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Games of Number Structures II
Reversed Difference

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Introduction

I proposed several materials for Clinical Mathematics Education in [1] such as dynamical graphs (representing cause and effect), strategy games (equivalence relations generated by simple basic relations), and various inverse problems in arithmetics (techiniques, skills, arts and structures in the world of numbers). I also developped in [2] the theory of dynamical graphs in the case of reduced divisor sums.

In this note, I will give a brief review of a theory of dynamical graphs (see [4] for details), and a detailed account in the case of Reversed Difference as an example.

§1. A Review of Dynamical Graphs

A graph $G = (V, E)$ is called *dynamical* (or simply a *dynagraph*), if the set of vertices $V = \{v\}$ is an at most countable (*i.e.* finite or countable) set, and the set of (oriented) edges $E \subset V \times V$ satisfies the following condition:

For any $v \in V$, there exists one and only one vertex $w \in V$ with $e = (v, w) \in E$.

An element v of V is called a *vertex*, and $e = (v, w)$ of E is called an *(oriented) edge* or *arrow*, where v is called a *source*, and w is called a *target* of the arrow e.

In a word, a dynamical graph is nothing but an at most countable oriented graph whose any vertex v has only one outgoing arrow from v.

Proposition 1 *The set $\mathcal{D}(V)$ of dyanamical graphs on V is bijective to the set $Map(V,V)$ of the maps of V to itself. The correspondence is given as follows.*

Given $f \in Map(V, V)$, take the set $E = \{(v, f(v)) \mid v \in V\}$ of pairs as the graph of f, then $G(f) = (V, E(f))$ is a dynamical graph.

Conversely, given a dynamical graph $G = (V, E)$, for any $v \in V$ we have only one vertex $w \in V$ with $(v, w) \in E$. So let $f(v) = w$. Denoting f by $f(G)$, we get that $G = G(f(G))$ and $f = f(G(f))$.

The mapping $f: V \to V$ gives a dynamical system on the discrete space V with discrete times:

$$\bar{f}: V \times \mathbb{N} \to V, \quad (v, n) \mapsto f^n(v),$$

where $\mathbb{N}$ denotes the set of all natural numbers.

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Two mappings $f, g: V \to V$ are called *isomorphic*, if there exists a bijection $\varphi: V \to V$ (called an *isomorphism*) satisfying then equality

$$\varphi(f^n(v)) = g^n(\varphi(v)), \qquad (\forall v \in V,\ n \in \mathbb{N}),$$

that is, the following diagram commutes:

$$\begin{array}{ccc} V \times \mathbb{N} & \xrightarrow{\bar{f}} & V \\ \downarrow{\scriptstyle \varphi \times id} & & \downarrow{\scriptstyle \varphi} \\ V \times \mathbb{N} & \xrightarrow{\bar{g}} & V \end{array}$$

This condition is equivalent with the single equality

$$\varphi \circ f = g \circ \varphi.$$

Isomorphic mappings are denoted by $f \cong g$, and the dynamical graphs $G(f)$ and $G(g)$ corresponding to isomorphic mappings $f, g: V \to V$ are called *isomorphic* and denoted by $G(f) \cong G(g)$.

Remark 1. The notion of equivalences of dynamical graphs can be weakened as an isomorphism of unoriented graphs, and be strengthened so that φ is an isomorhism only if $\varphi \circ f = f \circ \varphi$. We call the former a *weak* isomorphism, and the latter an *automorphism.*

Remark 2. An at most countable (unoriented) graph $G = (V, E)$ is called *dynamicalizable*, if there exists a suitable assignment of the directions of edges which makes G dynamical. The resulting dynamical graph $\overline{G} = (V, \overline{E})$ is called a *dynamicalization* of the graph G. Note that dynamicalizations are, in general, not unique.

The degree of a vertex of a graph is usualy defined as a number of connecting edges to this vertex. In the case of dynamical graphs, there exists only one outgoing edge for every vertex v, so we will define it the number of incoming edges of the vertex v, or of arrows with v as a target, *i.e.* $\deg(v) = |\{w \in V \mid (w, v) \in E\}|$. Note, for each vertex $v \in V$, its degree as of a graph G is larger by 1 than the degree as the dynanamicalization $\overline{G}$ of G, *i.e.*

$$\deg_G(v) = \deg_{\overline{G}}(v) + 1.$$

Now, we give a few definitions about dynamical graphs. Let $G = (V, E)$ and $G' = (V', E')$ be dynamical graphs. G' is called a *dynamical subgraph* (or simply *subgraph*) of G, if $V' \subset V$ and $E' \subset E$.

