Bull. Fac. Educ. Mie Univ. **45** (*Natur. Sci*): 7–25, 1994

Cusp Forms for $\Gamma_0(p)$ of weight 2

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Let p be a prime, and let $\mathfrak{S}_2(p)$ be the vector space of cusp forms for $\Gamma_0(p)$ of weight 2. As Hecke [7] conjectured and as Eichler [3], [4] proved, $\mathfrak{S}_2(p)$ is spanned by theta series of quaternary quadratic forms associated with a maximal order of some quaternion algebra (see also H. Hijikata, A. K. Pizer and T. R. Schemanske [8]).

There is a decomposition

$$\mathfrak{S}_2(p) = \mathfrak{S}_2^-(p) \oplus \mathfrak{S}_2^+(p)$$

where $\mathfrak{S}_2^-(p)$ (resp. $\mathfrak{S}_2^+(p)$) denotes the eigen space with eigen value -1 (resp. 1) of the operator associated with $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ which is involutive. In the present paper we are interested in $\mathfrak{S}_2^-(p)$. In our previous paper [11], we have constructed by using Hilbert modular forms, elliptic modular forms for $\Gamma_0(p)$ of weight 2 whose higher Fourier coefficients are computable and whose 0-th Fourier coefficients are essentially the values at 0 of some zeta functions of real quadratic fields. It is expected that there are arithmetic applications if we have simple formulas for the values at 0 of the zeta functions. Those elliptic modular forms are written as sums of the Eisenstein series and elements of $\mathfrak{S}_2^-(p)$, and for our purpose it is necessary to investigate the Fourier coefficients of modular forms in $\mathfrak{S}_2^-(p)$. This is the reason why we are interested in $\mathfrak{S}_2^-(p)$.

For small p we give basis of $\mathfrak{S}_2^-(p)$ explicitly which are constructed from Eisenstein series. One advantage of our basis is that their Fourier coefficients are easy to obtain, which is very important for our purpose. There is a table of the Fourier coefficients of some cusp forms for $\Gamma_0(N)$ of weight 2 in Modular Functions of One Variable IV (Lect. Notes in Math., **476** (1975)), and there is also a table of Hecke polynomials of the spaces of cusp forms for $\Gamma_0(p)$ of weight 2 in Doi and Miyake [2]. However no tables of the Fourier coefficients of basis of $\mathfrak{S}_2^-(p)$ can not be found in any literature. We supply the tables in the present paper, which might, the author hopes, benefit us.

The work is partly supported by Grant-in-Aid for Scientific Research on Priority Areas 231 "Infinite Analysis".

1. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$, the set of natural numbers, the ring of integers, the field of complex numbers respectively. For $n, k \in \mathbb{N}$, $\sigma_k(n)$ denotes $\sum_{0<d|n} d^k$, and if n is not integral we define $\sigma_k(n)$ to be 0. Let H denote the upper plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and let $GL_2^+(\mathbb{Z})$ denote the group of 2×2 integral matrices of positive determinant. The group acts on H by the usual modular substitution

$$z \longmapsto Mz = \frac{az+b}{cz+d}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}).$$

The center of $GL_2^+(\mathbb{Z})$, namely the group consisting of integral diagonal matrices, acts trivially on H, and so actually $GL_2^+(\mathbb{Z})$ acts via the quotient group $\overline{GL_2^+(\mathbb{Z})} = GL_2^+(\mathbb{Z})/(\text{the center})$. For a natural number N let

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, a \equiv d \equiv 1,\ c \equiv 0 \pmod N \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, c \equiv 0 \pmod N \right\},$$

and let $\Gamma_0(N)^*$ be the group generated by $\Gamma_0(N)$ and $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in GL_2^+(\mathbb{Z})$. If $\bar{\Gamma}_0(N)$, $\bar{\Gamma}_0(N)^*$ denote the subgroups of $\overline{GL_2^+(\mathbb{Z})}$ given as the natural surjective images of $\Gamma_0(N)$, $\Gamma_0(N)^*$ respectively, then the index $[\bar{\Gamma}_0(N); \bar{\Gamma}_0(N)^*]$ is equal to two. Let Γ be a subgroup of $GL_2^+(\mathbb{Z})$ commensurable with $SL_2(\mathbb{Z})$. The quotient space H/Γ has the natural compactification given by adding a finite number of points, which is called cusps. We denote by $(H/\Gamma)^*$, the compactified space, which is a smooth algebraic curve. Let f be a holomorphic function on H. Then f is called a modular form for Γ of weight $k \in \mathbb{N}$ if it satisfies

(i) $f|M = f$ for any $M \in \Gamma$,

(ii) f is holomorphic at each cusp,

where

$$(f|M)(z) = (ad-bc)^{k/2}(cz+d)^k f(z) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z})$$

with $(ad-bc)^{k/2} > 0$. We denote by $\mathfrak{M}_k(\Gamma)$, the space of modular forms for Γ of weight k. A modular form is called a cusp form if it vanishes at each cusp, and we denote by $\mathfrak{S}_k(\Gamma)$, the subspace of $\mathfrak{M}_k(\Gamma)$ consisting of cusp forms.

Let p be a prime, and let $\mathfrak{M}_k(p) := \mathfrak{M}_k(\Gamma_0(p))$, $\mathfrak{S}_k(p) := \mathfrak{S}_k(\Gamma_0(p))$. Let us recall the operators W_p, U_p on these spaces defined in Atkin and Lehner [1]. W_p is the one associated with the matrix $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$, i.e.,

$$f|W_p = f\left|\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}\right. = p^{-k/2}z^{-k}f\left(-\frac{1}{pz}\right)$$

Let $f(z) = \sum_{n=0}^{\infty} a_n \mathbb{e}(nz)$ be the Fourier expansion where $\mathbb{e}(x)$ denotes $\exp(2\pi\sqrt{-1}\,x)$. Then U_p is described in terms of the Fourier expansion, as

$$f|U_p = \sum_{n=0}^{\infty} a_{np}\mathbb{e}(nz).$$

These two operators commute with each other, and also with Hecke operators $T_m(p \nmid m)$. So the decomposition of $\mathfrak{M}_k(p)$ or $\mathfrak{S}_k(p)$ into eigen spaces gives that of Hecke modules. Let $\mathfrak{S}_k^-(p)$, $\mathfrak{S}_k^+(p)$ be the eigen subspaces in $\mathfrak{S}_k(p)$ with eigenvalue $-1, 1$ of W_p respectively, and let $\mathfrak{M}_k^-(p)$, $\mathfrak{M}_k^+(p)$ be the ones defined similarly for $\mathfrak{M}_k(p)$. By definition we have $\mathfrak{M}_k^+(p) = \mathfrak{M}_k(\Gamma_0(p)^*)$, $\mathfrak{S}_k^+(p) = \mathfrak{S}_k(\Gamma_0(p)^*)$. By Lemma 17 (iii) in [1], $f|W_p + f|U_p$ is a modular form for $SL_2(\mathbb{Z})$ of weight k.

