduce a variable $\zeta=z/w$, then the $V_{\mathfrak{g}}^{\vee}(\Lambda)$-valued function $z^{\Delta_4(\Lambda)}\Psi_{\varphi_2\otimes\varphi_1}(w, \zeta w)$ is independent of w, since by Remark 3.2, ii)

$$\left(w\frac{\partial}{\partial w}+z\frac{\partial}{\partial z}-\Delta_4(\Lambda)\right)\Psi_{\varphi_2\otimes\varphi_1}(w, z)=0.$$

So we abbreviate $z^{\Delta_4(\Lambda)}\Psi_{\varphi_2\otimes\varphi_1}(w, \zeta w)$ to $\Psi_{\varphi_2\otimes\varphi_1}(\zeta)$, then the $V_{\mathfrak{g}}^{\vee}(\Lambda)$-valued function $\Psi_{\varphi_2\otimes\varphi_1}(\zeta)$ (called *reduced 4-point function*) satisfies the joint system $KZ_1(\Lambda)$ and $lC_1(\Lambda)$ for $\mathscr{V}(\Lambda)$-valued functions $\Psi(\zeta)$ on $\boldsymbol{C}$:

$KZ_1(\Lambda)$
$$\left(\kappa\frac{d}{d\zeta}-\frac{\Omega_{12}+\kappa\Delta_4(\Lambda)}{\zeta}-\frac{\Omega_{23}}{\zeta-1}\right)\Psi(\zeta)=0$$

and

$lC_1(\Lambda)$
$$\sum_{m=0}^{L_1}\binom{L_1}{m}\zeta^m\Psi(\zeta)(u_4, X_\theta^m u_3, X_\theta^{L_1-m}u_2, |\lambda_1\rangle)=0,$$

$$\sum_{m=0}^{L_2}\binom{L_2}{m}\left(\frac{\zeta}{\zeta-1}\right)^m\Psi(\zeta)(u_4, X_\theta^m u_3, |\lambda_2\rangle, X_\theta^{L_2-m}u_1)=0,$$

$$\sum_{m=0}^{L_3}\binom{L_3}{m}\left(\frac{1}{1-\zeta}\right)^m\Psi(\zeta)(u_4, |\lambda_3\rangle, X_\theta^m u_2, X_\theta^{L_3-m}u_1)=0,$$

$$\sum_{|\mathfrak{m}|=L_4}\binom{L_4}{\mathfrak{m}}\Psi(\zeta)(|\lambda_4\rangle, X_\theta^{m_3}u_3, X_\theta^{m_2}u_2, X_\theta^{m_1}u_1)=0,$$

where $L_i=l-(\lambda_i, \theta)+1$ $(1\leqslant i\leqslant 4)$ and $\mathfrak{m}=(m_3, m_2, m_1)\in(\boldsymbol{Z}_{\geq 0})^3$.

By Proposition 4.1,

Proposition 4.3.

The solution space $\mathcal{S}_1(\Lambda)$ of the joint system $KZ_1(\Lambda)$ and $lC_1(\Lambda)$ is $\{\Psi_{\varphi_2\otimes\varphi_1}(\zeta);$ $\varphi_i\in\mathcal{V}(\mathbb{v}_i(\mu)),\ \mu\in P_l\}$.

Note. i) The system $KZ_2(\Lambda)$ of equations turn to a single differential equation $KZ_1(\Lambda)$, since $\Omega_{12}+\Omega_{13}+\Omega_{23}=-\kappa\Delta_4(\Lambda)$.

ii) The function $\Psi_{\varphi_2\otimes\varphi_1}(\zeta)$ has a convergent Laurent series expansion

$$\Psi_{\varphi_2\otimes\varphi_1}(\zeta)\ (\mathbb{u})=\zeta^{\Delta(\mathbb{v}_2(\mu))}\sum_{n\geq 0}\langle v(u_4)|\Phi_{\varphi_2,u_3}(n)\Phi_{\varphi_1,u_2}(-n)|u_1\rangle\zeta^n$$

in the region $\{|\zeta|<1\}$ and its initial term is $C^{12}_{\Lambda}(\varphi_2\otimes\varphi_1)$.

iii) The solution spaces $\mathcal{S}_i(\Lambda)$ of the joint systems $KZ_i(\Lambda)$ and $lC_i(\Lambda)$ $(i=1,2)$ are isomorphic with each other, and parametrized by the space $\sum_{\mu\in P_l}\mathcal{V}(\Lambda)^{12}_{\mu}$.

For each $\varphi_2\otimes\varphi_1\in\mathcal{V}(\Lambda)^{12}_{\mu}$, the associated solutions $\Psi_{\varphi_2\otimes\varphi_1}(\zeta)\in\mathcal{S}_1(\Lambda)$ and $\Psi_{\varphi_2\otimes\varphi_1}(w,z)\in\mathcal{S}_2(\Lambda)$ are related as

$$\Psi_{\varphi_2\otimes\varphi_1}(w,z)=z^{-\Delta_4(\Lambda)}\Psi_{\varphi_2\otimes\varphi_1}\left(\frac{z}{w}\right).$$

$\Psi_{\varphi_2\otimes\varphi_1}(\zeta)$ is a solution of the system $KZ_1(\Lambda)$ of differential equation with regular singularites only at $\zeta=0$, 1 and ∞ which is regularized as

$$\Psi_{\varphi_2\otimes\varphi_1}(\zeta)=\zeta^{\hat{\Delta}(\mathbb{v}_2(\mu))}\{C^{12}_{\Lambda}(\varphi_2\otimes\varphi_1)+O(\zeta)\},$$

where $O(\zeta)$ is a $V^{\vee}_{\mathfrak{g}}(\Lambda)$-valued holomorphic function near $\zeta=0$ and vanishes at $\zeta=0$.

For each $\mu\in P_+$, introduce the spaces

$$V^{\vee}_{\mathfrak{g}}(\Lambda)^{13}_{\mu}=\mathrm{Hom}_{\mathfrak{g}}(V_{\mu}\otimes V_{\lambda_2},\ V^{+}_{\lambda_4})\otimes\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_3}\otimes V_{\lambda_1},\ V_{\mu}),$$

and

$$V^{\vee}_{\mathfrak{g}}(\Lambda)^{23}_{\mu}=\mathrm{Hom}_{\mathfrak{g}}(V_{\mu}\otimes V_{\lambda_1},\ V^{+}_{\lambda_4})\otimes\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_3}\otimes V_{\lambda_2},\ V_{\mu}),$$

and the isomorphisms

$$C^{13}_{\Lambda}\colon\ \sum_{\mu\in P_+}V^{\vee}_{\mathfrak{g}}(\Lambda)^{13}_{\mu}\longrightarrow V^{\vee}_{\mathfrak{g}}(\Lambda);\qquad C^{23}_{\Lambda}\colon\ \sum_{\mu\in P_+}V^{+}_{\mathfrak{g}}(\Lambda)^{23}_{\mu}\longrightarrow V^{\vee}_{\mathfrak{g}}(\Lambda)$$

defined by

$$C^{13}_{\Lambda}(\varphi_2\otimes\varphi_1)\ (u_3\otimes u_2\otimes u_1)=\varphi_2(u_2\otimes\varphi_1(u_3\otimes u_1)),\quad \varphi_2\otimes\varphi_1\in V^{\vee}_{\mathfrak{g}}(\Lambda)^{13}_{\mu},$$

and

$$C^{23}_{\Lambda}(\varphi_2\otimes\varphi_1)\ (u_3\otimes u_2\otimes u_1)=\varphi_2(\varphi_1(u_3\otimes u_2)\otimes u_1),\quad \varphi_2\otimes\varphi_1\in V^{\vee}_{\mathfrak{g}}(\Lambda)^{23}_{\mu},$$

for $u_i\in V_{\lambda_i}$ $(i=1, 2, 3)$. Note that $V^{\vee}_{\mathfrak{g}}(\Lambda)^{13}_{\mu}=V^{\vee}_{\mathfrak{g}}(\Lambda)^{12}_{\mu}$ and $TC^{13}_{T\Lambda}$ on the space $V^{\vee}_{\mathfrak{g}}(\Lambda)^{13}_{\mu}$.

