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Double Lie Algebras on Simple Lie Algebras

Manabu SANAMI* and Yukihiro KANIE†

1 Introduction

In the begining of a course in abstract algebra, we encounter with the exercise of determining finite groups of a given order. For example, we get, as groups of order 4, the cyclic group $\boldsymbol{Z}_4$ and the Klein's four-group K_4. We may say that above two groups are all group structures which can be constructed on the four point set.

We want to consider a similar exercise for Lie algebras. Here, we shall introduce of the concept of *double Lie algebras*, that is, other Lie algebra structures on given Lie algebras. Precisely, a Lie algebra $\boldsymbol{h}$ is called the double Lie algebra on a given Lie algebra $\boldsymbol{g}$ if the underlying vector space of $\boldsymbol{h}$ coincides with that of $\boldsymbol{g}$.

The difficulty of the problem of determining groups of given orders grows rapidly as orders grow. Also in a Lie algebra case, it is practically impossible to determine double Lie algebras in general. Here we restrict ourselves to the case of projective double Lie algebras defined below:

Definition 1.1 [2] *Let $\boldsymbol{h}$ and $\boldsymbol{g}$ be finite dimensional Lie algebras over a field K with the same underlying vector space V. The Lie algebra $\boldsymbol{h}$ is called a projective double Lie algebra on $\boldsymbol{g}$ if and only if the relation $\mathrm{ad}(\boldsymbol{h}) \subset \mathrm{Der}(\boldsymbol{g})$ holds as subsets of the set of all linear transformations of V, where $\mathrm{ad}(\boldsymbol{h})$ denotes the set of inner derivations of $\boldsymbol{h}$ and $\mathrm{Der}(\boldsymbol{g})$ the set of all derivations of $\boldsymbol{g}$.*

The concept of projective double Lie algebras is introduced in [2]. This was deduced from the infinitesimal structure of a geodesic homogeneous local Lie loop in projective relation with a Lie group.

In this note, we shall investigate the class of projective double Lie algebras on simple Lie algebras.

Here, we give an example of projective double Lie algebras.

Example 1.1 *Let $\boldsymbol{g} = (V, [\ ,\ ]_{\boldsymbol{g}})$ be a Lie algebra over K. For $p \in K$, a Lie algebra $\boldsymbol{g}_p = (V, [\ ,\ ]_p)$ can be defined by $[x, y]_p = p[x, y]_{\boldsymbol{g}}$.*

Note that $\boldsymbol{g}_1 = \boldsymbol{g}$ and $\boldsymbol{g}_0$ is a commutative Lie algebra. (In fact, $\boldsymbol{g}_p$ is isomorphic to $\boldsymbol{g}$ as an abstract Lie algebra for $p \neq 0$.)

For $p \neq 0$, the relation $\mathrm{ad}(\boldsymbol{g}_p) = \mathrm{ad}(\boldsymbol{g}) \subset \mathrm{Der}(\boldsymbol{g})$ holds. On the other hand, when $p = 0$, $\mathrm{ad}(\boldsymbol{g}_0) = 0 \subset \mathrm{Der}(\boldsymbol{g})$. Hence, the Lie algebra $\boldsymbol{g}_p$ is a projective double Lie algebra on $\boldsymbol{g}$ for any $p \in K$.

In particular, on an arbitrary Lie algebra $\boldsymbol{g}$, the underlying space V can be considered as a commutative Lie algebra and projective double on $\boldsymbol{g}$. As a finite

* Toba National College
† Mie University, Faculty of Education

group analogue, the underlying set of the abelian group $\boldsymbol{Z}_4$ is nothing but the one of the group K_4.

The aim of this note is to show that the above example essentially exhausts the set of all projective double Lie algebras on a complex or real simple Lie algebra.

We denote by $\mathscr{R}$ the field of real numbers and $\mathscr{C}$ the field of complex numbers.