Let us assume $k = 2$. Then we have $W_p = -U_p$ as operators, since there are no nontrivial modular forms for $SL_2(\mathbb{Z})$ of weight 2. The curve $(H/\Gamma_0(p))^*$ has exactly two cusps, and by the residue theorem the 0-th Fourier coefficient at the cusp $\sqrt{-1}\,\infty$ of a modular form in $\mathfrak{M}_2(p)$ is $-p$ times the 0-th Fourier coefficient at the other cusp. In particular we have $\mathfrak{M}_2^+(p) = \mathfrak{S}_2^+(p)$. As is well-known, the genus of the algebraic curve $(H/\Gamma_0(p))^*$ is 0 for $p = 2, 3$, and is $(p-13)/12$, $(p-5)/12$, $(p-7)/12$ or $(p+1)/12$ according as $p \equiv 1, 5, 7, 11 \pmod{12}$ (see for example Shimura [10]). The curve $(H/\Gamma_0(p)^*)^*$ has $(H/\Gamma_0(p))^*$ as its double covering, and its genus is obtained by the Hurwitz formula if the number of the fixed points under the action of $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ is obtained, which was done by Fricke [5], Chap. 4 (see also Lehner and Newman [9]). Since $\dim \mathfrak{S}_2^-(p) = \dim \mathfrak{S}_2(p) - \dim \mathfrak{S}_2(\Gamma_0(p)^*)$, and since $\dim \mathfrak{S}_2(p)$, $\dim \mathfrak{S}_2(\Gamma_0(p)^*)$ are equal to the genera of curves $(H/\Gamma_0(p))^*$, $(H/\Gamma_0(p)^*)^*$ respectively, we obtain the following;

Lemma 1. *Let $p > 3$ be prime, and let $h(-p)$ be the class number of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-p})$. Then $\dim \mathfrak{S}_2^-(p)$ is given by*

$$\frac{1}{24}(p - \alpha_p) + \beta_p h(-p)$$

where $\alpha_p = 25, 17, 19, 11$ according as $p \equiv 1, 5, 7, 11 \pmod{12}$, and $\beta_p = 1/4$ $(p \equiv 1, 5 \pmod{12})$, $1/2$ $(p \equiv 7, 23 \pmod{24})$, 1 $(p \equiv 11, 19 \pmod{24})$.

We give a table of $\dim \mathfrak{S}_2^-(p)$ up to $p = 107$.

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
0	0	0	0	1	0	1	1	2	2	2	1	3	2	4

53	59	61	67	71	73	79	83	89	97	101	103	107
3	5	3	3	6	3	5	6	6	4	7	6	7

2. Let N be a natural number. Let $i, j \in \mathbb{Z}$. An Eisenstein series of weight $k \in \mathbb{N}$ is defined by

$$G_k(z, i, j; N) := \frac{\pi^k}{N^k} \sum_{c \equiv i, d \equiv j \pmod{N}} (cz+d)^{-k} |cz+d|^{-s} |_{s=0}$$

where the summation is taken over all $c, d \in \mathbb{Z}$ with $(c, d) \neq (0, 0)$ which satisfy the congruence. Obviously $G_k(z, i, j; N)$ is determined up to residue classes modulo N to which i, j belong, and further $G_k(z, -i, -j; N) = (-1)^k G_k(z, i, j; N)$. For $N > 1$, we define also other Eisenstein series

$$G_k(z; i; N) := \sum_{j=0}^{N-1} G_k(z, i, j; N),$$

$$G_2(z; N) := \frac{\pi^2}{2(1-N)} \{G_2(z, 0, 0; 1) - NG_2(Nz, 0, 0; 1)\}.$$

The Fourier expansions of $G_k(z, i, j; N)$ were given by Hecke [6]. From this, a simple calculation leads to the following;

Lemma 2. *Let* $N \in \mathbb{N}, > 1$. *Then*

$$G_1(z, 0, j; N) = \cot\frac{\pi j}{N} + 4\sum_{n=1}^{\infty}\left(\sum_{0<m|n} \sin\frac{2\pi jm}{N}\right)\mathrm{e}(nz) \qquad (j \not\equiv 0 \pmod N),$$

$$G_1(z, i, j; N) = -\sqrt{-1}\left(1 - \frac{2}{N}\langle i\rangle_N\right) - 2\sqrt{-1}\sum_{n=1}^{\infty}\left(\sum_{\substack{\mathbb{Z}\ni m|n \\ n/m \equiv i \pmod N}} \operatorname{sgn}(m)\mathrm{e}\left(\frac{jm}{N}\right)\right)\mathrm{e}\left(\frac{n}{N}z\right) \qquad (i \not\equiv 0 \pmod N),$$

$$G_1(z, i; N) = -\sqrt{-1}(N - 2\langle i\rangle_N) - 2\sqrt{-1}N\sum_{n=1}^{\infty}(\sum_{\substack{\mathbb{Z}\ni m|n \\ m \equiv i \pmod N}} \operatorname{sgn}(m))\mathrm{e}(z) \qquad (i \not\equiv 0 \pmod N),$$

$$G_2(z; N) = 1 + \frac{24}{N-1}\sum_{n=1}^{\infty}\left(\sigma_1(n) - N\sigma_1\left(\frac{n}{N}\right)\right)\mathrm{e}(nz).$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$G_k(z, i, j; N)|M = G_k(z, ai + cj, bi + dj; N),$$

from which it follows that $G_1(z, i, j; N)$ is in $\mathfrak{M}_1(\Gamma(N))$, and it is in $\mathfrak{M}_1(\Gamma_1(N))$ in particular if $i \equiv 0 \pmod N$. Further it follows that $G_1(z, i; N) \in \mathfrak{M}_1(\Gamma_1(N))$, and $G_2(z; N) \in \mathfrak{M}_2(\Gamma_0(N))$.

Let $N = p$, a prime. We note that

$$G_1(z, 0, j; p)|W_p = p^{-1/2}G_1(z, j; p).$$

We construct modular forms in $\mathfrak{S}_2^-(p)$ from Eisenstein series of weight 1 and $G_2(z; p)$. At first we consider the invariant elements under the action of $\Gamma_0(p)/\Gamma(p)$, of the vector subspace of $\mathfrak{M}_2(\Gamma(p))$ generated by products of two Eisenstein series of weight 1. It is easy to see that generators of the vector space are divided into three types;

$$\text{(I)} \qquad \sum_{i=1}^{p-1} G_1(z, 0, i; p)G_1(z, 0, id; p) \qquad (d \in \mathbb{Z}, p \nmid d),$$

$$\text{(II)} \qquad \sum_{i=1}^{p-1} G_1(z, 0, i; p)G_1(z, ci'; p) \qquad (c \in \mathbb{Z}, p \nmid c),$$

$$\text{(III)} \qquad \sum_{i=1}^{p-1}\sum_{j=0}^{p-1} G_1(z, i, j; p)\, G_1(z, ic, i'c + j; p) \qquad (c \in \mathbb{Z},\ p \nmid c),$$

where i' denotes the inverse of i modulo p. Obviously they are in $\mathfrak{M}_2(p)$, and modular forms of type (II) are in $\mathfrak{S}_2^+(p)$. In the present paper we consider only modular forms of type (I).