For convenience, we use the notations

$$V^{\vee}_{\mathfrak{g}}(\Lambda)^{(0)}_{\mu}=V^{\vee}_{\mathfrak{g}}(\Lambda)^{12}_{\mu},\ V^{\vee}_{\mathfrak{g}}(\Lambda)^{(1)}_{\mu}=V^{\vee}_{\mathfrak{g}}(\Lambda)^{23}_{\mu}\text{ and }V^{\vee}_{\mathfrak{g}}(\Lambda)^{(\infty)}_{\mu}=V^{\vee}_{\mathfrak{g}}(\Lambda)^{23}_{\mu}.$$

Choose bases $\{U_{\mu,k}^{(i)}(\mathbb{\Lambda})\}$ of $V_{\mathfrak{g}}^{\vee}(\mathbb{\Lambda})_{\mu}^{(i)}$, $i=0, 1, \infty$, such that $U_{\mu,k}^{(\infty)}(\Lambda)=TU_{\mu,k}^{(0)}(\Lambda)$. Then they form bases of $V_{\mathfrak{g}}^{\vee}(\mathbb{\Lambda})$ diagonalizing the operators Ω_{12}, Ω_{23} and Ω_{13}:

$$\Omega_{12}U_{\mu,k}^{(0)}=\kappa(\gamma_{\mu}^{(0)}-\Delta_4(\mathbb{\Lambda}))U_{\mu,k}^{(0)},\ \Omega_{23}U_{\mu,k}^{(1)}=\kappa\gamma_{\mu}^{(1)}U_{\mu,k}^{(1)},\ \Omega_{13}U_{\mu,k}^{(\infty)}=\kappa\gamma_{\mu}^{(\infty)}U_{\mu,k}^{(\infty)}$$

for any $\mu\in P_+$, where $\gamma_{\mu}^{(i)}$, $i=0, 1, \infty$ are constants given by

$$\gamma_{\mu}^{(0)}=\frac{1}{\kappa}\{\Delta_{\mu}-\Delta_{\lambda_1}-\Delta_{\lambda_2}\}+\Delta_4(\mathbb{\Lambda})=\frac{1}{\kappa}\{\Delta_{\mu}+\Delta_{\lambda_3}-\Delta_{\lambda_4}\}=\hat{\Delta}(\mathrm{v}_2(\mu)),$$

$$\gamma_{\mu}^{(1)}=\frac{1}{\kappa}\{\Delta_{\mu}-\Delta_{\lambda_2}-\Delta_{\lambda_3}\},\quad \gamma_{\mu}^{(\infty)}=\frac{1}{\kappa}\{\Delta_{\mu}-\Delta_{\lambda_1}-\Delta_{\lambda_3}\}\in \boldsymbol{Q}.$$

The system $KZ_1(\mathbb{\Lambda})$ is converted to $KZ_1(\mathbb{\Lambda})_1$ and $KZ_1(\mathbb{\Lambda})_{\infty}$ at 1 and ∞ as:

$$KZ_1(\mathbb{\Lambda})_1 \qquad \left(\frac{d}{d\xi}-\frac{\Omega_{23}}{\xi}-\frac{\Omega_{12}+\kappa\Delta_4(\mathbb{\Lambda})}{\xi-1}\right)\Psi(1-\xi)=0 \qquad (\xi=1-\zeta),$$

and

$$KZ_1(\mathbb{\Lambda})_{\infty} \qquad \left(\kappa\frac{d}{d\eta}-\frac{\Omega_{13}}{\eta}-\frac{\Omega_{23}}{\eta-1}\right)\Psi\left(\frac{1}{\eta}\right)=0 \qquad \left(\eta=\frac{1}{\zeta}\right)$$

for $V_{\mathfrak{g}}^{\vee}(\mathbb{\Lambda})$-valued functions $\Psi(\zeta)$ on $\zeta\in\boldsymbol{C}^*$.

Hence there are three bases $\{\Psi_{U_{\mu,k}^{(i)}(\mathbb{\Lambda})}^{(i)}(\zeta^{(i)})\}$ of $\mathscr{S}_1(\mathbb{\Lambda})$ which are regularlized at $\zeta=$i $\left(\zeta^{(0)}=\zeta,\ \zeta^{(1)}=\xi=1-\zeta,\ \zeta^{(\infty)}=\eta=\frac{1}{\zeta}\right)$ such that

$$\Psi_{U_{\mu,k}^{(0)}(\mathbb{\Lambda})}^{(0)}(\zeta^{(0)})=\Psi_{U_{\mu,k}^{(0)}(\mathbb{\Lambda})}(\zeta)=\zeta^{\gamma_{\mu}^{(0)}}\{C_{\mathbb{\Lambda}}^{12}(U_{\mu,k}^{(0)}(\mathbb{\Lambda}))+O(\zeta)\},$$
$$\Psi_{U_{\mu,k}^{(1)}(\mathbb{\Lambda})}^{(1)}(\zeta^{(1)})=(1-\zeta)^{\gamma_{\mu}^{(1)}}\{C_{\mathbb{\Lambda}}^{23}(U_{\mu,k}^{(1)}(\mathbb{\Lambda}))+O(1-\zeta)\},$$

and

$$\Psi_{U_{\mu,k}^{(\infty)}(\mathbb{\Lambda})}^{(\infty)}(\zeta^{(\infty)})=\zeta^{-\gamma_{\mu}^{(\infty)}}\left\{C_{\mathbb{\Lambda}}^{13}(U_{\mu,k}^{(\infty)}(\mathbb{\Lambda}))+O\left(\frac{1}{\zeta}\right)\right\}.$$

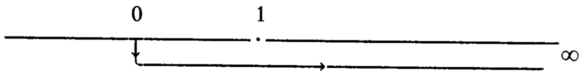
Then

Proposition 4.4. For any quadruple $\mathbb{\Lambda}\in(P_l)^4$, the isomorphism $C(\mathbb{\Lambda})$ is given by the connection isomorphism of the solutions regularized at $\zeta=0$ of the joint system $KZ_1(\mathbb{\Lambda})$ and $lC_1(\mathbb{\Lambda})$ to the solutions regularized at $\zeta=\infty$.

Proof. Recall the mapping $C(\mathbb{\Lambda})$ of $\mathscr{V}(\mathbb{\Lambda})$ to itself. For each $U\in\mathscr{V}(\mathbb{\Lambda})_{\mu}^{12}$, the solution $\Psi_U(w, z)\in\mathscr{S}_2(\mathbb{\Lambda})$ is absolutely convergent in $\mathscr{R}_2$. Denote by $\Psi_U(z, w)$ its analytic continuation along the path $\gamma(t)$. Then $T\Psi_U(z, w)$ is in $\mathscr{S}_2(T\mathbb{\Lambda})$, so it is expressed as $T\Psi_U(z, w)=\Psi_{\bar{U}}(w, z)$ for $\bar{U}=C(\mathbb{\Lambda})(U)\in\sum_{\mu\in P_l}\mathscr{V}(T\mathbb{\Lambda})_{\mu}^{12}$. Hence we get

$$T\Psi_U\left(\frac{1}{\zeta}\right)=\zeta^{-\Delta_4(\mathbb{\Lambda})}\Psi_{\bar{U}}(\zeta),$$

where $\Psi_U\left(\frac{1}{\zeta}\right)$ is the analytic continuation of $\Psi_U(\zeta)$ along the path $\bar{\gamma}(t)$ from 0 to the infinity figured below. In fact, the path $\gamma(t)$ from a point $(w, z)\in I_2$ to $(z, w)\in \bar{I}_2$ on M_2 corresponds a path from the point $\zeta=z/w$ in the set $J_1=\{\zeta\in \boldsymbol{R};\ 1>\zeta>0\}$ to the point $1/\zeta$ in the set $\bar{J}_1=\{\zeta\in\boldsymbol{R};\ \zeta>1\}$ on the manifold C^*. If z tends to zero, then the corresponding path tends to the path $\bar{\gamma}(t)$.



It is sufficient to note the equality

$$\gamma_\mu^{(\infty)}(\Lambda)+\Delta_4(\Lambda)=\gamma_\mu^{(0)}(T\Lambda)$$

among the exponents and the fact that the connection isomorphism of the equation $KZ_1(\Lambda)$ preserves the subspace $\mathscr{V}(\Lambda)$ in $V_{\mathfrak{g}}^{\vee}(\Lambda)$ because of the compatibility of the equations $KZ_1(\Lambda)$ and $IC_1(\Lambda)$. *q.e.d.*

4.3) Fusion Rule.

For each $\varphi_2\otimes\varphi_1\in\mathscr{V}(\Lambda)_\mu^{12}$, the composition $\Phi_{\varphi_2}(u_2;\ w)\Phi_{\varphi_1}(u_1;\ z)$ of the vertex operators is singular at $w=z$ and its behaviour near $w=z$ is described as below.

For each $\nu\in P_l$, introduce the space $\mathscr{V}(\Lambda)_\nu^{23}=\mathscr{V}\begin{pmatrix} & \nu & \\ \lambda_4^+ & & \lambda_1\end{pmatrix}\otimes\mathscr{V}\begin{pmatrix} & \lambda_3 & \\ \nu & & \lambda_2\end{pmatrix}$ and for $\varphi_2\otimes\varphi_1\in\mathscr{V}(\Lambda)_\nu^{23}$, define a "vertex operator" $\Phi^f_{\varphi_2\otimes\varphi_1}(z)$ of $\mathscr{H}_{\lambda_1}$ to $\hat{\mathscr{H}}_{\lambda_4^+}$ parametrized by $V_{\lambda_3}\otimes V_{\lambda_2}$ defined by

$$\Phi^f_{\varphi_2\otimes\varphi_1}(u_3\otimes u_2;\ z)=\Phi_{\varphi_2}(\varphi_1(u_3\otimes u_2);\ z)\qquad (u_i\in V_{\lambda_i}),$$

that is, the operator $\Phi^f_{\varphi_2\otimes\varphi_1}(v;z)$ satisfies the conditions (V2) and (V3) of vertex operators of weight ν with the exception that it is linear in $v\in V_{\lambda_3}\otimes V_{\lambda_2}$ in the condition (V1) (see §2.2).