As in Proposition 1, the set of dynamical subgraphs of $G = (V, E)$ is bijective to the set of invariant subsets of V under the mapping $f = f(G)$. For an f-invariant subset $W \subset V$, the dynamical graph $(W, E(f \mid W))$ is a subgraph of G, and vice versa.

For a vertex $v \in V$, the set

$$V^+(v) = \{w \in V \mid w = f^a(v) \text{ for some } a \geq 0\}$$

is called the *future* of v. For a subset $U \subset V$, $V^+(U) = \bigcup_{v \in U} V^+(v)$ is called the future of U.

For a vertex $v \in V$, the set

$$V^-(v) = \{w \in V | \, v = f^a(w) \text{ for some } a \geq 0\}$$

is called the *past* of v. For a subset $U \subset V$, $V^-(U) = \bigcup_{v \in U} V^-(v)$ is called the past of U.

A vertex $v \in V$ has a *life* n (and denoted by $\ell(v) = n$), if there exists a natural number n such that $v \in f^a(V)\,(0 \leq a \leq n-1)$ and $v \notin f^n(V)$. If such number n does not exist, then such vertex has an *infinite* life. The set $\mathcal{L}_\infty(G)$ of all verteces with an infinite life can be written as

$$\mathcal{L}_\infty(G) = \bigcap_{0 \leq a < \infty} f^a(V)$$

In general, we get $f(V) \subset V$, hence we get a sequence of the vertex sets of dynamical subgraphs:

$$V \supset f(V) \supset f^2(V) \supset f^3(V) \supset \cdots \supset \mathcal{L}_\infty(G).$$

The set $\mathcal{L}_n(G)$ of all verteces of life n coincides with the difference $f^{n-1}(V) \backslash f^n(V)$.

For example,

$$\mathcal{L}_1(G) = \{v \in V \,|\, \deg(v) = 0\}.$$

In particular, if G is a union of cycles(defined below), then $G = \mathcal{L}_\infty(G)$.

Remark 3. The life of a vertex measures a degree of shrinking of the world at each vertex in the course of time.

For a subset $U \subset V$, the minimal subgraph $(V^+(U), E')$ is called *generated by* U, if $u, v \in V^+(U)$ and $(u, v) \in E$ imply $(u, v) \in E'$. This will be denoted by $\langle U \rangle$, Note that $\langle U \rangle$ is nothing but the dynamical subgraph whose vertex set coincides with the future $V^+(U)$ of U, and of course, $G = \langle V \rangle$.

For a vertex v, the subgraph $\langle \{v\} \rangle$ is also simply denoted by $\langle v \rangle$, and then $\langle v \rangle$ is connected.

§1..1 Regular Dynamical Graphs

Similarly as in the ordinary graph theory, we can define paths, cycles, periods of cycles, connectivity, etc. For example, a subset $C = \{v_1, \cdots, v_p\}$ of (mutually different) verteces is called a *cycle*, if it satisfies

$$f(v_i) = \begin{cases} v_{i+1} & (i < p) \\ v_1 & (i = p). \end{cases}$$

And the number $p = p(C)$ is called the *period* of the cycle C. A cycle with a period 1 consists of a single vertex, and is also called a *fixed point*. The subgraph $\langle C \rangle$ generated by C is also called a cycle.

In dynamical graphs, there exist no paths connecting two cycles, and any connected component(*i.e.* maximal connected subgraph) contains at most one cycle. We call the cycle C a *limit cycle*, if its connected component has a vertex point other than C, or equivalently if the past of C is actually larger than C.

A dynamical graph $G = (V, E)$ is called *connected*, if $V^+(v) \cap V^+(w) \neq \phi$ for any verteces $v, w \in$

V. Maximal connected dynamical subgraphs are called *connected components* (or simply *components*), and if a set U of verteces is contained in a component G' of G, then G' is called the *component* of U and is denoted by G_U.

A connected component G' of a dynamical graph G is called *regular*, if it contains actually one cycle C. Then we say that any subsets or verteces of G' *belong to the cycle C* or *the C-family*}.

A dynamical graph G is called *regular*, if every component of G is regular.

Then we get easily the following.

Proposition 2 (*i*) *Any finite dynamical graph is regular.*

In the following, assume that G is regular.