Proposition 1. *Let $p > 2$ be a prime. Let $\langle i \rangle_p (i \in \mathbb{Z})$ denote the integer such that $0 \le \langle i \rangle_p \le p-1$, $\langle i \rangle_p \equiv i \pmod p$. Let $\delta_{a,b}$ $(a, b \in \mathbb{Z})$ be the arithmetic function defined by*

$$\delta_{a,b} = \begin{cases} 1 & (a \equiv b \pmod p),\ p \nmid a,\ p \nmid b) \\ -1 & (a \equiv -b \pmod p),\ p \nmid a,\ p \nmid b). \\ 0 & (\textit{otherwise}) \end{cases}$$

Then $\sum_{i=1}^{p-1} G_1(z, 0, i; p)\, G_1(z, 0, id; p) (d \in \mathbb{Z}, p \nmid d)$ has the Fourier expansion

$$-p(p-1) + \frac{4}{p}\sum_{i=1}^{p-1} i\langle id \rangle_p + 8\sum_{n=1}^{\infty}\Big\{\sum_{\substack{0<m|n \\ p\nmid m}} (p - \langle dm \rangle_p - \langle d'm \rangle_p) + p\sum_{k=1}^{n-1}\sum_{\substack{0<l|k \\ 0<m|n-k}} \delta_{l,dm}\Big\}\, \mathrm{e}(nz),$$

d' being the inverse of d mod p.

Proof. We have

$$\begin{aligned} &G_1(z, 0, i; p)\, G_1(z, 0, id; p) \\ &= \cot\frac{\pi i}{p}\cot\frac{\pi i d}{p} + 4\sum_{n=1}^{\infty}\Big\{\cot\frac{\pi i}{p}\sum_{0<m|n} \sin\frac{2\pi i d m}{p} + \cot\frac{\pi i d}{p}\sum_{0<m|n}\sin\frac{2\pi i m}{p} \\ &\qquad + 4\sum_{k=1}^{n-1}\sum_{\substack{0<l|k \\ 0<m|n-k}} \sin\frac{2\pi i l}{p}\sin\frac{2\pi i d l}{p}\Big\}\, \mathrm{e}(nz). \end{aligned}$$

At first we compute the constant term. Since $\cot\frac{\pi i}{p} = -\frac{2}{p}\sum_{k=1}^{p-1} k \sin\frac{2\pi i k}{p}$, we have

$$\begin{aligned} \cot\frac{\pi i}{p}\cot\frac{\pi i d}{p} &= \frac{4}{p^2}\sum_{k,l=1}^{p-1} kl \sin\frac{2\pi i k}{p}\sin\frac{2\pi i d l}{p} \\ &= \frac{2}{p^2}\sum_{k,l=1}^{p-1} kl\Big\{\cos\frac{2\pi i(k-dl)}{p} - \cos\frac{2\pi i(k+dl)}{p}\Big\}. \end{aligned}$$

Taking the sum over i, we have

$$\begin{aligned} \sum_{i=1}^{p-1}\cot\frac{\pi i}{p}\cot\frac{\pi i d}{p} &= \frac{2}{p}\sum_{k,l=1}^{p-1} kl\delta_{k,l} \\ &= -p(p-1) + \frac{4}{p}\sum_{i=1}^{p-1} i\langle id \rangle_p, \end{aligned}$$

which gives the constant term.

Now we compute the higher terms. We have

$$\begin{aligned}
&\cot\frac{\pi i}{p}\sum_{0<m|n}\sin\frac{2\pi idm}{p}\\
&=-\frac{2}{p}\sum_{k=1}^{p-1}k\sin\frac{2\pi ik}{p}\sum_{0<m|n}\sin\frac{2\pi idm}{p}\\
&=-\frac{1}{p}\sum_{k=1}^{p-1}\sum_{0<m|n}k\left\{\cos\frac{2\pi i(k-dm)}{p}-\cos\frac{2\pi i(k+dm)}{p}\right\}.
\end{aligned}$$

Then the summation over i is equal to

$$\begin{aligned}
&-\sum_{0<m|n}\sum_{k=1}^{p-1}k\delta_{k,dm}\\
&=\sum_{0<m|n}(\langle -dm\rangle_p-\langle dm\rangle_p)\\
&=\sum_{\substack{0<m|n\\ p\nmid n}}(p-2\langle dm\rangle_p).
\end{aligned}$$

Similarly we have

$$\sum_{i=1}^{p-1}\cot\frac{\pi i}{p}\sum_{0<m|n}\sin\frac{2\pi idm}{p}=\sum_{\substack{0<m|n\\ p\nmid m}}(p-2\langle d'm\rangle_p).$$

Finally we have

$$\begin{aligned}
&\sum_{i=1}^{p-1}\sum_{k=1}^{n-1}\sum_{\substack{0<l|k\\ 0<m|n-k}}\sin\frac{2\pi il}{p}\sin\frac{2\pi idl}{p}\\
&=\frac{1}{2}\sum_{k=1}^{n-1}\sum_{\substack{0<l|k\\ 0<m|n-k}}\sum_{i=1}^{p-1}\left\{\cos\frac{2\pi i(k-dm)}{p}-\cos\frac{2\pi i(k+dm)}{p}\right\}\\
&=\frac{p}{2}\sum_{\substack{0<l|k\\ 0<m|n-k}}^{n-1}\delta_{l,dm}.
\end{aligned}$$

We are done. q.e.d.

Corollary 1. *Let $d\in\mathbb{Z}$, $p\nmid d$. Then $\sum_{i=1}^{p-1}G_1(z,0,i;p)G_1(z,0,id;p)$ is in $\mathfrak{M}_2^-(p)$.*

Proof. We must show that $\sum_{i=1}^{p-1}G_1(z,0,i;p)G_1(z,0,id;p)$ has -1 as the eigenvalue of W_p. Since $(\sum_{i=1}^{p-1}G_1(z,0,i;p)G_1(z,0,id;o))|W_p=\frac{1}{p}\sum_{i=1}^{p-1}G_1(z,i;p)$ $G_1(z,id;p)$, it is enough to show the equality

$$\sum_{i=1}^{p-1} G_1(z, 0, i; p) G_1(z, 0, id; p) = -\frac{1}{p}\sum_{i=1}^{p-1} G_1(z, i; p) G_1(z, id; p),$$

which is done by comparing the Fourier expansions. $G_1(z, i; p) G(z, id; p)$ has the expansion

$$\begin{aligned}&-(p-2\langle i\rangle_p)(p-2\langle id\rangle_p)\\ &-2p\sum_{n=1}^{\infty}\{(p-2\langle i\rangle_p)\sum_{\substack{\mathbb{Z}\ni m|n\\ m\equiv id(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m)+(p-2\langle id\rangle_p)\sum_{\substack{\mathbb{Z}\ni m|n\\ m\equiv i(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m)\\ &\qquad+2p\sum_{k=1}^{n-1}(\sum_{\substack{\mathbb{Z}\ni m|k\\ m\equiv i(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m))(\sum_{\substack{\mathbb{Z}\ni m|n-k\\ m\equiv id(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m))\}\,\mathrm{e}(nz).\end{aligned}$$