Denote $\mathrm{w}_1(\nu)=\begin{pmatrix} & \lambda_3 & \\ \nu & & \lambda_2\end{pmatrix}$, then $\gamma_\nu^{(1)}=-\hat{\Delta}(\mathrm{w}_1(\nu))$. Then

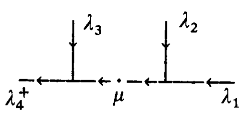
Theorem 4.5. (Short range expansion or Fusion rule)

i) Near $w=z$ $((w, z)\in\mathscr{R}_2)$,

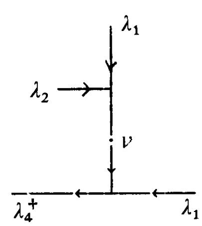
$$\Phi_{\varphi_2}(u_3;w)\Phi_{\varphi_1}(u_2;z)=\sum_{\nu\in P_l}(w-z)^{-\Delta(\mathrm{w}_1)}\ (\Phi^f_{\psi_\nu}(u_3\otimes u_2;\ z)+O(w-z))$$

$$\sim(w-z)^{-(\Delta_{\lambda_2}+\Delta_{\lambda_3})}\sum_{\nu\in P_l}(w-z)^{\Delta_\nu}\Phi^f_{\psi_\nu}(u_3\otimes u_2;\ z),$$

where $\psi_\nu\in\mathscr{V}(\Lambda)_\nu^{23}$, *and* $O(w-z)$ *is holomorphic near* $w=z$ *and vanishes at* $w=z$;



$$\sim\ \sum_{\nu\in P_l}(w-z)^{-\Delta(\mathrm{w}_1(\nu))}$$



In other words,

$$\frac{1}{2\pi\sqrt{-1}}\int_{C_{Z;0}} dw(w-z)^{\hat{\Delta}(w_1)-1}\Phi_{\varphi_2}(u_3;\ w)\Phi_{\varphi_1}(u_2;\ z)=\Phi^{f}_{\psi_\nu}(u_3\otimes u_2;\ z),$$

The value of $(w-z)^{-\hat{\Delta}(w_1)}$ *is chosen as it is positive for* $(w,\ z)\in\mathscr{R}_2\cap\mathscr{R}^2$.

ii) The fusion gives an isomorphism

$$\begin{array}{ccc} F(\Lambda):\ \mathscr{V}(\Lambda) & \longrightarrow & \mathscr{V}(\Lambda) \\ \wr\Big\uparrow C^{12}_{\Lambda} & & \wr\Big\uparrow C^{23}_{\Lambda} \\ \sum_{\mu\in P_l}\mathscr{V}(\Lambda)^{12}_{\mu} & \longrightarrow & \sum_{\nu\in P_l}\mathscr{V}(\Lambda)^{23}_{\nu} \end{array}$$

defined by

$$F(\Lambda)C^{12}_{\Lambda}(\varphi_2\otimes\varphi_1)=\sum_{\nu\in P_l}C^{23}_{\Lambda}(\psi_\nu)\qquad(\varphi_2\otimes\varphi_1\in\mathscr{V}(\Lambda)^{12}_{\mu}),$$

where ψ_ν *are the ones obtained in i).*

iii) The isomorphism $F(\Lambda)$ *is given by the connection of the solutions regularized at* $\zeta=0$ *of the joint system* $KZ_1(\Lambda)$ *and* $IC_1(\Lambda)$ *to the solutions regularized at* $\zeta=1$.

Proof. The operator $\Phi_{\varphi_2}(u_3;\ w)\Phi_{\varphi_1}(u_2;\ z)$ is determined by the 4-point function $\Psi_{\varphi_2\otimes\varphi_1}(w,\ z)\in\mathscr{T}_2(\Lambda)$ or $\Psi_{\varphi_2\otimes\varphi_1}(\zeta)\in\mathscr{T}_1(\Lambda)$. The initial term of the analytic continuation of $\Psi_{\varphi_2\otimes\varphi_1}(\zeta)$ along the standard path form 0 to 1 (the segment [0, 1]) is written as

$$\sum_{\nu\in P_l}(1-\zeta)^{\gamma^{(1)}_\nu}\{C^{23}_{\Lambda}(\psi_\nu)+O(1-\zeta)\},\quad \psi_\nu\in V^{\vee}_{\mathfrak{g}}(\Lambda)^{23}_{\nu}.$$

Now introduce the operator $\Xi_\nu(z)$: $V_{\lambda_3}\otimes V_{\lambda_2}\otimes\mathscr{H}_{\lambda_1}\to\hat{\mathscr{H}}^{+}_{\lambda_4}$ defined by

$$\Xi_\nu(u_3\otimes u_2;\ z)=\frac{1}{2\pi\sqrt{-1}}\int_{C_{z;0}}(w-z)^{-\gamma^{(1)}_\nu-1}\Phi_{\varphi_2}(u_3;\ w)\Phi_{\varphi_1}(u_2;\ z)dw,$$

where $C_{z;0}$ must be considered as a cyclce similarly as in §3.4. of [TK3]. Then

$$[X(m),\ \Xi_\nu(u_3\otimes u_2;\ z)]=z^m\{\Xi_\nu(Xu_3\otimes u_2;\ z)+\Xi_\nu(u_3\otimes Xu_2;\ z)\},$$

and

$$[L(m),\ \Xi_\nu(u_3\otimes u_2;\ z)]=z^m\left(z\frac{d}{dz}+(m+1)\Delta_\nu\right)\Xi_\nu(u_3\otimes u_2;\ z).$$

q.e.d.

4.4) $A^{(1)}_n$-case.

In the following, we restricted ourselves to the case where $\mathfrak{g}$ is of type A_n. Denote by $\mathscr{Y}^k$ the set of all Young diagrams $Y=[f_1, f_2,\cdots, f_k]$ with depth$(Y)\leqslant k$, where f_j means the number of nodes the j-th row of Y. To any Young diagram $Y=[f_1,\cdots, f_{n+1}]\in\mathscr{Y}^{n+1}$, define the dominant integral weight $\lambda(Y)\in P_{l(Y)}$, $l(Y)=f_1-f_{n+1}\in\boldsymbol{Z}_{\geq 0}$ and $m=m(Y)\in\boldsymbol{Q}$ by

$$\lambda(Y)=\sum_{j=1}^{n} b_j\bar{\Lambda}_j, \qquad b_j=f_j-f_{j+1} \qquad\qquad (1\leqslant j\leqslant n),$$

$$l(Y)=f_1-f_{n+1}=\sum_{j=1}^{n} b_j=(\lambda(Y),\ \theta),$$

$$m(Y)=\frac{1}{n+1}\sum_{k=1}^{n+1} f_k=\frac{|Y|}{n+1}=f_{n+1}+\frac{1}{n+1}\sum_{k=1}^{n} kb_k.$$

For each weight $\lambda\in P_l$, there is a Young diagram $Y\in\mathcal{Y}^{n+1}$ with $l=l(Y)$ and $\lambda=\lambda(Y)$. Sometimes we use the notations V_Y instead of $V_{\lambda(Y)}$ for $\mathcal{Y}^{n+1}$.

Note that for each i, f_i-m is independent of the choice of Young diagram expressions Y of λ. In fact, $f_i-m=f_i-f_{n+1}-\frac{1}{n+1}\sum_{k=1}^{n} kb_k=\sum_{k=i}^{n} b_k-\frac{1}{n+1}\sum_{k=1}^{n} kb_k$.

Introduce $\bar{\varepsilon}_j\in\mathfrak{h}^*$ defined by $\lambda(Y)+\bar{\varepsilon}_j=\lambda(Y+\bar{\varepsilon}_j)$, where the Young diagram $Y+\bar{\varepsilon}_j$ is

$$Y+\bar{\varepsilon}_j=[f_1,\cdots,\ f_{j-1},\ f_j+1,\ f_{j+1},\cdots,\ f_{n+1}].$$

Now we want to solve the fundamental problem for the case where $\lambda_2=\lambda_3=\bar{\Lambda}_1=\lambda(\square)$ in Λ. Note that $\Lambda=T\Lambda$ in this case. First we investigate the reduced equation $KZ_1(\Lambda)$ in detail for such quadruple Λ with $V_{\mathfrak{g}}^{\vee}(\Lambda)\neq 0$ and thereafter take the equation $lC_1(\Lambda)$ into account. Fix a quadruple $\Lambda=(\lambda_4,\ \lambda(\square),\ \lambda(\square),\ \lambda_1)$ with $V_{\mathfrak{g}}^{\vee}(\Lambda)\neq 0$ and introduce the sets $I(\Lambda)$ and $I_l(\Lambda)$

$$I(\Lambda)=\{\mu\in P_+;\ V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu}^{12}\neq\{0\}\}\supset I_l(\Lambda)=\{\mu\in P_l;\ \mathcal{V}(\Lambda)_{\mu}^{12}\neq\{0\}\}.$$

Note that $I_l(\Lambda)=I(\Lambda)\cap P_l$, since $V_{\mathfrak{g}}^{\vee}(\Lambda)_{\mu}^{12}=\mathcal{V}(\Lambda)_{\mu}^{12}$ for $\mu\in P_l$ by Proposition 2.10.

Write λ_1 as $\lambda_1=\sum_{j=1}^{n} b_j\bar{\Lambda}_j=\lambda(Y)$ with $Y=[f_1,\cdots,\ f_{n+1}]$. First note that $\#I(\Lambda)=\dim V_{\mathfrak{g}}^{\vee}(\Lambda)\leqslant 2$. And $\dim V_{\mathfrak{g}}^{\vee}(\Lambda)=2$ if and only if

(D2) $\qquad \lambda_4^{\dagger}=\lambda_1+\bar{\varepsilon}_i+\bar{\varepsilon}_j, \qquad b_{i-1}\geqslant 1,\ b_{j-1}\geqslant 1 \qquad (i<j).$

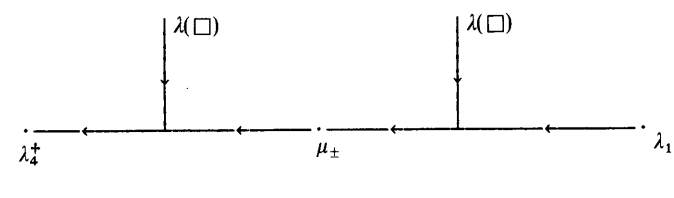
In this case, $I(\Lambda)=\{\mu_+=\lambda_1+\bar{\varepsilon}_i,\ \mu_-=\lambda_1+\bar{\varepsilon}_j\in P_+\}$. Introduce the number $d(\Lambda)=j-i+f_i-f_j$.