First in this note, we shall show that each projective double Lie algebra on a simple Lie algebra $\boldsymbol{g}$ is obtained from a linear transformation of $\boldsymbol{g}$ which satisfies the *condition* (D) (given in §3). The Key of this work is a property of linear transformations with the condition (D), which we call Key Lemma (the proof will be given §4):

Key Lemma *If a linear transformation φ of a complex or real simple Lie algebra $\boldsymbol{g}$ satisfies the condition (D), then φ belongs to the centroid of $\boldsymbol{g}$.*

This lemma and the Jacobson's results for the centroid of a simple Lie algebra give us the following main result of this note. (c.f. §3.)

Main Result (1) *If a Lie algebra $\boldsymbol{g}$ is simple over $\mathscr{C}$, then any projective double Lie algebra on $\boldsymbol{g}$ is the Lie algebra $\boldsymbol{g}_p$ in Example* 1.1 *for some complex number p.*
(2) *Let $\boldsymbol{g}$ be a simple Lie algebra over $\mathscr{R}$.*

(i) *If $\boldsymbol{g}$ admits no complex Lie algebra structures, then any projective double Lie algebra on $\boldsymbol{g}$ is $\boldsymbol{g}_p$ for some real number p.*

(ii) *Assume that $\boldsymbol{g}$ admits a complex Lie algebra structure J. Then any projective double Lie algebra on $\boldsymbol{g}$ is $(\boldsymbol{m}_p)_{\mathscr{R}}$ for some complex number p, where $\boldsymbol{m}$ is the complex Lie algebra obtained from $\boldsymbol{g}$ by the complex structure J.*

Here we use the following notations: Let F be a field and K a subfield of F, or F is an extension field of K. For a Lie algebra $\boldsymbol{g}$ over K, $\boldsymbol{g}^F$ denotes the scalar extenson of $\boldsymbol{g}$ to F. Conversely, for a Lie algebra $\boldsymbol{m}$ over F, $\boldsymbol{m}_K$ denotes the scalar restriction of $\boldsymbol{m}$ to K.

In the paper [2], it was asserted that Lie algebras in Example 1.1 exhaust the set of projective double Lie algebras on any odd-dimensional real simple Lie algebra, whose proof in [2] has an error about invariant forms unfortunately. We avoid here this false fact in that proof, and generalise the result.

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2 Preliminaries

For a finite dimensional Lie algebra $\boldsymbol{g} = (V, [\ ,\ ])$ over a field K, the *centroid* $\boldsymbol{\Gamma}(\boldsymbol{g})$ of $\boldsymbol{g}$ is defined to be the set of all linear transformations of V which commute every inner derivation of $\boldsymbol{g}$, that is;

$$\boldsymbol{\Gamma}(\boldsymbol{g}) = \{\varphi\colon V \to V \mid \varphi \text{ is linear and } \varphi \cdot \mathrm{ad}\, x = \mathrm{ad}\, x \cdot \varphi \text{ for any } x \in V\}.$$

Note that the centroid $\boldsymbol{\Gamma}(\boldsymbol{g})$ contains naturally the base field K as the subset which consists of $p \cdot id\,(p \in K)$. The Lie algebra $\boldsymbol{g}$ is called *central* if its centroid coincides with its base field.

The following was shown in Chapter X of [1];

Theorem 2.1 (Jacobson) (1) *If a Lie algebra $\boldsymbol{g}$ is simple over any field K, then the centroid $\boldsymbol{\Gamma}(\boldsymbol{g})$ of $\boldsymbol{g}$ becomes a commutative field. Moreover, the Lie algebra $\boldsymbol{g}$ is defined over the field $\boldsymbol{\Gamma}(\boldsymbol{g})$ by the scalar multiplication $\varphi \cdot x = \varphi(x)$ for $x \in \boldsymbol{g}$, $\varphi \in \boldsymbol{\Gamma}(\boldsymbol{g})$, and the Lie algebra $\boldsymbol{g}$ is central simple over $\boldsymbol{\Gamma}(\boldsymbol{g})$.*