Then

$$-\sum_{i=1}^{p-1}(p-2\langle i\rangle_p)(p-2\langle id\rangle_p)=p^2(p-1)-4\sum_{i=1}^{p-1} i\langle id\rangle_p,$$

which verifies the above equality for the 0-th coefficients. Further we have

$$\begin{aligned}&\sum_{i=1}^{p}(p-2\langle i\rangle_p)\sum_{\substack{\mathbb{Z}\ni m|n\\ m\equiv id(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m)\\ &=p\sum_{i=1}^{p}\sum_{\substack{\mathbb{Z}\ni m|n\\ m\equiv id(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m)-2\sum_{i=1}^{p} i\sum_{\substack{\mathbb{Z}\ni m|n\\ m\equiv id(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m)\\ &=-2\sum_{0<m|n}\sum_{i=1}^{p-1} i\delta_{m,id}\\ &=2\sum_{\substack{0<m|n\\ p\nmid n}}(p-2\langle d'm\rangle_p),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{p}(p-2\langle id\rangle_p)\sum_{\substack{\mathbb{Z}\ni m|n\\ m\equiv i(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m)\\ &=2\sum_{\substack{0<m|n\\ p\nmid n}}(p-2\langle dm\rangle_p),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{p-1}(\sum_{\substack{\mathbb{Z}\ni m|k\\ m\equiv i(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m))(\sum_{\substack{\mathbb{Z}\ni m|n-k\\ m\equiv id(\mathrm{mod}\, p)}}\mathrm{sgn}\,(m))\\ &=\sum_{i=1}^{p-1}(\sum_{0<l|k}\delta_{l,i})(\sum_{0<m|n-k}\delta_{m,id})\end{aligned}$$

$$= 2 \sum_{\substack{0 < l | k \\ 0 < m | n-k}} \delta_{l,md}.$$

This shows that the corresponding higher terms are also equal, and our assertion follows. q.e.d.

As stated above, U_p and $-W_p$ are equal as operators on $\mathfrak{M}_2(p)$. Hence we have the following;

Corollary 2. *Let a_n denote the n-th Fourier coefficient of* $\sum_{i=1}^{p-1} G_1(z, 0, i; p) \times G_1(z, 0, id; p)$. *Then* $a_{pn} = a_n$.

3. Let $f \in \mathfrak{M}_2(p)$. By the residue theorem, if the 0-th Fourier coefficient of f at some cusp is zero, then the 0-th coefficient at the other cusp is also zero and hence it is a cusp form. The Eisenstein series $G_2(z; p)$ is obviously in $\mathfrak{M}_2^-(p)$. Then the difference of $\sum_{i=1}^{p-1} G_1(z, 0, i; p) G_1(z, 0, id; p)$ and $G_2(z; p)$ with 0 as its 0-th Fourier coefficient is in $\mathfrak{S}_2^-(p)$ by Corollary to Proposition 1.

Let g denote the genus of the curve $(H/\Gamma_0(p))^*$. Since the degree of the canonical divisor is $2g-2$ and since the number of the cusps of $(H/\Gamma_0(p))^*$ is 2, the number of zeros of a cusp form of weight 2 at cusps is at most $2g (= 2g - 2 + 2)$ including multiplicities. A cusp form in $\mathfrak{S}_2^-(p)$ has the same vanishing order at every cusp, and hence it vanishes if all the first g Fourier coefficients of its do. In other words the linear independence can be checked by examining the first g Fourier coefficients. By a direct computation, we obtain the following;

Theorem. *The cusp forms obtained above span $\mathfrak{S}_2^-(p)$ for prime $p < 1000$ except for $p = 389, 433, 563, 571, 643, 709, 997$.*

Remark. In the above exceptional cases, the cusp forms span a subspace of codimension one in $\mathfrak{S}_2^-(p)$ for $p \neq 997$, and that of codimension two for $p = 997$.

4. We give the table of basis of $\mathfrak{S}_2^-(p)$ for prime $p \leq 107$, $p = 163$ and the first thirty Fourier coefficients of them. We treat 163 in anticipation of some applications because $\mathbb{Q}(\sqrt{-163})$ is of class number one. Basis are in the form

$$a\left(\sum_{i=1}^{p} G_1(z, 0, i; p) G_1(z, 0, id; p) + bG_2(z; p)\right).$$

The indication of the table is as follows;

p | d | a | b |

the first thirty Fourier coefficients except the 0-*th coefficient*:

11	3	5/88	−6											
1	−2	−1	2	1	2	−2	0	−2	−2	1	−2	4	4	−1
−4	−2	4	0	2	2	−2	−1	0	−4	−8	5	−4	0	2

17	3	1/34	−20											
1	−1	0	−1	−2	0	4	3	−3	2	0	0	−2	−4	0
−1	1	3	−4	2	0	0	4	0	−1	2	0	−4	6	0
19	4	3/152	−22											
1	0	−2	−2	3	0	−1	0	1	0	3	4	−4	0	−6
4	−3	0	1	−6	2	0	0	0	4	0	4	2	6	0
23	3	11/184	−42											
3	−2	1	−1	−4	−8	2	1	6	10	−8	7	9	6	−16
−6	10	−4	−6	−6	−14	−2	3	15	5	−6	−1	−8	−9	18
—	4	11/184	−34											
4	1	−6	−5	2	−7	10	−6	8	6	−18	13	12	8	−14
3	6	2	−8	−8	−26	−10	4	20	−8	3	6	−18	−12	2
29	3	7/232	−72											
2	−1	1	0	−2	−4	2	−5	−2	1	3	7	0	6	−1
6	−6	−6	12	0	−6	2	−8	−6	−8	7	3	−14	2	4
—	5	7/232	−52											
3	2	−2	−7	−3	1	10	−4	−10	−2	8	0	7	2	2
9	−16	−2	18	7	−2	3	−26	5	−12	0	8	−14	3	−1
31	4	5/248	−66											
2	1	−2	−1	2	−6	−4	0	6	1	4	−4	−2	3	−2
−3	6	13	0	−1	−6	2	−2	10	−8	−6	−20	7	10	−6
—	12	5/248	2											
1	−2	4	−3	1	2	−7	5	−7	−2	2	−2	4	4	4
6	8	−6	5	−3	−8	−4	−16	0	−4	2	0	11	10	2
37	5	3/296	−56											
1	0	1	−2	0	0	−1	0	−2	0	3	−2	−4	0	0
4	6	0	2	0	−1	0	6	0	−5	0	−5	2	−6	0
41	3	5/164	−156											
3	−1	2	1	−2	−4	4	−5	−1	−6	−4	−6	2	2	2
13	−6	7	10	6	−4	−2	12	0	−7	6	−10	8	−10	6
—	6	1/164	−100											
1	1	0	−1	−2	−2	2	1	1	−2	−2	−4	2	4	2
−1	−2	1	4	−2	−4	4	4	2	−1	−2	−2	2	−2	6
—	12	5/164	−12											
1	3	−6	7	−4	−8	8	−5	3	−12	12	−2	−6	14	14
−9	−2	9	−10	−8	−18	6	−16	0	11	2	10	16	10	22
43	4	7/344	−134											
3	2	−2	0	4	−6	−4	−4	−3	−2	1	0	7	2	2
−12	19	−2	−10	0	−2	10	−5	12	−5	14	8	0	6	−8
—	5	7/344	−82											
1	3	−3	0	−1	−2	1	−6	−1	4	5	0	7	−4	−4
−4	11	−3	−8	0	4	1	−11	4	−11	7	12	0	9	2
47	3	23/376	−210											
7	−2	5	3	−4	−8	10	−10	−1	−12	−8	−11	6	−16	−16
10	11	20	2	18	17	22	−16	−17	33	18	−19	24	−20	−2
—	4	23/376	−162											