The case (D2) is divided into two cases $(\mathrm{D2})_i$ such that $\#I_l(\Lambda)=i$ $(i=1,\ 2)$:

$(\mathrm{D2})_2$ $\qquad \varepsilon_0<1$: $j-i<n$, or "$j-i=n$ & $(\theta,\ \lambda_1)<l$", then $I_l(\Lambda)=I(\Lambda)$,

$(\mathrm{D2})_1$ $\qquad \varepsilon_0=1$; $j-i=n$ & $(\theta,\ \lambda_1)=l$, then $I_l(\Lambda)=\{\lambda_-=\lambda_1+\bar{\varepsilon}_{n+1}=\lambda_1-\bar{\Lambda}_n\}$,

where $\varepsilon_0=\varepsilon_0(\Lambda)=\dfrac{d(\Lambda)+1}{\kappa}$.



Moreover $\dim V_g^\vee(\Lambda)=1$, if and only if either of the following conditions (D1) holds:

$(\mathrm{D1})_1$ $\quad \lambda_4^+=\lambda_1+2\bar{\varepsilon}_i;\ b_{i-1}\geqslant 2$

$(\mathrm{D1})_2$ $\quad \lambda_4^+=\lambda_1+\bar{\varepsilon}_i+\bar{\varepsilon}_{i+1};\ b_{i-1}\geqslant 1,\ b_i=0$

And $I(\Lambda)=\{\lambda_1+\bar{\varepsilon}_i\}$. Note that one of the conditons $(\mathrm{D1})_i$ implies $\sharp I_l(\Lambda)=1$.

Denote by (D0) the case where $V_g^\vee(\Lambda)=0$, i.e. $I(\Lambda)=\phi$.

Case (D2)

First we get three bases $\{U_\pm^{(0)}\}$, $\{U_\pm^{(1)}\}$ and $\{U_\pm^{(\infty)}\}$ of $V_g^\vee(\Lambda)$ such that they diagonalize the operator Ω_{12}, Ω_{23} and Ω_{13} respectively:

$$\Omega_{12}U_\pm^{(0)}=\kappa(\gamma_\pm^{(0)}-\Delta_4(\Lambda))U_\pm^{(0)},\ \Omega_{23}V_\pm^{(1)}=\kappa\gamma_\pm^{(1)}U_\pm^{(1)},\ \Omega_{13}U_\pm^{(\infty)}=\kappa\gamma_\pm^{(\infty)}U_\pm^{(\infty)},$$

and

$$\gamma_+^{(0)}=\frac{1}{\kappa}\left\{j-f_j+m-\frac{n}{n+1}\right\},\ \gamma_+^{(1)}=\frac{n}{\kappa(n+1)},\ \gamma_+^{(\infty)}=\frac{1}{\kappa}\{f_i-i-m+1\},$$

$$\gamma_-^{(0)}=\frac{1}{\kappa}\left\{i-f_i+m-\frac{n}{n+1}\right\},\ \gamma_-^{(1)}=\frac{-(n+2)}{\kappa(n+1)},\ \gamma_-^{(\infty)}=\frac{1}{k}\{f_j-j-m+1\}.$$

The differences $\gamma^{(i)}=\gamma_+^{(i)}-\gamma_-^{(i)}$ $(i=0,\ 1,\ \infty)$ are not integers: in fact,

$$\gamma^{(0)}=\gamma^{(\infty)}=\frac{d}{\kappa}\quad (d=d(\Lambda)),\quad \gamma_+^{(1)}=\frac{2}{\kappa}\quad (\kappa=l+g).$$

The transformation matrices $S^{(i,k)}$ between the bases $\{U_\pm^{(i)}\}$ and $\{U_\pm^{(k)}\}$ are explicitly given as

$$(U_+^{(k)},\ U_-^{(k)})=(U_+^{(i)},\ U_-^{(i)})S^{(i,k)},$$

where

$$S^{(0,1)}=S^{(1,0)}=\begin{pmatrix} A & B\\ B & -A\end{pmatrix},\ S^{(0,\infty)}=S^{(\infty,0)}=\begin{pmatrix} -A'' & B''\\ B'' & A''\end{pmatrix}\in O(2)\backslash SO(2),$$

$$S^{(\infty,1)}={}^tS^{(1,\infty)}=\begin{pmatrix} A' & -B'\\ B' & A'\end{pmatrix}\in SO(2),$$

and the constants $A\sim B''$ are given as

$$A=A'=\left(\frac{d-1}{2d}\right)^{1/2},\ A''=\frac{1}{d},\ B=B'=\left(\frac{d+1}{2d}\right)^{1/2}\ \text{and}\ B''=\frac{\sqrt{(d+1)\ (d-1)}}{d}.$$

For this computation, we take the orthonormal basis $u_\pm^{(i)}$ of the space of highest weight vectors of $V_\square\otimes V_\square\otimes V_\lambda$ of weight λ_4^+ diagonalizing Ω_{21}, Ω_{32} and Ω_{31} as follows.

$$u_+^{(0)}=\frac{1}{c_+^0}\{u_j\otimes u_i\otimes|\lambda\rangle+\sum_{k=i+1}^{j-1}u_k\otimes u_{k,\lambda}-\frac{1}{d(\Lambda)}u_i\otimes u_j\otimes|\lambda\rangle$$

$$+u_i\otimes\sum_{k=1}^{j-1}u_k\otimes w_{k,\lambda}+\sum_{k=1}^{i-1}u_k\otimes u_{k,\lambda}\},$$

$$u_-^{(0)}=\frac{1}{c_-^0}\{u_i\otimes u_j\otimes|\lambda\rangle+\sum_{k=1}^{i-1}u_k\otimes u'_{k,\lambda}\},$$

$$u_+^{(1)}=\frac{1}{c_+^1}\{(u_j\otimes u_i+u_i\otimes u_j)\otimes|\lambda\rangle+\sum_{\mu>\bar\varepsilon_i+\bar\varepsilon_j}v_\mu\otimes w_\mu\},$$

$$u_-^{(1)}=\frac{1}{c_-^1}\{(u_j\otimes u_i-u_i\otimes u_j)\otimes|\lambda\rangle+\sum_{\mu>\bar\varepsilon_i+\bar\varepsilon_j}v'_\mu\otimes w'_\mu\},$$

$$u_+^{(\infty)}=T_{32}u_+^{(0)}\text{ and }u_-^{(\infty)}=T_{32}u_-^{(0)},$$

where $c_\pm^0$ and $c_\pm^1$ are positive constants and $v_{k,\lambda}$, $v'_{k,\lambda}\in(V_\Box\otimes V_\lambda)_{\lambda+\bar\varepsilon_j+\bar\varepsilon_i-\bar\varepsilon_k}$, $w_{k,\lambda}\in V_{\lambda,\lambda+\bar\varepsilon_j-\bar\varepsilon_k}$, v_μ, $v'_\mu\in(V_\Box\otimes V_\Box)_\mu$, w_μ, $w'_\mu\in V_{\lambda,\lambda+\bar\varepsilon_i+\bar\varepsilon_j-\mu}$ and T_{32} is the linear isomorphism T_{32}: $V_\Box\otimes V_\Box\otimes V_\lambda\to V_\Box\otimes V_\Box\otimes V_\lambda$ defined by $T_{32}(u_3\otimes u_2\otimes u_1)=(u_2\otimes u_3\otimes u_1)$.

Similarly as Proposition 4.3 in [TK], we get the fundamental solutions of the linear differential equation $KZ_1(\Lambda)$ with regular singular points at $\zeta=0$, 1 and ∞ by means of the Gauss' hypergeometric function $F(\alpha,\beta,\gamma;\zeta)$:

Proposition 4.6.

Let $\Psi_\pm^{(i)}(\zeta)$ be the fundamental solutions of the equation $KZ_1(\Lambda)$ regularized at $\zeta=i$ ($i=0$, 1, ∞):

$$(\Psi_+^{(i)}(\zeta),\ \Psi_-^{(i)}(\zeta))=(U_+^{(i)},\ U_-^{(i)})\begin{pmatrix}\varphi_{++}^{(i)}(\zeta) & \varphi_{-+}^{(i)}(\zeta)\\ \varphi_{+-}^{(i)}(\zeta) & \varphi_{--}^{(i)}(\zeta)\end{pmatrix}.$$

Then

(i)

$$\varphi_{++}^{(0)}(\zeta)=\zeta^{\gamma_+^{(0)}}(1-\zeta)^{\gamma_+^{(1)}}F(\alpha,\beta,\gamma^{(0)};\zeta);$$
$$\varphi_{+-}^{(0)}(\zeta)=c_+^{(0)}\zeta^{1+\gamma_+^{(0)}}(1-\zeta)^{\gamma_+^{(1)}}F(\alpha+1,\beta+1,2+\gamma^{(0)};\zeta);$$
$$\varphi_{-+}^{(0)}(\zeta)=c_-^{(0)}\zeta^{1+\gamma_-^{(0)}}(1-\zeta)^{\gamma_-^{(1)}}F(-\alpha+1,-\beta+1,2-\gamma^{(0)};\zeta);$$
$$\varphi_{--}^{(0)}(\zeta)=\zeta^{\gamma_-^{(0)}}(1-\zeta)^{\gamma_-^{(1)}}F(-\alpha,-\beta,-\gamma^{(0)};\zeta).$$