(*ii*) *Any vertex v of infinite life belongs to a cycle. Hence the set $\mathcal{L}_\infty(G)$ is a disjoint union of cycles.*

(*iii*) *If the degree of every vertex is 1, then G itself is a union of cycles.*

Proof. Since $|V| < \infty$, there is a finite maximum life $k = \max_{v \in V,\ \ell(v)<\infty} \ell(v)$. Then we get

$$\bigcap_{0 \le a < \infty} f^a(V) = \bigcap_{0 \le a \le k} f^a(V) = f^k(V).$$

Any vertex $v \in f^k(V)$ is of life ∞, so $v \in f^a(V)$ for any $a \in \mathbb{N}$. Since $|V| < \infty$, there is a vertex w such that $v = f^{a_1}(w) = f^{a_2}(w)$ for some $a_1 < a_2$. We may assume that $v \neq f^a(w)$ for $a_1 < a < a_2$. Then by putting $b = a_2 - a_1$, we get

$$f^b(v) = f^{b+a_1}(w) = f^{a_2}(w) = v,$$

hence v generates a cycle C with a period b. □

Now we define the notion of the *height* of verteces in regular components. Let v be a vertex belonging to a cycle C. Define the height $\mathrm{ht}(v) = \mathrm{ht}_C(v)$ as

$$\mathrm{ht}(v) = \begin{cases} 0 & (v \in C) \\ n & (f^n(v) \in C,\ f^m(v) \notin C \text{ for any } 0 \leqq m < n). \end{cases}$$

Let $G' = (V', E')$ be a regular component of a dynamical graph G, with the cycle C. Let $\mathcal{F}_k(C)$ be the set of all verteces belonging to V' of height k, *i.e.*

$$\mathcal{F}_k(C) = \mathcal{F}_k(C;\, G) = \{v \in V'(\text{or } V) \mid \mathrm{ht}_C(v) = k\}.$$

Then $\mathcal{F}(C) = \bigcup_{k \geqq 0} \mathcal{F}_k(C)$ is a disjoint union, nothing but G', and symbolically G' is drawn as

$$\circlearrowleft C = \mathcal{F}_0(C) \leftarrow \mathcal{F}_1(C) \leftarrow \cdots \leftarrow \mathcal{F}_k(C) \leftarrow \mathcal{F}_{k+1}(C) \leftarrow \cdots$$

Now we give an invariant for dynamical graphs. Let $G = (V, E)$ be a dynamical graph, then define the *degree characteristic* of G as the vector

$$\mathbb{D}_G = (D_G(0),\ D_G(1),\ D_G(2),\ D_G(3),\ \cdots),$$

where

$$D_G(i)\,(=D(i)) = |\{v \in V \mid \deg(v) = i\}|$$

is the number of verteces of degree i. We can write it also as a sum

$$\mathbb{D}_G = \sum_{i \geqq 0} D_G(i)\mathbb{k}_i,$$

where $\mathbb{k}_i$ is the vector with 1 in the i-th component and 0 in all other components.

If there is a number i such that $D_G(j) = 0$ for any $j > i$, then we write it briefly as

$$\mathbb{D}_G = (D_G(0),\ D_G(1),\ \cdots,\ D_G(i-1),\ D_G(i)).$$

In the case of finite graphs G, we get

$$\sum_{i \geqq 0} D_G(i) = |V| \text{ and } \sum_{i \geqq 0} iD_G(i) = |V|,$$

where the latter equality is well known as $\sum_{i \geqq 0} iD_G(i) = |E|$ in graph theory, but $|E| = |V|$ for finite dynamical graphs.

We have many examples even in the case where $V \subset \mathbb{N}$, which will be assumed in the rest of this article. In this case, we say G is a *dynamical graph of numbers*. In the next subsection, we will give few examples illustrating the notions defined above.

§1..2 Some Examples of Dynamical Graphs of Numbers

Examples will be given as pairs of a set V of verteces and a map f on V.

Example 1 (Addition Graph). At first, we give the most trivial example.

Let $V = \mathbb{N}$. And for any $a \in \mathbb{N}$, let $f(x) = x + a$ $(x \in V)$ and denote the corresponding dynamical graph by $A^a = G(f)$. The dynamical graph A^1 is drawn as follows.

$$A^1: 0 \to 1 \to 2 \to \cdots \to n \to n+1 \to \cdots.$$

Hence, for $n \geqq 1$,

$$\mathcal{L}_n(A^1) = \{n-1\},\ \mathcal{L}_\infty(A^1) = \emptyset \text{ and } \mathbb{D}_{A^1} = (1, \infty) = \mathbb{k}_0 + \infty\, \mathbb{k}_1.$$

The graph A^1 is connected and not regular.