10 7 −6 1 14 −18 −12 −11 −8 −4 −18 −19 48 −13 10
11 −4 22 16 −40 −2 −8 10 −21 34 52 −14 54 −22 30
— | 5 | 23/376 | −102 |
8 1 9 −13 2 4 −5 −18 −11 6 4 −6 20 −15 −38
18 −17 −10 22 14 26 12 8 −3 64 14 −2 11 −36 −22
— | 6 | 23/376 | −130 |
12 13 2 −8 −20 −17 4 −4 18 −14 6 −55 30 12 −34
27 −14 31 10 −2 16 −28 12 −39 96 44 −26 5 −54 36
53 | 3 | 13/424 | −272 |
4 −1 3 2 −2 −4 6 −5 0 −6 −4 −5 4 −8 −8
−6 −6 0 15 12 −2 14 5 6 −6 −1 17 16 −10 2
— | 5 | 13/424 | −136 |
2 6 −5 1 −1 −2 3 −9 −13 −3 11 4 2 9 9
−16 10 −13 14 −7 −1 7 −4 3 −16 6 2 −5 −31 1
— | 6 | 13/424 | −164 |
7 8 2 −3 −10 −7 4 −12 0 −4 6 −12 7 12 −14
−17 9 −26 36 8 −10 18 −14 17 −17 8 46 2 −37 10
59 | 3 | 29/472 | −342 |
9 −2 7 5 −4 −8 14 −10 1 −12 −8 −9 10 −16 −16
−11 −12 −26 6 1 27 34 −16 18 18 30 −17 11 38 10
— | 4 | 29/472 | −262 |
13 10 −6 4 20 −18 −12 −8 −5 2 −18 −42 8 −36 −7
26 31 14 28 −34 39 4 −36 −32 55 24 −31 −26 −16 8
— | 5 | 29/472 | −222 |
15 16 2 −11 3 6 4 −36 −37 −20 6 −15 36 12 −17
1 38 −24 −19 −37 16 18 −46 −28 117 50 −38 −1 −14 −22
— | 6 | 29/472 | −202 |
16 19 6 −4 −20 −11 12 −21 −24 −31 18 −16 50 7 −22
−26 −2 −14 −28 −24 48 54 −22 −55 90 92 −56 −32 16 −37
— | 25 | 29/472 | 42 |
5 −14 20 −23 30 2 11 −12 −22 32 −56 53 −46 4 4
−19 32 −66 42 −51 −14 6 −54 68 39 −22 26 77 5 12
61 | 7 | 5/488 | −136 |
1 3 −1 2 −4 −3 −2 0 3 −2 7 −7 −1 −1 −1
1 3 −1 5 −3 −3 16 −1 −5 −4 −13 10 1 5 7
— | 8 | 5/488 | −112 |
2 1 3 −1 −3 −1 −4 0 6 −4 9 −9 −2 −2 −7
2 1 3 −5 4 −11 7 3 −5 2 −1 10 7 5 −1
— | 17 | 5/488 | −32 |
2 1 −2 4 2 −6 1 5 −4 −9 9 −4 −12 −2 8
−8 −4 3 10 −6 4 12 3 0 2 −1 0 −3 0 4
67 | 4 | 11/536 | −342 |
5 4 −2 2 8 −6 −4 −2 −1 2 −6 −14 4 −12 −12
−10 13 8 1 12 6 −18 21 −8 1 12 2 −6 7 −14
— | 5 | 11/536 | −234 |
4 1 5 −5 2 4 −1 −6 −3 −5 4 −9 1 −3 −3

3	6	2	−19	3	−4	1	8	−13	−8	3	−16	4	32	2
—	7	11/536	−130											
1	3	4	−4	−5	1	−3	−7	2	7	1	−5	3	2	2
9	−4	−5	−2	−2	−1	3	13	−6	−24	−2	−15	1	19	−5
71	3	35/568	−506											
11	−2	9	7	−4	−8	18	−10	3	−12	−8	−7	14	−16	−16
−9	−12	−26	10	−28	2	−24	−16	−5	26	42	−15	56	15	22
—	4	35/568	−386											
16	13	−6	7	26	−18	−12	−5	−2	8	−18	−42	14	−36	−36
−29	8	−6	40	7	22	16	−36	15	76	−28	10	−14	60	−73
—	6	7/568	−290											
4	5	2	0	−4	−1	4	−3	−4	−12	−8	−7	14	12	−2
−9	2	−5	10	−7	−12	−10	−2	−5	12	14	6	14	8	−6
—	8	5/568	−266											
3	4	2	1	−2	−4	−6	−5	4	4	6	−6	2	−8	−13
−7	4	2	15	6	−4	−2	−8	−10	3	6	10	8	0	−14
—	16	35/568	−66											
6	18	−11	7	1	2	−22	−15	8	3	2	−42	14	4	−31
−24	38	−11	50	42	−18	6	4	10	11	−28	30	−14	5	−58
—	21	35/568	2											
3	9	−23	21	18	−34	24	10	−31	19	−34	14	−28	2	2
−47	54	−23	−10	−14	96	−32	2	40	58	−14	15	−42	55	−29
73	5	3/292	−284											
1	3	−2	1	0	0	2	−3	−5	0	0	−2	−4	0	0
1	6	−3	8	−6	2	6	6	−6	1	−6	4	−4	−6	6
—	7	1/292	−204											
1	1	0	1	0	−2	0	1	−1	−2	2	−2	−2	0	2
−1	0	−3	−2	−4	0	4	6	0	−1	0	2	−2	0	2
—	16	3/584	−16											
1	0	1	−2	3	0	2	−3	−2	3	−3	1	−7	3	0
−2	0	−3	8	0	−1	−6	3	3	−2	−6	−2	−1	6	−3
79	4	13/632	−482											
6	5	−2	3	10	−6	−4	−1	0	4	−6	−14	6	−12	−12
−9	4	−13	−10	5	10	−5	14	−4	4	31	6	−28	24	−10
—	5	13/632	−402											
7	8	2	−3	3	6	4	−12	−13	−4	6	−12	−6	−14	−14
−4	22	0	−3	−5	16	18	−1	−22	9	8	−32	−24	28	−16
—	7	13/632	−198											
5	2	7	−4	4	−5	1	−3	0	−1	−5	−3	−8	3	3
−1	12	−13	−17	−11	−9	11	3	1	−1	2	5	−19	20	−17
—	8	13/632	−322											
8	11	6	4	−4	−8	−14	−10	0	1	18	−10	8	−16	−16
−25	14	−13	4	11	−4	28	−16	−14	−12	24	8	−20	32	−48
—	29	13/632	−46											
3	−4	12	−5	−8	10	−2	−7	13	−24	23	6	−23	20	−6
−11	28	−26	8	−4	−34	30	−32	−2	2	−30	−10	12	−14	8