(ii)

$$\varphi_{++}^{(1)}(\zeta)=\zeta^{\gamma_+^{(0)}}(1-\zeta)^{\gamma_+^{(1)}}F(\alpha,\beta,\gamma^{(1)};1-\zeta);$$
$$\varphi_{+-}^{(1)}(\zeta)=c_+^{(1)}\xi^{\gamma_+^{(0)}}(1-\zeta)^{1+\gamma_+^{(1)}}F(\alpha+1,\beta+1,2+\gamma^{(1)};1-\zeta);$$
$$\varphi_{-+}^{(1)}(\zeta)=c_-^{(1)}\zeta^{\gamma_-^{(0)}}(1-\zeta)^{1+\gamma_-^{(1)}}F(-\alpha+1,-\beta+1,2-\gamma^{(1)};1-\zeta);$$
$$\varphi_{--}^{(1)}(\zeta)=\zeta^{\gamma_-^{(0)}}(1-\zeta)^{\gamma_-^{(1)}}F(-\alpha,-\beta,-\gamma^{(1)};1-\zeta).$$

(iii)

$$\varphi_{++}^{(\infty)}(\zeta)=\zeta^{-\gamma_+^{(\infty)}}\left(1-\frac{1}{\zeta}\right)^{\gamma_+^{(1)}}F\left(\alpha,\beta,\gamma^{(\infty)};\frac{1}{\zeta}\right);$$

$$\varphi^{(\infty)}_{+-}(\zeta)=c^{(\infty)}_{+}\zeta^{-1-\gamma^{(\infty)}_{+}}\left(1-\frac{1}{\zeta}\right)^{\gamma^{(1)}_{+}}F\left(\alpha+1,\ 1+\beta,\ 2+\gamma^{(\infty)};\ \frac{1}{\zeta}\right);$$

$$\varphi^{(\infty)}_{-+}(\zeta)=c^{(\infty)}_{-}\zeta^{-1-\gamma^{(\infty)}_{-}}\left(1-\frac{1}{\zeta}\right)^{\gamma^{(1)}_{-}}F\left(1-\alpha,\ 1-\beta,\ 2-\gamma^{(\infty)};\ \frac{1}{\zeta}\right);$$

$$\varphi^{(\infty)}_{--}(\zeta)=\quad \zeta^{-\gamma^{(\infty)}_{-}}\left(1-\frac{1}{\zeta}\right)^{\gamma^{(1)}_{-}}F\left(-\alpha,\ -\beta,\ -\gamma^{(\infty)};\ \frac{1}{\zeta}\right),$$

where

$$\alpha=\frac{d+1}{\kappa},\ \beta=\frac{1}{\kappa},\ c^{(0)}_{\pm}=c^{(\infty)}_{\pm}=-\frac{\sqrt{d+1}\sqrt{d-1}}{d(\kappa\pm d)}\ \text{and}\ c^{(1)}_{\pm}=-\frac{\sqrt{d+1}\sqrt{d-1}}{2(\kappa\pm 2)}.$$

Case (D1).

Since $\dim V^{\vee}_{\mathfrak{g}}(\mathbb{\Lambda})=1$, the choice of basis vectors of $V^{\vee}_{\mathfrak{g}}(\mathbb{\Lambda})$ is not of importance. But by the compatibility with the case (D2), we choose basis vectors $\{U^{(i)};\ i=0,\ 1,\ \infty\}$ of $V^{\vee}_{\mathfrak{g}}(\mathbb{\Lambda})$ such that

$$U^{(0)}=U^{(1)}=U^{(\infty)}\ \text{for}\ (D1)_1;\ U^{(0)}=U^{(1)}=-U^{(\infty)}\ \text{for}\ (D1)_2.$$

The exponents $\gamma^{(0)}$, $\gamma^{(1)}$ and $\gamma^{(\infty)}$ of the equation $KZ_1(\mathbb{\Lambda})$ at $\zeta=0,\ 1,\ \infty$ are given as

$$(\mathrm{D1})_1\qquad \gamma^{(0)}=\frac{1}{\kappa}\left\{i-f_i+m-\frac{n+2}{n+1}\right\},\qquad \gamma^{(1)}=\frac{n}{\kappa(n+1)},\qquad \gamma^{(\infty)}=\frac{1}{\kappa}\{f_i-i-m+1\},$$

$$(\mathrm{D1})_2\qquad \gamma^{(0)}=\frac{1}{\kappa}\left\{i-f_i+m+\frac{1}{n+1}\right\},\qquad \gamma^{(1)}=\frac{-(n+2)}{\kappa(n+1)},\qquad \gamma^{(\infty)}=\frac{1}{\kappa}\{f_i-i-m+1\},$$

Then we get

Proposition 4.6'.

The fundamental solution $\Psi^{(i)}(\zeta)=U^{(i)}\varphi^{(i)}(\zeta)$ of the equation $KZ_1(\mathbb{\Lambda})$ normalized at $\zeta=i$ $(i=0,\ 1,\ \infty)$ is given as

$$(\mathrm{D1})_1\qquad \varphi^{(0)}(\zeta)=\varphi^{(1)}(\zeta)=\zeta^{\gamma^{(0)}}(1-\zeta)^{\gamma^{(1)}},\quad \varphi^{(\infty)}(\zeta)=q^{\frac{-n}{2n+2}}\varphi^{(0)}(\zeta),$$

and

$$(\mathrm{D1})_2\qquad \varphi^{(0)}(\zeta)=\varphi^{(1)}(\zeta)=\zeta^{\gamma^{(0)}}(1-\zeta)^{\gamma^{(1)}},\quad \varphi^{(\infty)}(\zeta)=-q^{\frac{n+2}{2n+2}}\varphi^{(0)}(\zeta).$$

where the exponents $\gamma^{(i)}$ are correspponding ones and $q=\exp(2\pi\sqrt{-1}/\kappa)$.

4.5) Connection Matrices for $\Lambda=(\lambda_4,\ \lambda(\square),\ \lambda(\square),\ \lambda_1)$.

Now take an intermediate edge μ for a quadruple $\mathbb{\Lambda}=(\lambda_4,\ \lambda(\square),\ \lambda(\square),\ \lambda_1)$ with $I(\mathbb{V})\neq\phi$. We want to know the analytic continuation of the reduced 4-point function $\Psi_\mu(\zeta)$ along the path $\bar{\gamma}(t)$.

For the case (D1), we get easily the connection (scalar) matrix $K(\mathbb{\Lambda})$ of the fundamental solution $\Psi^{(0)}(\zeta)$ at $\zeta=0$ to $\Psi^{(\infty)}(\zeta)$ at $\zeta=\infty$ of the equation $KZ_1(\mathbb{\Lambda})$: $S^{(0,\infty)}\varphi^{(0)}(\zeta)=\varphi^{(\infty)}(\zeta)K(\mathbb{\Lambda})$ as follows:

$$(\mathrm{D1})_1 \qquad K(\Lambda)=q^{\frac{n}{2(n+1)}}=q\cdot q^{\frac{-(n+2)}{2(n+1)}}$$

$$(\mathrm{D1})_2 \qquad K(\Lambda)=-q^{\frac{-(n+2)}{2(n+1)}}\quad\left(q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)\right).$$

Now we deal with the case (D2). By the same arguments in the Appendix II in our paper [TK], we get the connection matrix $K(\Lambda)=\begin{pmatrix}K_+^+ & K_-^+\\ K_+^- & K_-^-\end{pmatrix}$ of the fundamental solutions $(\Psi_+^{(0)}, \Psi_-^{(0)})$ at $\zeta=0$ to $(\Psi_+^{(\infty)}, \Psi_-^{(\infty)})$ at $\zeta=\infty$ of the equation $KZ_1(\Lambda)$:

$$(\Psi_+^{(0)}, \Psi_-^{(0)})=(\Psi_+^{(\infty)}, \Psi_-^{(\infty)})\begin{pmatrix}K_+^+ & K_-^+\\ K_+^- & K_-^-\end{pmatrix}.$$

Proposition 4.7.

$$K_+^+=-q^{\frac{-1}{2}\left(d+\frac{1}{n+1}\right)}\,\frac{\gamma}{\beta}\,\frac{\Gamma(\gamma)\Gamma(-\gamma)}{\Gamma(\beta)\Gamma(-\beta)},\qquad K_-^+=q^{\frac{-1}{2(n+1)}}\left(\frac{\gamma^2}{\alpha\varepsilon}\right)^{1/2}\frac{\Gamma(-\gamma)\Gamma(-\gamma)}{\Gamma(-\alpha)\Gamma(-\varepsilon)},$$

$$K_+^-=\ q^{\frac{-1}{2(n+1)}}\left(\frac{\gamma^2}{\alpha\varepsilon}\right)^{1/2}\frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\varepsilon)},\qquad K_-^-=q^{\frac{1}{2}\left(d-\frac{1}{n+1}\right)}\,\frac{\gamma}{\beta}\,\frac{\Gamma(\gamma)\Gamma(-\gamma)}{\Gamma(\beta)\Gamma(-\beta)},$$

where $d=d(\Lambda)$, $\alpha=\dfrac{d+1}{\kappa}$, $\beta=\dfrac{1}{\kappa}$, $\gamma=\dfrac{d}{\kappa}$, $\varepsilon=\dfrac{d-1}{\kappa}$ *and* $q=\exp(2\pi\sqrt{-1}/\kappa)$.

Hence by Propositions 4.1 and 4.7, we get

Proposition 4.8. *Let* $\Lambda=(\lambda_4, \lambda(\square), \lambda(\square), \lambda_1)$ *with* $I(\Lambda)\neq\phi$. *Then*

$(D1)\qquad C(\Lambda)=C_\mu^{\bar\mu}(\Lambda)=K(\Lambda)$, *where* $\mu\in I(\Lambda)$ *and* $\bar\mu\in I(\bar\Lambda)$.