For $a > 1$, the graph A^a is no longer connected, and the number of connected components is a and any components are not regular. For $a = 2$, the graph A^2 is drawn as follows.

$$0 \to 2 \to 4 \to \cdots \to 2n \to 2n+2 \to \cdots$$
$$1 \to 3 \to 5 \to \cdots \to 2n-1 \to 2n+1 \to \cdots$$

and for $n \geqq 1$,

$$\mathcal{L}_n(A^a) = \{i \mid (n-1)a \leqq i < na\},\ \mathcal{L}_\infty(A^a) = \emptyset \text{ and } \mathbb{D}_{A_1} = (a, \infty).$$

Let k be a positive integer, and consider the above graphs modulo k. That is, let $V = I_k = \{i \in \mathbb{N} \mid 0 \leq i < k\}$ and $f(x) \equiv x + a \pmod{k}$ for $x \in I_k$. The corresponding dynamical graph $G(f)$ is regular and denoted by A_k^a.

If k and a are mutually prime (*i.e.* $(k, a) = 1$), then A_k^a itself is a cycle and for $n > 0$

$$\mathcal{L}_n(A_k^a) = \emptyset,\ \ \mathcal{L}_\infty(A_k^a) = A_k^a \text{ and } \mathbb{D}_{A_k^a} = (0, k) = k\,\mathbb{k}_1.$$

For example, A_{10}^3 is drawn as follows.

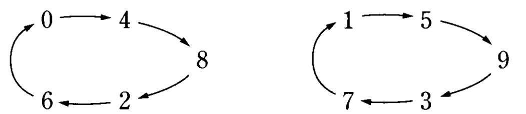
$$\begin{array}{ccccccccc} 0 & \rightarrow & 3 & \rightarrow & 6 & \rightarrow & 9 & \rightarrow & 2 \\ \uparrow & & & & & & & & \downarrow \\ 7 & \leftarrow & 4 & \leftarrow & 1 & \leftarrow & 8 & \leftarrow & 5 \end{array}$$

If k and a are not mutually prime (*i.e.*} $(k, a) = d > 1$), then the graph A_k^a has d connected components which are cycles C_i $(1 \leqq i \leqq d)$. For $n > 0$

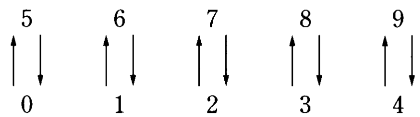
$$\mathcal{L}_n(A_k^a) = \emptyset,\ \ \mathcal{L}_\infty(A_k^a) = I_k,\ \mathcal{F}(C_i) = C_i \text{ and } \mathbb{D}_{A_k^a} = (0, k) = k\,\mathbb{k}_1.$$

For example, A_{10}^4 and A_{10}^5 are drawn as follows.

A_{10}^4:



A_{10}^5:



Example 2 (Multiplication Graph).

Let $V = \mathbb{N}$. And for any $a \in \mathbb{N}$, let $f(x) = ax$ $(x \in V)$ and denote the corresponding dynamical graph by $M^a = G(f)$.

The dynamical graph M^1 is isomorphic with A^0, and its every vertex is a fixed point(cycle with period 1). Hence for $n > 0$

$$\mathcal{L}_n(M^1) = \emptyset,\ \ \mathcal{L}_\infty(M^1) = \mathbb{N},\ \mathcal{F}(\{n\}) = \{n\} \text{ and } \mathbb{D}_{M^1} = (0, \infty).$$

For $a > 1$, the dynamical graph M^a has one cycle(fiexed point) and infinite components, each of which is isomorphic with A^1 and so not regular. We get

$$\mathcal{L}_n(M^a) = \{N \in \mathbb{N} \mid a^m \mid N\ (m < n),\ a^n \nmid N\},$$
$$\mathcal{L}_\infty(M^a) = \{0\},\ \mathcal{F}(\{0\}) = \{0\} \text{ and } \mathbb{D}_{M^a} = (\infty, \infty) = \infty\,\mathbb{k}_0 + \infty\,\mathbb{k}_1.$$

In particular, $\mathcal{L}_1(M^2)$ consists of all odd integer > 0, and M^2 is drawn as follows.

$1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 32 \rightarrow 64 \rightarrow 128 \rightarrow \cdots$

$3 \rightarrow 6 \rightarrow 12 \rightarrow 24 \rightarrow 48 \rightarrow 96 \rightarrow 192 \rightarrow \cdots$

$\vdots$

$m = 2n+1 \rightarrow 2m \rightarrow 4m \rightarrow 8m \rightarrow 16m \rightarrow 32m \rightarrow \cdots$

$\vdots$

Let k be a positive integer, and consider the above graphs modulo k. That is, $V = I_k = \{i \in \mathbb{N} \mid 0 \le i < k\}$ and $f(x) \equiv ax \pmod{k}$ for $x \in I_k$. The corresponding dynamical graph $G(f)$ is regular and denoted by M^a_k. It always contains the fixed point $\{0\}$.