83 | 3 | 41/664 | −702 |
13 −2 11 9 −4 −8 22 −10 5 −12 −8 −5 18 −16 −16
−7 −12 −26 14 −28 6 −24 −16 −40 −7 −28 −13 31 21 34
— | 4 | 41/664 | −534 |
19 16 −6 10 32 −18 −12 −2 1 14 −18 −42 20 −36 −36
−26 14 −38 −30 −22 −7 28 5 −8 97 −22 63 −2 −4 −26
— | 5 | 41/664 | −378 |
7 21 −13 8 1 2 15 −18 −32 3 2 −9 −25 4 −37
−29 44 −14 17 7 60 6 4 −31 53 7 −7 −59 46 −29
— | 6 | 41/664 | −394 |
24 31 14 4 −20 1 28 −9 −16 −60 −40 −66 8 43 2
6 22 −48 70 −58 30 3 −80 −77 6 24 −24 −50 146 6
— | 7 | 41/664 | −366 |
25 34 18 11 −14 −28 −5 6 −3 −42 −28 −79 22 26 26
37 −1 −50 8 −98 21 −2 −56 −58 37 66 16 −35 94 −4
— | 9 | 41/664 | −126 |
16 7 23 −11 14 −13 5 −47 3 42 −13 −3 −22 15 −26
−37 1 −32 −8 16 20 −39 15 17 4 −66 −16 35 29 −78
89 | 3 | 11/356 | −812 |
7 −1 6 5 −2 −4 12 −5 3 −6 −4 −2 10 −8 −8
−3 −6 −13 8 −14 4 −12 −8 −20 −3 −14 −6 −4 −10 −2
— | 5 | 11/712 | −512 |
6 7 2 −2 3 6 4 −9 −10 −2 6 −8 −4 −10 −10
−12 −2 −8 10 −1 5 18 12 −14 −1 10 −2 −16 −18 −8
— | 6 | 11/356 | −452 |
13 17 8 3 −10 2 16 −3 −7 −30 −20 −32 6 4 −18
7 14 1 18 −26 20 −16 26 −78 7 18 −8 −20 −50 12
— | 7 | 11/356 | −316 |
5 15 −2 13 −14 −6 −4 9 −1 −20 −6 −36 4 10 10
23 2 −3 12 −32 6 −18 32 −52 1 12 2 16 −4 30
— | 9 | 11/712 | −392 |
7 10 6 5 −2 −4 −10 −16 −8 5 18 −2 10 −8 −8
−25 −17 −24 8 8 15 32 14 −20 −25 −14 −6 −4 12 −2
— | 24 | 11/356 | −76 |
9 5 −8 19 10 −24 6 25 −15 −14 20 −12 −6 −26 18
15 −58 −23 48 −62 −20 16 −4 −32 15 −18 30 −2 6 54
97 | 5 | 1/97 | −532 |
3 1 4 −3 2 4 0 −3 −1 −2 4 −4 2 −8 0
−3 −2 −3 −4 −2 0 4 8 −4 −11 14 −8 −8 2 0
— | 7 | 1/97 | −492 |
5 7 4 3 −2 −4 0 3 1 −6 −4 −20 −10 0 8
11 10 −5 −4 −14 0 4 16 −20 −21 −14 −8 0 22 8
— | 9 | 1/194 | −312 |
1 3 0 3 −2 0 −4 −1 −3 −2 4 0 6 −4 0
−5 −2 −5 4 −2 −4 8 8 −4 −5 6 −4 0 6 0
— | 10 | 1/97 | −220 |

1 3 4 −1 −2 4 −8 −9 −3 2 4 4 14 −8 0
−9 −6 −9 4 2 0 20 0 4 −9 26 −16 −8 6 −8
101 | 3 | 25/808 | −1056 |
8 −1 7 6 −2 −4 14 −5 4 −6 −4 −1 12 −8 −8
−2 −6 −13 10 −14 6 −12 −8 −20 −2 −14 −5 −2 −10 −24
— | 6 | 5/808 | −580 |
3 4 2 1 −2 1 4 0 −1 −6 −4 −6 2 2 −8
−7 −1 2 10 −4 1 −2 2 −5 8 −4 0 −2 0 −14
— | 7 | 25/808 | −416 |
13 14 2 16 3 −19 4 20 −6 −16 6 −36 −18 −13 12
28 −16 −18 35 −54 −9 18 −13 −45 3 −54 20 −22 −10 11
— | 8 | 25/808 | −352 |
11 8 19 2 −9 7 −12 −10 18 −27 7 −17 4 −11 −11
16 −2 4 45 −38 −23 −4 −11 −65 41 12 −10 16 −20 −33
— | 9 | 5/808 | −220 |
2 1 3 −1 2 −1 1 −5 1 1 −1 1 −2 3 −2
−3 −4 −2 5 −1 −6 −3 3 −5 2 −1 0 7 0 −6
— | 12 | 25/808 | −192 |
6 18 −1 17 −14 −3 −2 −10 3 −17 −3 −7 −16 19 −31
11 8 −16 45 −23 −8 −9 19 −65 36 2 40 −14 30 −18
— | 22 | 25/808 | −116 |
13 14 2 −9 28 6 4 −30 −6 34 6 −36 −43 37 12
−47 −66 −18 60 −4 −84 −32 62 30 3 −54 20 78 40 −39
103 | 4 | 17/824 | −834 |
8 7 −2 5 14 −6 −4 1 2 8 −6 −14 10 −12 −12
−7 8 −11 −10 −4 −16 −18 −12 −30 10 13 14 −11 36 −2
— | 5 | 17/824 | −606 |
3 9 −5 4 1 2 7 −6 −12 3 2 −1 −9 4 −13
−9 3 −19 −8 −10 11 23 4 −7 25 −10 1 −2 22 −5
— | 7 | 17/824 | −434 |
4 12 −1 11 −10 −3 −2 9 1 −13 −3 −24 −12 −6 −6
22 21 3 −5 −19 9 8 28 −49 5 −19 7 −14 1 −1
— | 8 | 17/824 | −522 |
11 16 10 9 −2 −4 −14 −5 7 −6 −4 −32 1 −8 −8
1 28 4 −1 −14 −22 −12 9 −54 1 −14 32 4 24 −24
— | 10 | 17/824 | −254 |
9 10 2 12 3 −11 4 −1 −19 −8 23 −3 −10 12 −5
−27 9 −23 −24 −13 16 1 −5 −4 24 −30 −14 −6 32 −32
— | 19 | 17/824 | −122 |
7 4 11 −2 −9 16 5 −14 6 −27 16 9 −21 15 −19
−38 41 −16 4 −12 −31 31 32 −5 13 −46 −9 35 6 −23
107 | 3 | 53/856 | −1190 |
17 −2 15 13 −4 −8 30 −10 9 −12 −8 −1 26 −16 −16
−3 −12 −26 22 −28 14 −24 −16 −40 −3 −28 −9 −2 −20 −48
— | 4 | 53/856 | −902 |
25 22 −6 16 44 −18 −12 4 7 26 −18 −42 32 −36 −36