$(D2)_1\qquad C(\Lambda)=C_{\mu_-}^{\bar\mu_-}(\Lambda)=K_-^-(\Lambda)$, *where* $\bar\mu_-=\mu_-=\lambda_1-\bar\Lambda_n$.

$(D2)_2\qquad C(\Lambda)=(C_\mu^{\bar\mu}(\Lambda))_{\bar\mu,\mu\in I(\Lambda)}=K(\Lambda)$ as 2×2-*matrices*.

Remark. In the case $(\mathrm{D2})_2$, all entries of the matrix $C(\Lambda)=K(\Lambda)$ do not vanish. In the case $(\mathrm{D2})_1$, $\varepsilon_0=1$ implies $K_-^+(\Lambda)=0$, hence the matrix $K(\Lambda)$ is of the form $\begin{pmatrix}* & 0\\ * & *\end{pmatrix}$.

Now recall the notion of *q-integers* for $q\in\boldsymbol{C}^*$: for each integer $i\in\boldsymbol{Z}$, introduce the q-integer $[i]=[i]_q$ defined by

$$[i]_q=\begin{cases}\dfrac{q^i-1}{q-1} & (q\neq 1)\\[2ex] i & (q=1).\end{cases}$$

Then

Lemma 4.9.

i) $[0]_q=0$, $[1]_q=1$ *and* $[2]_q=1+q$.

ii) $[-i]_q=-q^{-i}[i]_q$ *and* $[i]_{1/q}=q^{1-i}[i]_q$ $(i\in\boldsymbol{Z})$.

iii) $[i]_q=0$, *if and only if* $q^i=1.\left([\kappa]_q=0 \text{ if } q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right).\right)$

iv) $\lim_{q\to 1}[i]_q = i$ *for any* $i\in \boldsymbol{Z}$.

Then in the case $(D2)_2$, the matrix $K(\Lambda)$ can be symmetrized by means of q-integers:

Proposition 4.10. *For each quadruple* $\Lambda=(\lambda_4, \lambda(\square), \lambda(\square), \lambda_1)$ *satisfying the condition* $(D2)_2$,

$$K(\Lambda)=q^{\frac{-(n+2)}{2(n+1)}}\begin{pmatrix}\gamma_+^{-1} & \\ & \gamma_-^{-1}\end{pmatrix}\begin{bmatrix}\dfrac{-1}{[d]} & \dfrac{\sqrt{q[d-1][d+1]}}{[d]} \\ \dfrac{\sqrt{q[d-1][d+1]}}{[d]} & \dfrac{q^d}{[d]}\end{bmatrix}\begin{pmatrix}\gamma_+ & \\ & \gamma_-\end{pmatrix}$$

where

$$\gamma_\pm=\frac{\Gamma(\pm\gamma)}{(\Gamma(\pm\alpha)\Gamma(\pm\varepsilon))^{1/2}},\quad q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)\ \text{and}\ [i]=[i]_q.$$

We can get the connection matrix (=scalar) $K(\Lambda)$ in the cases $(D2)_1$ and $(D1)$:

Proposition 4.10. *Let* $q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)$.

i) $(D2)_1$ $\quad K(\Lambda)=K_-^-(\Lambda)=q^{\frac{-(n+2)}{2(n+1)}+\kappa-1}/[\kappa-1]=-q^{\frac{-(n+2)}{2(n+1)}}$

ii) $(D1)_1$ $\quad K(\Lambda)=q\cdot q^{\frac{-(n+2)}{2(n+1)}}$

iii) $(D1)_2$ $\quad K(\Lambda)=-q^{\frac{-(n+2)}{2(n+1)}}$

Remark. The values in ii) are also obtained from the ones in the case $(D2)_2$. Let $\Lambda=((\lambda_1+\varepsilon_i+\varepsilon_{i+1})^+, \lambda(\square), \lambda(\square), \lambda_1)$, then $d=1$, so $K_+^-(\Lambda)=K_-^+(\Lambda)=0$ and $K_+^+(\Lambda)=-q^{\frac{-(n+2)}{2(n+1)}}$

§5. Monodromy Representations of Braid Groups.

In this section, we construct representations of braid groups on the spaces of multi-correlation functions, and show that they give the same representations of Hecke algebras constructed by H. Wenzl. Fix an integer $N\geqslant 2$ throughout this section.

5.1) Braid Groups and Mondromy Representations.

Denote by $\bar{X}_N$ the quotient manifold X_N by the N-th symmetric group $\mathfrak{S}_N$. The fundamental group of $\bar{X}_N$ is the *braid group with N strings of the manifold* $\boldsymbol{C}$, that is, the *classical braid groupp of Artin*, and is denoted by B_N. It is well-known that the group B_N has a system $\{b_i;\ 1\leqslant i\leqslant N-1\}$ of generators with the fundamental relations

$$\text{(BR)}\qquad b_ib_j=b_jb_i\ \ (|i-j|\geqslant 2)\quad \text{and}\quad b_ib_{i+1}b_i=b_{i+1}b_ib_{i+1}\ \ (1\leqslant i\leqslant N-2).$$

These generators b_i are represented by the curves on $\boldsymbol{C}$ defined by

$$b_i(t)=\left(N,\ N-1,\cdots,\ i+\frac{1}{2}(1+e^{\pi\sqrt{-1}t}),\ i+\frac{1}{2}(1-e^{\pi\sqrt{-1}t}),\cdots,\ 2,\ 1\right),$$

$t\in[0, 1]$. Here we take $(N,\cdots, 2, 1)$ as a base point of X_N.

See our previous paper [TK] for monodromy representations of B_N on the space of multivalued holomorphic functions on X_N.

For each τ and $\lambda \in P_l$, introduce the space

$$\mathscr{V}(\lambda, N; \tau)=\sum_{\mu} \mathscr{V}(\lambda, N; \tau)_{\mu} \qquad \boldsymbol{\mu}=(\mu_{N-1}, \cdots, \mu_1)\in (P_l)^{N-1}$$

where

$$\mathscr{V}(\lambda, N; \tau)_{\mu}=\mathscr{V}\left(\left(\begin{smallmatrix} & \lambda & \\ \tau & & \mu_{N-1}\end{smallmatrix}\right)\right)\otimes\cdots\otimes\mathscr{V}\left(\left(\begin{smallmatrix} & \lambda & \\ \mu_i & & \mu_{i-1}\end{smallmatrix}\right)\right)\otimes\cdots\otimes\mathscr{V}\left(\left(\begin{smallmatrix} & \lambda & \\ \mu_1 & & 0\end{smallmatrix}\right)\right),$$

and the weight $\tau \in P_l$ is called a *target edge*. Then $\mathscr{V}(\lambda, N; \tau)$ is isomorphic with the space $\mathscr{V}\!\mathit{er}(\Lambda_\tau)$ of $(N+1)$-point functions, *i.e.* with the space

$$\mathscr{V}(\Lambda_\tau)=\sum_{\mu\in P_l^{N-1}} \mathscr{V}\left(\left(\begin{smallmatrix} & \tau^+ & \\ 0 & & \tau\end{smallmatrix}\right)\right)\otimes\mathscr{V}(\lambda, N; \tau)_{\mu},$$

where $\Lambda_\tau=\Lambda_\tau(\lambda)=(\tau^+, \lambda, \cdots, \lambda)\in(P_l)^{N+1}$.

Now recall $\kappa=l+g$ and $q=\exp\left(\dfrac{2\pi\sqrt{-1}}{\kappa}\right)$. Consider the systems $KZ(N, \lambda; \tau)$ and $lC(N, \lambda; \tau)$ of equations for $V_{\mathfrak{g}}^{\vee}(\Lambda_\tau(\lambda))$-valued functions on the manifold X_N:

$$KZ(N, \lambda; \tau) \qquad \left(\kappa\frac{\partial}{\partial z_i}-\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Omega_{ik}}{z_i-z_k}\right)\Psi(z_N, \cdots, z_1)=0 \qquad (1\leqslant i\leqslant N)$$

and for any $u_k\in V_{\lambda_k}(\lambda_{N+1}=\tau^+,\ \lambda_i=\lambda\ (1\leqslant i\leqslant N))$,

$$lC(N, \lambda; \tau) \quad \sum_{\mathfrak{m}_i}\binom{L_i}{\mathfrak{m}_i}\prod_{\substack{k=1\\k\neq i}}^{N}(z_k-z_i)^{-m_k}\Psi(z)\ (u_{N+1}, X_\theta^{m_N}u_N, \cdots, |\lambda\rangle, \cdots, X_\theta^{m_1}u_1)=0$$

for $1\leqslant i\leqslant N$, and

$$\sum_{\mathfrak{m}_{N+1}}\binom{L_{N+1}}{\mathfrak{m}_{N+1}}\Psi(\mathbb{Z})\ (|\lambda_{N+1}\rangle, X_\theta^{m_N}u_N, \cdots, X_\theta^{m_1}u_1)=0$$

where $\mathfrak{m}_i=(m_N, \cdots, \hat{m}_i, \cdots, m_1)\in(\boldsymbol{Z}_{\geq 0})^{N-1}$ $(1\leqslant i\leqslant N)$ and $\mathfrak{m}_{N+1}=(m_N, \cdots, m_1)\in(\boldsymbol{Z}_{\geq 0})^N$ with $|\mathfrak{m}_i|=L_i=l-(\lambda_i, \theta)+1$ $(1\leqslant i\leqslant N+1)$.