(2) *Let $\boldsymbol{g}$ be a Lie algebra over a field K. Assume that $\boldsymbol{g}$ is central simple and F is any extension field of K. Then the scalar extension $\boldsymbol{g}^F$ of $\boldsymbol{g}$ is central simple over F.*

It is well known that the only finite-dimensional field extension of $\mathscr{C}$ is $\mathscr{C}$-itself and any algebraic extension field of $\mathscr{R}$ is either $\mathscr{R}$-itself or is isomorphic to $\mathscr{C}$. These facts and the above theorem immediately give the following remark:

Remark 2.1 (1) *Every finite dimensional simple Lie algebra over $\mathscr{C}$ is central.*
(2) *Let $\boldsymbol{g}$ be a finite dimensional simple Lie algebra over $\mathscr{R}$.*

(i) *If $\boldsymbol{g}$ is central ($\boldsymbol{\Gamma}(\boldsymbol{g}) = \mathscr{R}$), then its complexification $\boldsymbol{g}^{\mathscr{C}}$ is also simple and central over $\mathscr{C}$.*

(ii) *By the definition, $\boldsymbol{g}$ is not central if and only if $\boldsymbol{\Gamma}(\boldsymbol{g})$ is isomorphic to $\mathscr{C}$. Since $\boldsymbol{g}$ can be considered as an algebra over $\boldsymbol{\Gamma}(\boldsymbol{g})$, if $\boldsymbol{g}$ is not central then $\boldsymbol{g}$ admits a complex Lie algebra structures. In other words, there exists a Lie algebra $\boldsymbol{m}$ over $\mathscr{C}$ such that $\boldsymbol{g} = \boldsymbol{m}_{\mathscr{R}}$. Note, in this case, that the complexification $\boldsymbol{g}^{\mathscr{C}}$ is isomorphic to the double $\boldsymbol{m} \oplus \boldsymbol{m}$ of $\boldsymbol{m}$ as a complex Lie algebra.*

3 Key Lemma and Main Theorem

At first, we consider a way to construct new Lie algebra structures on a given Lie algebra by using a linear transformation.

Let $\boldsymbol{g} = (V, [\ ,\ ])$ be a Lie algebra over a field K and $\varphi: V \to V$ a linear transformation of V. Then we can define the bilinear map $[\ ,\ ]_\varphi: V \times V \to V$ by $[x, y]_\varphi = [\varphi(x), y]$.

It is easy to check the condition that the new bracket $[\ ,\ ]_\varphi$ gives a Lie algebra structure on V:

Lemma 3.1 *The bilinear map $[\ ,\ ]_\varphi$ defined above gives a Lie algebra structure on V if and only if the linear transformation φ satisfies the following condition (D) for any $x, y \in \boldsymbol{g}$;*

$$(D): \begin{cases} (1) \quad [\varphi(x), y] = [x, \varphi(y)]. \\ (2) \quad \varphi([\varphi(x), y]) = [\varphi(x), \varphi(y)]. \end{cases}$$

Definition 3.1 *Let $\boldsymbol{g} = (V, [\ ,\ ])$ be a Lie algebra over a field K. When a linear transformation φ of $\boldsymbol{g}$ satisfies the condition (D), we denote by $\boldsymbol{g}_\varphi$ the Lie algebra $(V, [\ ,\ ]_\varphi)$, and call it the* canonical *Lie algebra on $\boldsymbol{g}$ for φ.*

Remark 3.1 *Let $\boldsymbol{g} = (V, [\ ,\ ])$ be a Lie algebra over a field K.*

(1) *When* $\varphi = p \cdot id\,(p \in K)$, *the canonical Lie algebra* $\boldsymbol{g}_{\varphi}$ *for* φ *is nothing but the Lie algebra* $\boldsymbol{g}_p$ *in Example* 1.1.

(2) *If* φ *belongs to the centroid of* $\boldsymbol{g}$, *then* φ *satisfies the condition* (D).