−20	26	−32	−30	−10	−48	−54	−36	−90	33	−10	−7	22	61	−2
—	5	53/856	−658											
20	7	27	−19	14	28	1	−18	−5	−11	28	−23	15	−50	3
−16	−11	−15	−24	−61	4	31	3	−19	37	45	−101	7	−36	9
—	6	53/856	−650											
32	43	22	12	−20	13	44	3	−8	−60	−40	−58	24	26	−80
−68	−60	−24	110	−34	70	−67	26	−41	38	125	−98	−10	6	−134
—	7	53/856	−410											
21	10	31	−12	20	−13	9	−3	8	7	−13	−48	−24	−26	27
15	7	−29	−4	−19	36	14	−79	−12	−38	34	−61	−43	−6	−25
—	8	53/856	−346											
11	33	−9	24	13	−27	−18	6	−16	39	−27	−63	−5	−1	−1
−30	39	5	8	38	−19	−81	−54	−29	−30	38	16	33	118	−56
—	9	53/856	−542											
35	52	34	33	−2	−4	−38	−58	−22	−6	−4	−27	66	−8	−8
−81	−6	−66	11	−14	−46	−12	45	−126	25	92	−84	52	96	−130
163	4	27/1304	−2134											
13	12	−2	10	24	−6	−4	6	7	18	−6	−14	20	−12	−12
−2	18	−6	−10	6	−16	−18	−12	−30	25	6	−20	−28	12	−36
—	5	27/1304	−1610											
5	15	−7	8	3	6	13	−6	−16	9	6	5	−11	12	−15
−17	9	−21	−8	−6	−2	18	−15	−24	−7	−6	−16	−17	−12	−18
—	7	27/1304	−1058											
11	6	17	−4	12	−3	7	3	8	9	−3	−16	−8	−6	21
−10	9	−30	4	−24	1	−9	−6	12	−10	3	−19	−5	33	9
—	8	9/1304	−906											
2	6	−1	5	3	−3	−2	3	−1	9	−3	−7	1	−6	−6
−1	0	−3	−5	3	1	0	−6	−6	8	12	−1	−5	−3	−18
—	10	9/1304	−726											
5	6	2	8	3	−3	4	3	−7	0	6	−4	−2	3	−6
−7	9	−12	−8	3	7	0	3	3	−7	−6	−7	−17	−3	−18
—	11	27/1304	−794											
8	24	5	29	−6	15	−17	12	−31	−18	−12	8	4	−24	3
5	36	−39	25	−42	−14	−9	3	−6	5	12	−31	−38	24	−18
—	12	9/1304	−570											
4	3	7	1	6	3	−4	−3	−2	0	−6	−5	2	−3	15
−2	9	−6	8	−12	−7	−9	−3	−3	7	−3	−11	−1	12	9

We give the n-th Fourier coefficients of basis of $\mathfrak{S}_2^-(p)$ for prime $p \leq 107$, $p = 163$ where n is a prime between 30 and 100, namely $n = 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$.

11	3	5/88	−6											
7	3	−8	−6	8	−6	5	12	−7	−3	4	−10	−6	15	−7
17	3	1/34	−20											
4	−2	−6	4	0	6	−12	−10	4	−4	−6	12	−4	10	2
19	4	3/152	−22											

−4	2	−6	−1	−3	12	−6	−1	−4	6	−7	8	12	12	8
23	3	11/184	−42											
−3	4	5	0	1	−14	4	10	−16	29	35	−2	−34	−16	30
—	4	11/184	−34											
18	−2	−8	0	−6	−4	20	−16	−14	46	32	−32	−38	−36	62
29	3	7/232	−72											
1	−8	14	11	5	−4	8	−2	−4	−10	8	−1	0	−2	−14
—	5	7/232	−52											
−16	−12	42	20	18	−27	26	4	−20	−8	12	2	−14	18	−42
31	4	5/248	−66											
2	−4	14	−2	−4	−12	0	−6	16	4	8	−10	−12	10	−14
—	12	5/248	2											
1	−2	7	−6	−12	4	−5	−28	8	27	−6	10	14	−10	13
37	5	3/296	−56											
−4	1	−9	8	3	−3	12	8	−4	−15	11	−10	9	6	8
41	3	5/164	−156											
16	−6	3	−8	−6	2	0	6	4	26	−18	30	−8	10	14
—	6	1/164	−100											
8	−6	1	−4	−8	2	0	6	2	12	−10	8	0	6	2
—	12	5/164	−12											
12	−12	1	4	−2	14	−20	2	−12	2	24	10	24	−30	−22
43	4	7/344	−134											
−9	−12	−7	3	18	29	−2	18	15	−22	−30	2	35	−24	−7
—	5	5/344	−82											
−3	−18	−7	1	6	5	4	13	19	−12	−3	−4	21	−15	−7
47	3	23/376	−210											
−6	13	18	−14	7	10	6	−26	16	−25	58	31	36	−14	42
—	4	23/376	−162											
−48	12	6	26	10	−58	−44	22	82	−62	50	18	12	−66	106
—	5	23/376	−102											
−20	5	14	−16	8	−5	−49	−33	38	−45	86	19	28	−85	25
—	6	23/376	−130											
−30	42	−2	22	12	−42	−16	−38	−12	−10	60	86	88	−70	26
53	3	13/424	−272											
−2	−4	−14	20	−16	4	6	−12	12	−11	10	−5	37	−4	2
—	5	13/424	−136											
25	37	−20	−29	−34	2	16	−19	45	−51	31	17	38	24	1
—	6	13/424	−164											
42	45	−44	−4	−54	7	4	−60	60	−68	50	−12	94	58	−29
59	3	29/472	−342											
−2	52	1	−10	−32	−7	9	36	−26	10	−30	−5	2	56	−46
—	4	29/472	−262											
68	30	−34	−8	−72	6	13	110	−44	−21	−24	54	−10	10	−118
—	5	29/472	−222											
74	48	−66	22	−92	−2	15	118	−24	36	−108	−18	−132	74	−96
—	6	29/472	−202											