Introduce the transposition operators T_{ij} of $V_{\mathfrak{g}}^{\vee}(\Lambda_\tau(\lambda))$ $(1\leqslant i, j\leqslant N)$ defined by

$$(T_{ij}\varphi)\ (v\otimes u_N\otimes\cdots\otimes u_j\otimes\cdots\otimes u_i\otimes\cdots\otimes u_1)=\varphi(v\otimes v_N\otimes\cdots\otimes u_i\otimes\cdots\otimes u_j\otimes\cdots\otimes u_1)$$

for $v\in V_\tau^{+}$, $u_j\in V_\lambda$ $(1\leqslant j\leqslant N)$. Then $T_{ij}(1\leqslant i, j\leqslant N)$ preserves the subspace $\mathscr{V}(\Lambda_\tau(\lambda))$ $\cong\mathscr{V}(\lambda, N; \tau)$.

By Proposition 4.2, we can define a monodromy representation of B_N on the space $\mathscr{V}(\lambda, N; \tau)$ as follows: for each i and $\varphi_N\otimes\cdots\otimes\varphi_1\in\mathscr{V}(\lambda, N; \tau)$

$$b_i(\varphi_N\otimes\cdots\otimes\varphi_1)=\sum_{\mu\in P_l^{N-1}} C_\mu\psi_N\otimes\cdots\otimes\psi_1,\ \psi_N\otimes\cdots\otimes\psi_1\in\mathscr{V}(\lambda, N; \tau)_\mu,$$

if the analytic continuation of the $(N+1)$-point function $\Phi_{v\otimes\varphi_N\otimes\cdots\otimes\varphi_1}(\mathbb{z})$, $\mathbb{z}=(z_{N+1}, z_N, \cdots, z_1)\in\mathscr{R}_{N+1}$, along the path γ_i is written as

$$\sum_{\mu\in P_l^{N-1}} C_\mu T_{i,i+1}\Phi_{\psi_N\otimes\cdots\otimes\psi_1}(\mathbb{z}),\quad \psi_N\otimes\cdots\otimes\psi_1\in\mathscr{V}(\lambda, N; \tau)_\mu,$$

where

$$\gamma_i(t)=\Bigl(z_{N+1},\ z_N,\ z_{N-1},\cdots,\ \frac{z_{i+1}+z_i}{2}+e^{\pi\sqrt{-1}t}\frac{z_{i+1}-z_i}{2},$$

$$\frac{z_{i+1}+z_i}{2}-e^{\pi\sqrt{-1}t}\ \frac{z_{i+1}-z_i}{2},\cdots,\ z_1\Bigr),\ t\in[0,\ 1].$$

5.2) Iwahori Hecke Algebras and Monodromy Representations.

For $q\in \boldsymbol{C}^*$, the Iwahori Hecke algebra $H_N(q)$ of type A_{N-1} is defined as a $\boldsymbol{C}$-algebra with a system $\{T_1,\cdots,\ T_{N-1}\}$ of generators and the fundamental relations as

$$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}\ \ (1\leqslant i\leqslant N-2);\ T_iT_j=T_jT_i\ \ (|i-j|\geqslant 2);\ (T_i-q)(T_i+1)=0.$$

In the following, we restrict ourselves to the case $\lambda=\lambda(\square)$ and omit $\lambda=\lambda(\square)$ in the notations of §5.1.

Let $W(N;\tau)$ be the solution space of the joint system $KZ(N;\tau)$ and $lC(N;\tau)$. Then by Theorem 3.4, the space $W(N;\ \tau)$ has a basis $\{\Psi_{\mathbb{p}}(z_N,\cdots,\ z_1);\ \mathbb{p}\in\mathscr{P}_l(N;\ \tau)\}$ defined as follows: Let

$$\mathscr{P}_l(N;\ \tau)=\{\mathbb{p}=(\lambda_N,\cdots,\lambda_1,\lambda_0);\ \lambda_N=\tau,\ \lambda_0=0,\ \lambda_i\in P_l\ \ (1\leqslant i\leqslant N)$$

$$\lambda_i=\lambda_{i-1}+\bar{\varepsilon}_j \text{ for some } j\}.$$

For each $\mathbb{p}\in\mathscr{P}_l(N;\ \tau)$, define the $V_{\mathfrak{g}}^{\vee}(\Lambda_\tau)$-valued, multi-valued holomorphic function $\Psi_{\mathbb{p}}(z_N,\cdots,\ z_1)$ on X_N by

$$\Psi_{\mathbb{p}}(z_N,\cdots,z_1)\ \ (v,u_N,\cdots,u_1)=\langle v(v)|\Phi_{\mathbb{v}_N}(u_N;z_N)\cdots\Phi_{\mathbb{v}_1}(u_1;z_1)|\mathrm{vac}\rangle$$

for $v\in V_{\tau^+}$ and $u_i\in V_{\square}\ (1\leqslant i\leqslant N)$, where the vertex $\mathbb{v}_i=\mathbb{v}_i(\mathbb{p})$ is defined as $\mathbb{v}_i=\begin{pmatrix} & \lambda(\square) & \\ \lambda_i & & \lambda_{i-1}\end{pmatrix}$ $(1\leqslant i\leqslant N)$. Note that $\dim\mathscr{V}(\mathbb{v}_i)=1$ and the initial term $\varphi_{\mathbb{v}_i}\in\mathscr{V}(\mathbb{v}_i)$ of $\Phi_{\mathbb{v}_i}$ is taken as the fixed basis vector of $\mathscr{V}(\mathbb{v}_i)$.

The braid group B_N acts on this space $W(N;\tau)$ as monodromies. The commutation relations of vertex operators gives a factorization of this monodromy representation $(\pi_{N,\tau},\ W(N;\tau))$. The $\mathfrak{S}_N$-module structure of the space $V_{\mathfrak{g}}^{\vee}(\Lambda)$ is defined by

$$(\sigma\varphi)(u_N,\cdots,\ u_1)=\varphi(u_{(N)\sigma},\cdots,\ u_{(1)\sigma})\qquad (\varphi\in V_{\mathfrak{g}}^{\vee}(\Lambda),\ \sigma\in\mathfrak{S}_N),$$

and the B_N-module structure on the space of $V_{\mathfrak{g}}^{\vee}(\Lambda)$-valued functions on X_N is defined in §5.1. By Propositions 4.10, 4.10′ and 5.1, we will give this representation $\pi=\pi_{N,\tau}$ explicitly.

For each i $(1\leqslant i\leqslant N-1)$, the action $\pi(b_i)$ of the generator b_i of the group B_N on the space $W(N;\ \tau)$ is given as follows.

At first, divide the set $\mathscr{P}_l(N;\ \tau)$ into the six parts: Let $\mathbb{p}=(\lambda_N,\ \lambda_{N-1},\cdots,\ \lambda_i,\cdots,\ \lambda_1,\ \lambda_0)\in\mathscr{P}_l(N;\ \tau)$, $\lambda_N=\tau$, $\lambda_0=0$:

$$\mathbb{p}\in\mathscr{P}_i^a(N;\ \tau)\longleftrightarrow\lambda_{i+1}=\lambda_{i-1}+2\bar{\varepsilon}_j\quad\text{for some j}$$

$\mathfrak{p}\in\mathscr{P}_i^b(N;\ \tau)\longleftrightarrow\lambda_{i+1}=\lambda_{i-1}+\bar{\varepsilon}_j+\bar{\varepsilon}_{j+1}$ and $\langle\lambda_{i-1},\ H_j\rangle=0$ for some j

$\mathfrak{p}\in\mathscr{P}_i^c(N;\ \tau)\longleftrightarrow\lambda_{i+1}=\lambda_{i-1}+\bar{\varepsilon}_1+\bar{\varepsilon}_{n+1}$ and $(\lambda_{i-1},\ \theta)=l$

$\mathfrak{p}\in\mathscr{P}_i^d(N;\ \tau)\longleftrightarrow\lambda_{i+1}=\lambda_{i-1}+\bar{\varepsilon}_j+\bar{\varepsilon}_k$ for some $j>k$ with $1<j-k<n$

$\mathfrak{p}\in\mathscr{P}_i^e(N;\ \tau)\longleftrightarrow\lambda_{i+1}=\lambda_{i-1}+\bar{\varepsilon}_j+\bar{\varepsilon}_{j+1}$ and $\langle\lambda_{i-1},\ H_j\rangle>0$ for some j

$\mathfrak{p}\in\mathscr{P}_i^f(N;\ \tau)\longleftrightarrow\lambda_{i+1}=\lambda_{i-1}+\bar{\varepsilon}_1+\bar{\varepsilon}_{n+1}$ and $(\lambda_{i-1},\ \theta)<l$

Then the operation $\pi(b_i)$ is given on the basis vectors $\{\Psi_{\mathfrak{p}};\ \mathfrak{p}\in\mathscr{P}_l(N;\ \tau)\}$ as:

a) If $\mathfrak{p}\in\mathscr{P}_i^a(N;\ \tau)$, $\qquad \pi(b_i)\Psi_{\mathfrak{p}}=q\ q^{\frac{-(n+2)}{2(n+1)}}\Psi_{\mathfrak{p}}$.

b, c) If $\mathfrak{p}\in\mathscr{P}_i^b(H;\ \tau)\cup\mathscr{P}_i^c(N;\ \tau)$, $\qquad \pi(b_i)\Psi_{\mathfrak{p}}=-q^{\frac{-(n+2)}{2(n+1)}}\Psi_{\mathfrak{p}}$.