If a linear transformation φ of a Lie algebra $\boldsymbol{g}$ satisfies the condition (D), then the relation $\mathrm{ad}(\boldsymbol{g}_{\varphi}) \subset \mathrm{ad}(\boldsymbol{g})$ holds, and the canonical Lie algebra $\boldsymbol{g}_{\varphi}$ is a projective doulbe Lie algebra on $\boldsymbol{g}$. Conversely, the following lemma holds.

Lemma 3.2 *Let* $\boldsymbol{g}$ *be a finite dimensional Lie algebra over* K. *If* $\boldsymbol{g}$ *has a trivial center and no outer derivations, then any projective double Lie algebra on* $\boldsymbol{g}$ *is the canonical Lie algebra* $\boldsymbol{g}_{\varphi}$ *for some linear transformation* φ.

Proof. By the assumption, $\mathrm{ad}(\boldsymbol{h}) \subset \mathrm{Der}(\boldsymbol{g}) = \mathrm{ad}(\boldsymbol{g})$ for a projective double Lie algebra $\boldsymbol{h}$ on $\boldsymbol{g}$. And the adjoint representation $\mathrm{ad}_{\boldsymbol{g}}\colon \boldsymbol{g} \to \mathrm{ad}(\boldsymbol{g})$ is bijective since $\boldsymbol{g}$ has a trivial center. Hence we can define the composition $\varphi = \mathrm{ad}_{\boldsymbol{g}}^{-1} \circ \mathrm{ad}_{\boldsymbol{h}}$, and so the Lie algebra $\boldsymbol{h}$ coincides with $\boldsymbol{g}_{\varphi}$. In fact

$$\begin{aligned}[x, y]_{\varphi} &= [\varphi(x), y]_{\boldsymbol{g}} = [\mathrm{ad}_{\boldsymbol{g}}^{-1}(\mathrm{ad}_{\boldsymbol{h}}(x)), y]_{\boldsymbol{g}} \\ &= \mathrm{ad}_{\boldsymbol{g}}(\mathrm{ad}_{\boldsymbol{g}}^{-1}(\mathrm{ad}_{\boldsymbol{h}}(x)))(y) = \mathrm{ad}_{\boldsymbol{h}}(x)(y) = [x, y]_{\boldsymbol{h}}.\end{aligned}$$

q.e.d.

Since any complex or real simle Lie algebras have trivial centers and no outer derivations, we get

Corollary *Let* $\boldsymbol{g}$ *be a finite dimensional Lie algebra over* $\mathscr{C}$ *or* $\mathscr{R}$. *If* $\boldsymbol{g}$ *is simple, then any projective double Lie algebra on* $\boldsymbol{g}$ *is the canonical Lie algebra* $\boldsymbol{g}_{\varphi}$ *for some* φ.

The following lemma is the key of this note (It will be proved in §4).

Key Lemma *Let* $\boldsymbol{g}$ *be a finite dimensional* simple *Lie algebra over* $\mathscr{C}$ *or* $\mathscr{R}$. *If a linear transformation* φ *of* $\boldsymbol{g}$ *satisfies the condition* (D), *then* φ *belongs to the centroid of* $\boldsymbol{g}$.

We translate this Key Lemma into the following theorem. It is easy to see that this theorem is equivalent to Main Result in §1.

Theorem 3.1 (1) *Let* $\boldsymbol{g}$ *be a finite dimensional simple Lie alebra over* $\mathscr{C}$. *Then, any projective double Lie algebra on* $\boldsymbol{g}$ *is the Lie algebra* $\boldsymbol{g}_p$ *in Example* 1.1 *for some complex number* p.

(2) *Let* $\boldsymbol{g}$ *be a finite dimensional simple Lie algebra over* $\mathscr{R}$.

(*i*) *If* $\boldsymbol{g}$ *is central, then any projective double Lie algebra on* $\boldsymbol{g}$ *is the Lie algebra* $\boldsymbol{g}_p$ *for some real number* p.