If k and a are mutually prime (*i.e.* $(k, a) = 1$), then for $n > 0$

$$\mathcal{L}_n(M^a_k) = \emptyset,\ \mathcal{L}_\infty(M^a_k) = I_k,\ \mathcal{F}(M^a_k) = M^a_k,\ \mathbb{D}_{M^a_k} = (0, k) = k\,\mathbb{k}_1,$$

and the graph M^a_k is a sum of cycles.

In the case where k and a are not mutually prime, there are various types of graph structures.

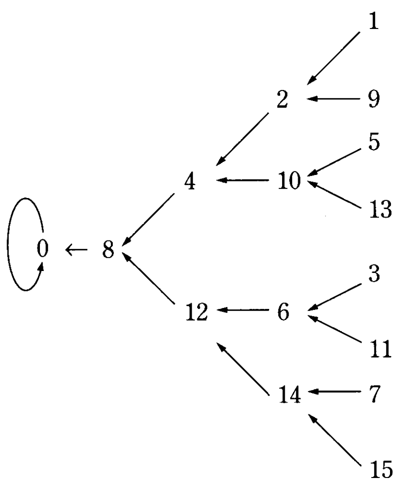
For example, the graph $M^2_{2^k}$ will be a tree, if we omit the arrow from 0. Such dyna-graph will be called a tree-like graph. Rigorously, a dyna-graph G is called *tree-like*, if periods of any cycles are 1.

The fixed point $C = \{0\}$ is a unique cycle in $M^2_{2^k}$, and for any n $(0 < n \leqq k)$,

$$\mathcal{L}_n(M^2_{2^k}) = \mathcal{F}_{k+1-n}(C) = \{N \in I_{2k} \mid 2^m \mid N\ (\mathrm{m} < \mathrm{n}),\ 2^n \nmid \mathrm{N}\},$$

$$\mathcal{L}_\infty(M^2_{2^k}) = C,\ \mathcal{F}(C) = M^2_{2^k} \text{ and } \mathbb{D}_{M^2_{2^k}} = (2^{k-1}, 0, 2^{k-1}) = 2^{k-1}\,\mathbb{k}_0 + 2^{k-1}\,\mathbb{k}_2.$$

In particular, $\mathcal{L}_1(M^2_{2^k})$ consists of all odd integer > 0. For example, M^2_{16} is drawn as follows.



The graph $M^p_{p^2}$ is also tree-like, and $\mathcal{L}_\infty(M^p_{p^2}) = \{0\}$, $\mathcal{L}_2(M^p_{p^2}) = \{ip \mid 1 \leqq i < p\}$,

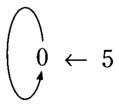
$$\mathcal{L}_1(M^p_{p^2}) = \{n \in I_{p^2} \mid p \nmid n\}, \text{ and } \mathbb{D}_{M^p_{p^2}} = p(p-1)\,\mathbb{k}_0 + p\mathbb{k}_p.$$

For the mixed type such as $k = 10 = 2 \times 5$ or $100 = 2^2 \times 5^2$, the structures of graphs will be more complicated. M^2_{10} has two cycles $C_0 = \{0\}$ and $C_1 = \{2, 4, 8, 6\}$, and

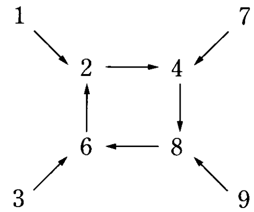
$$\mathcal{F}_1(C_0) = \{5\},\ \mathcal{F}_1(C_1) = \{1, 7, 9, 3\},\ \mathbb{D}_{M^2_{10}} = \{5, 0, 5\} = 5\,\mathbb{k}_0 + 5\,\mathbb{k}_2.$$

$I_{10} = \mathcal{L}_1(M^2_{10}) \cup \mathcal{L}_\infty(M^2_{10})$, $\mathcal{L}_1(M^2_{10}) = \{\text{odd numbers}\}$, $\mathcal{L}_\infty(M^2_{10}) = \{\text{even numbers}\}$.
The graph M^2_{10} is drawn as as follows.