d) If $\mathfrak{p}\in\mathscr{P}_i^d(N;\ \tau)$, there is only one $\mathfrak{p}'\in\mathscr{P}_i^d(N;\ \tau)$ such that $\lambda_h=\lambda'_h$ for any $h\neq i$ and $\lambda_i-\lambda'_i=\pm(\varepsilon_j-\varepsilon_k)$. We define the action $\pi(b_i)$ for which $\boldsymbol{C}\Psi_{\mathfrak{p}}+\boldsymbol{C}\Psi_{\mathfrak{p}}'$ is invariant. We modify the notations as $\mathfrak{p}_\pm=(\tau,\ \lambda_{N-1},\cdots,\ \lambda_{i+1},\ \lambda_i^\pm,\ \lambda_{i-1},\cdots,\ \lambda_1)$, where $\lambda_i^+=\lambda_{i-1}+\bar{\varepsilon}_k$ and $\lambda_i^-=\lambda_{i-1}+\bar{\varepsilon}_j(k<j)$. Then the action $\pi(b_i)$ on $\boldsymbol{C}\Psi_{\mathfrak{p}_+}+\boldsymbol{C}\Psi_{\mathfrak{p}_-}$ is given as $\pi(b_i)=K(\lambda_{i+1},\ \lambda(\square),\ \lambda(\square),\ \lambda_{i-1})$:

$$\pi(b_i)|_{C\Psi_{\mathfrak{p}_+}+C\Psi_{\mathfrak{p}_-}}=q^{\frac{-(n+2)}{2(n+1)}}\begin{pmatrix}\gamma_+^{-1} & \\ & \gamma_-^{-1}\end{pmatrix}\begin{bmatrix}\dfrac{-1}{[d]} & \dfrac{\sqrt{q[d+1][d-1]}}{[d]}\\ \dfrac{\sqrt{q[d+1][d-1]}}{[d]} & \dfrac{q^d}{[d]}\end{bmatrix}\begin{pmatrix}\gamma_+ & \\ & \gamma_-\end{pmatrix}$$

where $d=k-j+f_j-f_k$, $\lambda_{i-1}=\lambda([f_1,\cdots,\ f_{n+1}])$ and

$$\gamma_\pm=\frac{\Gamma\left(\pm\dfrac{d}{\kappa}\right)}{\left(\Gamma\left(\dfrac{d+1}{\pm\kappa}\right)\Gamma\left(\left(\dfrac{d-1}{\pm\kappa}\right)\right)\right)^{1/2}}.$$

e) Let $k=j+1$, then $\pi(b_i)$ is the same as in case c).

f) Let $j=1$ and $k=n+1$, then $\pi(b_i)$ is the same as in case d).

In each case, $\{q,\ -1\}$ are only possible eigenvalues of the operators $q^{\frac{n+2}{2(n+1)}}\pi(b_i)$. Thus the actions $q^{\frac{n+2}{2(n+1)}}\pi(b_i)$ on the space $W(N;\ \tau)$ satisfy the relations of the Hecke algebra $H_N(q)$.

Theorem 5.1.

The monodromy representation $q^{\frac{n+2}{2(n+1)}}\pi_{N,\tau}$ *of the braid group* B_N on the space W(N; τ) *gives a representation of the Hecke algebra* $H_N(q)$, *where* $q=\exp\left(\dfrac{2\pi\sqrt{-1}}{\kappa}\right)$.

5.3) Wenzl's Representations of Hecke Algebra.

H. Wenzl[W] constructed irreducible representations $(\pi_Y,\ V_Y)$ of Hecke algebras $H_N(q)$ for any q not being roots of unity, parametrized by the set $\mathscr{Y}_N$ of all Young diagrams on N nodes. If $q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)$ with $\kappa\geqslant 4$, he also constructed irreducible representations $(\pi_Y^{(g,\kappa)}, V_Y^{(g,k)})$ of $H_N(q)$ parametrized by the set $\mathscr{Y}^{(g,k)}$ of all (g, κ)-diagrams on N nodes. Note that the representations $\pi_Y^{(g,\kappa)}$ are unitarizable as representations of the group B_N.

In this paragraph, we show that our representation $(\pi_{N,\tau}, W(N; \tau))$ of the Hecke algebra $H_N(q)$ $\left(q=\exp\left(\frac{2\pi\sqrt{-1}}{\kappa}\right)\right)$ is equivalent to the representation $(\pi_Y^{(g,\kappa)}, V_Y^{(g,\kappa)})$ and all Wenzl's unitarizable ones are thus obtained. (Recall that $\kappa=l+g$ and $g=n+1$ in our case.)

Let $\mathscr{Y}_N^g$ be the set of all Young diagrmas Y on N nodes with depth$(Y)\leqslant g$. For each $Y=[f_1,\cdots, f_g]\in\mathscr{Y}_N^g$, put $l(Y)=f_1-f_g$. Introduce the set $\mathscr{Y}_N^{(g,\kappa)}$ of all (g, κ)-diagrams on N nodes, defined by $\mathscr{Y}_N^{(g,\kappa)}=\{Y\in\mathscr{Y}_N^g;\ l(Y)\leqslant\kappa-g(=l)\}$.

We shall write $Y'<Y$, if the Young diagram Y' can be obtained by taking away appropriate nodes of Y. For each $Y\in\mathscr{Y}_N^{(g,\kappa)}$, let

$$\mathscr{P}_l(Y)=\{\not{p}=(Y_{(N)},\cdots, Y_{(1)});\ Y_{(i)}\in\mathscr{Y}_i^{(g,\kappa)},\ Y_{(i)}<Y_{(i+1)},\ Y_{(N)}=Y\}.$$

H. Wenzl defines an irreducible representation $(\pi_Y^{(g,\kappa)}, V_Y^{(g,\kappa)})$ of the algebra $H_N(q)$ for each $Y\in\mathscr{Y}_N^{(g,\kappa)}$, where $V_Y^{(g,\kappa)}$ has the form $\bigoplus_{\not{p}\in\mathscr{P}_l(Y)} \mathbf{C}v_{\not{p}}$. This gives a unitary representation of the group B_N.

Note that for each N, the weight $\lambda(Y)$ determines the Young diagram $Y\in\mathscr{Y}_N^g$ uniquely. For each $\not{p}=(Y_{(N)},\cdots,Y_{(1)})\in\mathscr{P}_l(Y)$ with $Y\in\mathscr{Y}_N^{(g,\kappa)}$, let $K(\not{p})=(\tau, \lambda(Y_{(N-1)}),\cdots, \lambda(Y_{(1)}), 0)\in\mathscr{P}_l(N; \tau)$ with $\tau=\lambda(Y)$. Then the mapping K gives a bijection of $\mathscr{P}_l(Y)$ with $\mathscr{P}_l(N; \lambda(Y))$.

Define the mapping $K\colon V_Y^{(g,\kappa)}\longrightarrow W(N; \lambda(Y))$ by

$$K(\vec{v}_{\not{p}})=\gamma(\not{p})\Psi_{K(\not{p})}\qquad \text{for}\quad \not{p}=(Y_{(N)},\cdots, Y_{(1)})\in\mathscr{P}^l(Y),$$

where $\gamma(\not{p})\in\mathbb{C}$ is determined by the following condition up to a constant multiple: Let $\not{p}$ and $\not{p}'$ be different at only one position, i.e. $\not{p}=(Y_{(N)},\cdots, Y_{(1)})$ and $\not{p}'=(Y'_{(N)},\cdots, Y'_{(1)})$ with $Y'_{(i)}=Y'_{(i)}$ and $Y_{(h)}=Y'_{(h)}$ for all $h\neq i$. Assume that $Y_{(i)}=Y_{(i-1)}+\bar{\varepsilon}_j$ and $Y'_{(i)}=Y_{(i-1)}+\bar{\varepsilon}_K$ with $j<k$. Put $d=f_k-f_j+k-j$, then

$$\text{(C)}\qquad \Gamma\left(-\frac{d}{\kappa}\right)\left(\Gamma\left(\frac{d+1}{\kappa}\right)\Gamma\left(\frac{d-1}{\kappa}\right)\right)^{\frac{1}{2}}\gamma_{\not{p}}=\Gamma\left(\frac{d}{\kappa}\right)\left(\Gamma\left(\frac{d+1}{-\kappa}\right)\Gamma\left(\frac{d-1}{-\kappa}\right)\right)^{\frac{1}{2}}\gamma_{\not{p}'}.$$

Here we fix $\gamma_{\not{p}_0}$ at some path $\not{p}_0$, say at the maximal $\not{p}_0$, then $\gamma_{\not{p}}$'s are uniquely determined for all $\not{p}\in\mathscr{P}_l(Y)$.

Then the mapping K *intertwines Wenzl's representations* $(\pi_Y^{(g,\kappa)}, V_Y^{(g,\kappa)})$ and our $(\pi, W(N; \tau))$:

Proposition 5.2. *For each* $Y\in\mathscr{Y}_N^{(g,k)}$, *set* $\tau=\lambda(Y)$. *Then*

$$K\ \pi_Y = q^{\frac{n+2}{2(n+1)}} \pi_{N,\tau} K.$$

Note. If we construct the theory for $l \notin \boldsymbol{Q}$ as in Remark 3.7, we get the monodromy representations of the Hecke algebra $H_N(q)$, $q=\exp\left(\frac{2\pi\sqrt{-1}}{l+g}\right)$, which are isomorphic to the representations (π_Y, V_Y) parametrized by $Y \in \mathscr{Y}_N^{(g)}$.

The result in this section is announced in [K], but the existence condition of vertex operators are falsely presented there. Therem 2.7 in this paper is the correct version.

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