(*ii*) *When* $\boldsymbol{g}$ *is not central, there exists a Lie algebra* $\boldsymbol{m}$ *over* $\mathscr{C}$ *such that* $\boldsymbol{g} = \boldsymbol{m}_{\mathscr{R}}$. *Then, any projective double Lie algebra on* $\boldsymbol{g}$ *is the Lie algebra* $(\boldsymbol{m}_p)_{\mathscr{R}}$ *for some complex number* p.

Proof. (1) By Key Lemma and Remark 2.1 (1), for any linear transformation φ satisfying the condition (D), there exists a complex number p such that $\varphi = p \cdot id$. Hence, Corollary of Lemma 3.2 and Remark 3.1 (1) imply the assertion.

(2) (i) is obtained by the same argument in the case (1). (ii) The existence of

$\boldsymbol{m}$ is due to Remark 2.1 (2). Note that the base field of $\boldsymbol{m}$ is identified with the centroid of $\boldsymbol{g}$. Hence, the result follows from the similar argument to (1). *q.e.d.*

4 Proof of Key Lemma

For the proof of Key Lemma, we prepare a few lemmas about the condition (*D*).

Lemma 4.1 *Let $\boldsymbol{g}$ be a Lie algebra over a field K. Assume that a linear transformation φ of $\boldsymbol{g}$ satisfies the condition (D).*

(1) *If φ has a non-zero eigenvalue $\lambda \in K$, then the eigenspace $\boldsymbol{g}(\lambda) = \{x \in \boldsymbol{g} \mid \varphi(x) = \lambda \cdot x\}$ is an ideal of $\boldsymbol{g}$.*

(2) *If φ is non-zero and nilpotent, then there exists a non-zero element $z \in \boldsymbol{g}$ such that* $(\operatorname{ad} z)^2 = 0$.

Proof. (1) Take an element $x \in \boldsymbol{g}(\lambda)$, then $[x, y] \in \boldsymbol{g}(\lambda)$ for any $y \in \boldsymbol{g}$. In fact,

$$\varphi([x, y]) = \varphi\left(\left[\frac{1}{\lambda}\varphi(x), y\right]\right) = \frac{1}{\lambda}[\varphi(x), \varphi(y)] = \frac{1}{\lambda}[\varphi^2(x), y] = \lambda[x, y].$$

(2) Since φ is nilpotent, we can take an element v such that $\varphi(v) \neq 0$ and $\varphi^2(v) = 0$. Then $z = \varphi(v)$ is the desired element. In fact,

$$(\operatorname{ad} z)^2(x) = [\varphi(v), [\varphi(v), x]] = [v, \varphi[\varphi(v), x]] = [v, [\varphi^2(v), x]] = 0.$$

q.e.d.

Lemma 4.2 *Let $\boldsymbol{g}$ be a finite dimensional simple Lie algebra over $\mathscr{C}$ or $\mathscr{R}$. If a linear transformation φ of $\boldsymbol{g}$ satisfies the condition (D), then φ is surjective or nilpotent.*

Proof. First, assume that the base field is $\mathscr{C}$. Take an eigenvalue λ of φ, then $\lambda \in \mathscr{C}$. If $\lambda \neq 0$, then the Lie algebra $\boldsymbol{g}$ coincides with the eigenspace $\boldsymbol{g}(\lambda)$ due to Lemma 4.1 (1) and the simplicity of $\boldsymbol{g}$. This implies that φ is surjective. If every eigenvalue of φ is zero, then φ is nilpotent.

Now assume that the base field is real.

(1) When $\boldsymbol{g}$ is central, $\boldsymbol{g}^{\mathscr{C}}$ is also simple as a Lie algebra over $\mathscr{C}$ by Remark 2.1. Hence, the assertion follows from the same argument for the complexification $\varphi^{\mathscr{C}}: \boldsymbol{g}^{\mathscr{C}} \to \boldsymbol{g}^{\mathscr{C}}$.

(ii) Let $\boldsymbol{g}$ be not central. Then, due to Remark 2.1 (2), $\boldsymbol{g} = \boldsymbol{m}_{\mathscr{R}}$ for some complex simple Lie algebra $\boldsymbol{m}$ and that $\boldsymbol{g}^{\mathscr{C}}$ is isomorophic to $\boldsymbol{m} \oplus \boldsymbol{m}$ as a complex Lie algebra.