$\mathcal{F}(C_0)$



$\mathcal{F}(C_1)$

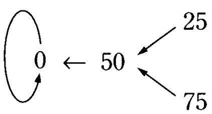


The graph M^2_{100} has three cycles C_0, C_1, C_2, where

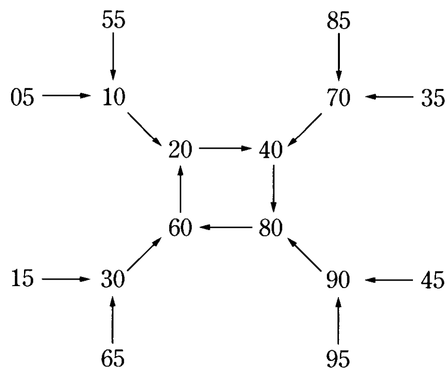
$$C_0 = \{0\},\ C_1 = \{20, 40, 80, 60\},$$
$$C_2 = \{n \in I_{100} \mid n \equiv 0 \pmod 4,\ n \not\equiv 0 \pmod 5\},$$
$$\mathcal{F}(C_0) = \{n \in I_{100} \mid n \equiv 0 \pmod{25}\},$$
$$\mathcal{F}(C_1) = \{n \in I_{100} \mid n \equiv 0 \pmod 5,\ n \not\equiv 0 \pmod{25}\},$$
$$\mathcal{F}(C_2) = \{n \in I_{100} \mid n \not\equiv 0 \pmod 5\},\ \mathbb{D}_{M^2_{100}} = (50, 0, 50) = 50\,\mathbb{k}_0 + 50\,\mathbb{k}_2,$$
$$I_{100} = \mathcal{L}_1(M^2_{100}) \cup \mathcal{L}_2(M^2_{100}) \cup \mathcal{L}_\infty(M^2_{100}),$$
$$\mathcal{L}_1(M^2_{100}) = \{n \not\equiv 0 \pmod 2\},$$
$$\mathcal{L}_2(M^2_{100}) = \{n \equiv 0 \pmod 2,\ n \not\equiv 0 \pmod 4\},$$
$$\mathcal{L}_\infty(M^2_{100}) = \{n \equiv 0 \pmod 4\}.$$

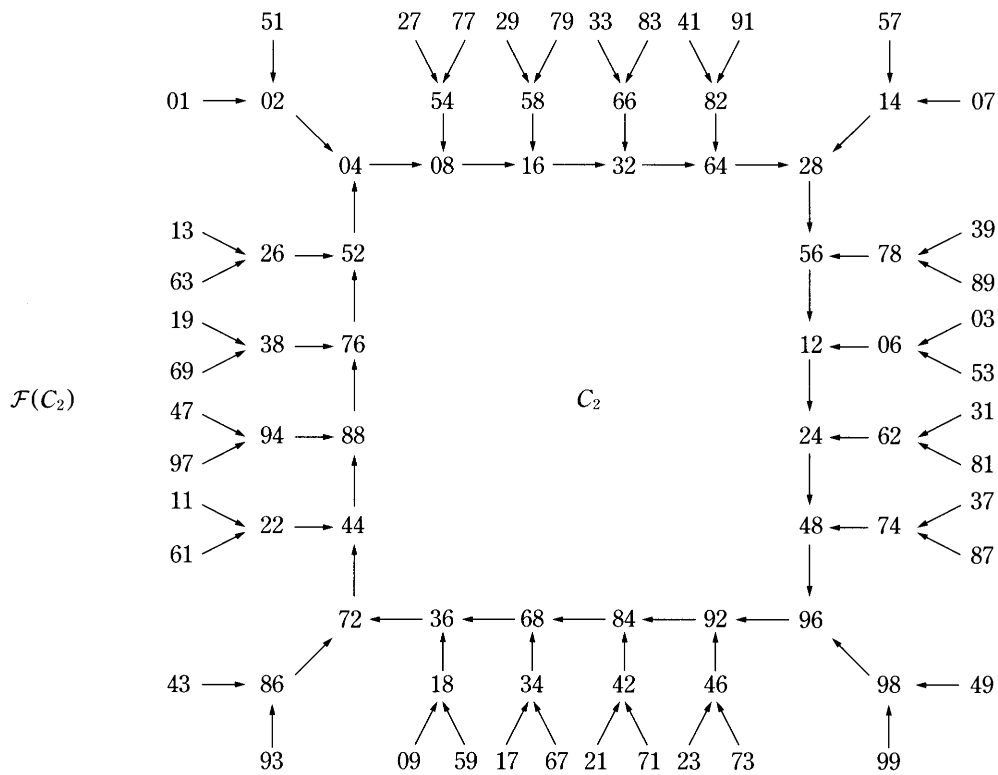
M^2_{100} is drawn as follows.

$\mathcal{F}(C_0)$



$\mathcal{F}(C_1)$





Example 3 (*Polynomial Graph*). Let $V = \mathbb{N}$. And for any polynomial $p(x) \in \mathbb{N}[x]$, let $f(x) = p(x)$ $(x \in \mathbb{N})$, and denote the corresponding dynamical graph $G(f)$ by $P(p(x))$.