48 86 −82 8 −102 −6 16 122 −14 50 −92 −54 −106 106 −56
— | 25 | 29/472 | 42 |
44 16 94 −12 8 38 5 20 −124 −46 −36 52 14 44 −148
61 | 7 | 5/488 | −136 |
−13 3 −3 −1 8 −6 0 1 −12 17 −6 −5 4 0 18
— | 8 | 5/488 | −112 |
−1 −9 −6 −7 6 −2 20 2 6 9 −7 15 −12 −10 16
— | 17 | 5/488 | −32 |
−6 6 14 −12 −14 −2 15 2 1 14 8 −5 8 10 −4
67 | 4 | 11/536 | −342 |
−16 3 12 −22 −13 28 3 −20 5 −14 7 −40 24 −1 6
— | 5 | 11/536 | −234 |
18 9 25 −33 −17 18 −24 −27 4 −42 32 −43 17 −14 18
— | 7 | 11/356 | −130 |
21 5 20 −22 −7 21 −6 −15 1 −49 8 −30 −4 13 21
71 | 3 | 35/568 | −506 |
2 −2 42 −6 −32 −36 −40 52 −22 11 44 −30 14 45 28
— | 4 | 35/568 | −386 |
22 −22 −28 4 −2 −46 −90 12 −32 16 134 20 −56 40 98
— | 6 | 7/568 | −290 |
−12 12 14 −20 −18 −8 −26 −4 6 4 16 −16 14 24 0
— | 8 | 5/568 | −266 |
−4 9 −4 −8 −6 −18 −20 6 −6 3 22 0 7 20 14
— | 16 | 35/568 | −66 |
−18 18 −28 −51 8 −26 −60 −48 −12 6 59 25 14 15 28
— | 21 | 35/568 | 2 |
26 −96 −14 −8 74 −48 40 −94 −6 3 82 −40 −98 −115 14
73 | 5 | 3/292 | −284 |
2 −10 6 −10 6 18 −6 −16 −4 −6 1 2 −24 6 −16
— | 7 | 1/292 | −204 |
2 0 −2 0 8 4 −2 −8 2 0 1 0 −4 4 −6
— | 16 | 3/584 | −16 |
−4 −4 6 2 6 6 −6 −13 14 3 1 −7 −9 −12 −7
79 | 4 | 13/632 | −482 |
10 −6 44 −22 2 −14 −4 −18 −34 16 −24 6 −16 46 −10
— | 5 | 13/632 | −402 |
−10 6 34 −30 50 14 −22 −8 −57 −16 −15 7 −36 58 −3
— | 7 | 13/632 | −198 |
−9 8 28 −14 19 10 1 24 −11 −17 −7 5 −22 34 −30
— | 8 | 13/632 | −322 |
22 −34 50 −12 20 42 −40 −24 −80 4 −32 8 −56 96 4
— | 29 | 13/632 | −46 |
−21 10 −4 2 40 58 −2 30 −4 −44 14 3 −60 49 8
83 | 3 | 41/664 | −702 |
6 84 −28 −2 −32 46 −40 27 −18 −48 −22 −26 13 −60 44
— | 4 | 41/664 | −534 |

34 148 19 16 -72 -40 -8 -52 62 -108 -70 -38 19 -94 58
— | 5 | 41/664 | -378 |
-22 61 -34 62 -115 -32 -31 24 -16 12 -97 -55 7 -108 30
— | 6 | 41/664 | -394 |
30 174 -58 72 -78 -16 -118 176 -172 -76 -28 -130 24 -300 220
— | 7 | 41/664 | -366 |
62 212 -16 34 -30 -44 -99 156 -104 -86 46 -132 25 -292 236
— | 9 | 41/664 | -126 |
61 157 -25 48 -52 -38 17 49 -60 -78 -46 -32 16 -118 174
89 | 3 | 11/356 | -812 |
26 2 -14 0 28 -18 -20 -6 36 20 -10 32 -28 7 -18
— | 5 | 11/712 | -512 |
38 30 -12 22 2 -6 -58 -2 12 -30 -7 -4 -46 6 -28
— | 6 | 11/356 | -452 |
64 54 -26 22 52 -46 -100 -30 48 12 38 -16 -96 13 -134
— | 7 | 11/356 | -316 |
28 14 12 22 -2 -16 -52 -20 10 8 18 4 -64 5 -82
— | 9 | 11/712 | -392 |
48 24 8 44 -16 15 -86 -6 36 -24 -32 10 -72 7 -29
— | 24 | 11/356 | -76 |
24 -54 26 0 -52 2 -10 30 -48 -12 -82 148 -58 9 46
97 | 5 | 1/97 | -532 |
0 -6 14 12 24 10 12 2 -28 16 -34 8 4 -34 3
— | 7 | 1/97 | -492 |
16 -18 2 12 32 14 20 6 -20 32 -30 0 44 -62 5
— | 9 | 1/194 | -312 |
-4 6 -18 0 0 6 12 -14 -8 12 -10 8 12 -18 1
— | 10 | 1/97 | -220 |
0 6 -14 4 8 22 28 -10 -28 24 -30 32 12 -38 1
101 | 3 | 25/808 | -1056 |
6 4 -14 52 -16 -18 30 -4 19 26 -8 -10 22 -30 -16
— | 6 | 5/808 | -580 |
-4 -1 -14 22 -6 2 0 6 14 6 -8 10 2 -20 14
— | 7 | 25/808 | -416 |
41 19 -54 72 -1 -48 30 56 34 11 -38 15 -58 -30 -26
— | 8 | 25/808 | -352 |
-23 -57 12 84 3 44 10 -18 73 -33 -36 55 -1 -60 28
— | 9 | 5/808 | -220 |
-1 1 -6 8 1 -2 5 4 11 9 -2 5 -7 -10 1
— | 12 | 25/808 | -192 |
-33 3 -48 14 -37 24 -40 22 83 7 -6 105 -71 -60 63
— | 22 | 25/808 | -116 |
16 44 -54 22 24 2 -20 56 84 86 12 40 -158 -80 -26
103 | 4 | 17/824 | -834 |
-16 32 30 -22 44 24 -30 -14 -34 -2 -20 -40 -8 -28 -32
— | 5 | 17/824 | -606 |

11	29	24	-21	25	-8	-7	-18	-17	29	-33	19	-20	-70	-12
—	7	17/824	-434											
9	-1	32	-11	-12	-5	19	-24	-17	-18	-10	14	-4	-48	1
—	8	17/824	-522											
-22	10	71	-26	18	16	14	-32	-34	-58	-2	-21	6	-64	-44
—	10	17/824	-254											
-18	36	55	22	7	27	-21	-37	0	-15	-31	-28	-43	-40	32
—	19	17/824	-122											
-31	-23	22	-15	-21	4	63	26	17	-57	-9	33	-24	-84	-28
107	3	53/856	-1190											
14	63	25	6	-32	70	-40	100	-10	-48	92	-18	-56	-60	-30
—	4	53/856	-902											
58	102	-10	146	-19	-28	-90	66	-102	-108	154	-120	33	-82	12
—	5	53/856	-658											
57	71	-61	32	59	73	-72	180	-18	-44	49	-43	-175	104	-107
—	6	53/856	-650											
70	50	72	30	-54	138	12	288	-50	-240	354	-90	-280	-88	-150
—	7	53/856	-410											
89	109	-19	76	1	127	-118	136	50	28	123	-16	-197	35	-9
—	8	53/856	-346											
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—	9	53/856	-542											
60	58	-14	56	90	35	-20	209	-270	-236	364	-168	-134	-30	-174
163	4	27/1304	-2134											
-16	8	33	5	57	27	24	23	-34	-9	-10	14	39	36	59
—	5	27/1304	-1610											
-2	55	48	4	-3	27	30	40	-11	9	-35	-86	-12	72	-23
—	7	27/1304	-1058											
-26	-14	-24	-29	15	27	66	34	-89	63	31	16	60	18	52
—	8	9/1304	-906											
10	-5	3	7	15	9	3	25	-8	0	-5	-2	6	9	16
—	10	9/1304	-726											
7	1	12	-14	-12	27	12	31	-20	9	1	-32	6	27	4
—	11	27/1304	-794											
13	7	12	-53	60	0	-60	118	31	36	-83	-89	-84	72	82
—	12	9/1304	-570											
-16	-1	-3	-4	12	-9	6	14	-25	36	-10	14	12	18	14

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