Take an eigenvalue λ of φ. If λ is not real, then its complex conjugate $\bar{\lambda}(\neq \lambda)$ is also an eigenvalue. So, we get that $\boldsymbol{g}^{\mathscr{C}} \supset \boldsymbol{g}^{\mathscr{C}}(\lambda) \oplus \boldsymbol{g}^{\mathscr{C}}(\bar{\lambda})$ as vector spaces. We know that the eigenspaces $\boldsymbol{g}^{\mathscr{C}}(\lambda)$ and $\boldsymbol{g}^{\mathscr{C}}(\bar{\lambda})$ are ideals of $\boldsymbol{g}^{\mathscr{C}}$ due to Lemma 4.1 (1) and $\boldsymbol{g}^{\mathscr{C}} \cong \boldsymbol{m} \oplus \boldsymbol{m}$. Thus the Lie algebras $\boldsymbol{g}^{\mathscr{C}}(\lambda)$ and $\boldsymbol{g}^{\mathscr{C}}(\bar{\lambda})$ are both isomorphic to $\boldsymbol{m}$ and $\boldsymbol{g}^{\mathscr{C}} = \boldsymbol{g}^{\mathscr{C}}(\lambda) \oplus \boldsymbol{g}^{\mathscr{C}}(\bar{\lambda})$. This implies that $\varphi^{\mathscr{C}}$ is surjective and so is φ.

When every eigenvalue of φ is real, the assertion follows from the same arguments on the complex case. *q.e.d.*

Lemma 4.3 *Let $\boldsymbol{g}$ be a finite dimensional complex* semi simple *Lie algebra. Then* *there are no non-zero nilpotent linear transformations of* $\boldsymbol{g}$ *which satisfy the condition* (D).

Proof. Due to Lemma 4.1 (2), it is sufficent to show that there are no non-zero elements $z \in \boldsymbol{g}$ such that $(\operatorname{ad} z)^2 = 0$.

Decompose any non-zero element $z \in \boldsymbol{g}$ as $z = h + \sum_{\alpha \in \Delta} a_\alpha X_\alpha$ by a fixed root space decomposition of the semi simple Lie algebra $\boldsymbol{g}$. Then we can choose a root vector w of $\boldsymbol{g}$ such that $(\operatorname{ad} z)^2(w) \neq 0$. *q.e.d.*

Now, we are at the stage to prove Key Lemma.

Let φ be a linear transformation in Key Lemma. Then, by Lemma 4.2, φ is surjective or nilpotent.

If φ is surjective, then we get $\varphi([x, y]) = [x, \varphi(y)]$ due to the second equality of the condition (D). Thus we get that φ is in the centroid of $\boldsymbol{g}$.

Let φ be nilpotent.

If the base field of $\boldsymbol{g}$ is $\mathscr{C}$, then $\varphi = 0$ due to Lemma 4.3, in particular $\varphi \in \boldsymbol{\Gamma}(\boldsymbol{g})$.

When $\boldsymbol{g}$ is a Lie algebra over $\mathscr{R}$, we can see that the complexification $\varphi^{\mathscr{C}}$ satisfies the condition (D) and is also nilpotent as a transformation of $\boldsymbol{g}^{\mathscr{C}}$. On the other hand, Remark 2.1 implies that $\boldsymbol{g}^{\mathscr{C}}$ is semi simple over $\mathscr{C}$. Then we get $\varphi^{\mathscr{C}} = 0$ by Lemma 4.3, hence $\varphi = 0$.

This completes the proof.

References

[1] N. Jacobson, Lie Algebras, Interscience Pub., 1962.

[2] M. Sanami and M. Kikkawa, *A Class of Double Lie Algebras on Simple Lie Algebras and Projectivity of Simple Lie Groups*, Mem. Fac. Sci., Shimane Univ. **25** (1991), 39–44.