Then the former examples are particular cases of polynoimal graphs: addition graphs $A^a = P(x + a)$ and multiplication graphs $M^a = P(ax)$.

The finite version is similar. Let k be a positive integer. Let $V = I_k$ and $f(x) \equiv p(x) \pmod{k}$ for $x \in I_k$. The corresponding dynamical graph $G(f)$ is regular and denoted by $P_k(p(x))$. Then $A_k^a = P_k(x+a)$ and $M_k^a = P_k(ax)$.

In some sense, this is the most general case:

Proposition 3 *Let $G = (V, E)$ be a finite dynamical graph with $n = |V|$, then G is isomorphic with a polynomial graph on I_n.*

Proof. Any finite set is bijective with the section $I_n = \{i \mid 0 \leqq i < n\}$ of $\mathbb{N}$. Any function f on I_n can be extended to a polynomial $F(x)$ such that $F(i) = f(i)$, $i \in I_n$ (polynomial interpolation). □

Example 4 (*Reduced Divisor Sum Graph*). Let $V = \mathbb{N}_{>0}$, and

$$f(i) = \begin{cases} \sum_{k|i} k - i & (i > 1) \\ 1 & (i = 1) \end{cases}$$

The dynamical graph G corresponding to the map f is discussed in [2] (see it for drawings of some parts).

Many facts on perfect numbers, abundant numbers, deficient numbers, amicable numbers, prime numbers, etc. can be described in terms of G.

C_0 = {1} is a fixed point, and $\mathcal{F}_1(C_0)$ = {prime numbers} by the definition itself of prime numbers. It is easily seen that 2, 5 has a life 1 ($2, 5 \in \mathcal{L}_1(G)$). If we assume that Goldbach's conjecture for even integers holds, then any odd number > 6 has an infinite life, in particular, any prime number $p(\neq 2, 5)$ belongs to $\mathcal{L}_\infty(G) \cap \mathcal{F}_1(C_0)$.

Other fixed points are perfect numbers such as 6, 28, 496, $\cdots$ (denote the i-th perfect number by pf_i, and $C_{pf_i} = \{pf_i\}$. Amicable numbers such as C_{am_1} = {220, 284} make cycles with peiord 2. For example, C_{pf_1} = {6}, C_{pf_3} = {496}, C_{am_1} are limit cycles, but C_{pf_2} = {28} is not.

As for the futures, for $v < 276$ the subgraph $\langle v \rangle$ generated by v is finite, and belongs to the cycle C_0 or C_{pf_1} or C_{pf_3} or C_{am_1}. But I can't determine whether $\langle 276 \rangle$ is regular or not. It seems to me that the graph $\langle 276 \rangle$ grows unboundedly, so is not regular. For $v \leqq 1000$, there are 12 verteces v = 276, 306, 396, 552, 564, 660, 696, 828, 840, 888, 966, 996 for which $\langle v \rangle$ may not be regular.

If v is small, for example, $v \leqq 1000$, most of them(about more than 965) belogs to the cycle C_0. So in [2], we say that for a prime number p, a vertex $v \in V^-(p)$ belongs to the family p, and we study the structures of the C_0-component by statistical treatment.

If you want to use the notation of families in this aritcle, you can modify the dynamical graph. Replace V with $\{v \in \text{in } \mathbb{N} \mid v > 1\}$, and change the values of f for prime numbers p as $f(p) = p$. Then $C_\mathrm{p} = \{p\}$ is also a fixed point.

Similarly, we can consider transformations of dynamical graphs, which will be convenient and useful for the study of the relations of dynamical graphs (see [4] for detailes).

For illustraing the complexity of this dynamical graph, we note several facts.

$$\mathcal{F}(12161) \cap I_{1000} = \{120, 240, 504\}, \ \mathcal{F}_{11}(12161) \cap I_{1000} = \{120\},$$
$$|\mathcal{F}(321329) \cap I_{1000}| = 8, \ 318 \in \mathcal{F}_{34}(321329), 330, 498 \in \mathcal{F}_{33}(321329),$$
$$|\mathcal{F}(59) \cap I_{100}| = 1, |\mathcal{F}(59) \cap I_{200}| = 5, |\mathcal{F}(59) \cap I_{500}| = 16, |\mathcal{F}(59) \cap I_{1000}| = 45,$$
$$138 \in \mathcal{F}_{177}(59), 150 \in \mathcal{F}_{176}(59), 222 \in \mathcal{F}_{175}(59),$$
$$168, 234 \in \mathcal{F}_{174}(59), 312 \in \mathcal{F}_{173}(59), 528 \in \mathcal{F}_{172}(59).$$

138 is the heighest vertex of height 177 among verteces of finite height in I_{1000}.

§2. Reversed Difference

Let Z = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} be the set of figures, and $\mathbb{N}$ be the set of all natural numbers.

For $k > 0$, consider the set

$$\mathbb{N}_k = \{n \in \mathbb{N} \mid 0 \leq n < 10^k\} = I_{10^k}$$

of k-figures as the set of verteces of our dynamical graph.

Then, by noting the isomorphism

$$\varphi: Z^k \ni (a_k, a_{k-1}, \cdots, a_1) \mapsto \sum_{i=1}^{k} a_i 10^{i-1} \in \mathbb{N}_k$$

of Z^k with $\mathbb{N}_k$, we can define the inversion in $\mathbb{N}_k$ through the order reversion

$$\bar{\ }: Z^k \ni (a_k,\ a_{k-1},\ \cdots,\ a_1) \mapsto (a_1,\ a_2,\ \cdots,\ a_k) \in Z^k$$

in Z^k, as

$$\bar{n} = \varphi(\overline{\varphi^{-1}(n)}) \quad (n \in \mathbb{N}_k).$$

Now we consider the dynamical system on $\mathbb{N}_k$ defined by the map

$$f_k(n) = | n - \bar{n} | \quad (n \in \mathbb{N}_k).$$

For $k = 4$, this game has been familiar with mathematicians, like as a folklore. Someone called it Kakutani's game, and someone Kac's game, since they introduced it to Japanese mathematicians as a kind of recreation mathematics.

Denote by G_k the dynamical graph corresponding to the map f_k. Before dealing with the graph G_4, we study the structure of the graphs G_k for $k = 1, 2, 3$.

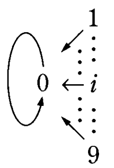
§2..1 Case of $k = 1$

In the case of $k = 1$, $V = \mathbb{N}_1$ and the map $f = f_1$: $V \to V$ is given as

$$f(d) = | d - d | = 0, \quad \text{hence} \quad C_0 = \{0\} = f(V).$$

C_0 is the unique cycle, and the graph $G_1 = G(f)$ is connected. So they are drawn as follows.

C_0: 

$\mathbb{N}_1 = \mathcal{F}(C_0)$: 

$(1 \le i \le 9)$

§2..2 Case of $k = 2$

In the case of $k = 2$, $V = \mathbb{N}_2$ and the map $f = f_2$: $V \to V$ is given as

$$f(x) = |(10c + d) - (10d + c)| = 9|c - d|, \quad (x = 10c + d \in V).$$

Hence V is decomposed as a sum of invariant subsets:

$$V = \tilde{I}_0 \cup \tilde{I}_1,$$

where

$$\tilde{I}_0 = \{10c + d \mid c = d\} = 11Z \quad \text{and} \quad \tilde{I}_1 = \{10c + d \mid c \neq d\}.$$

The subgraph $f(G_2)$ consists of 10 verteces, and is obtained from $G = G_2$ by dropping out 90 verteces of life 1.

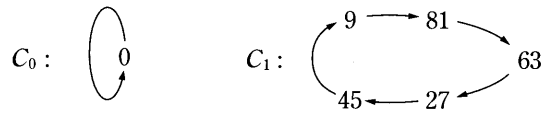
Denote by I_i the f-image $f(\tilde{I}_i)$ $(i = 0, 1)$, then

$$I_0 = C_0 = \{0\} \quad \text{and} \quad I_1 = f(\tilde{I}_1) = 9Z^{\times} = \{9, 18, 27, 36, 45, 54, 63, 72, 81\},$$

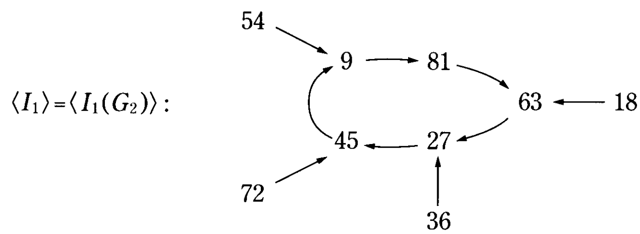
where $Z^{\times} = Z \setminus \{0\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Since elements $10c + d$ of I_1 satisfy $c + d = 9$, hence their images are odd. In fact, since $|c - d| = |c - (9 - c)| = |2c - 9|$ is odd, $f(10c + d) = 9|c - d|$ is also odd. Let $C_1 = I_1 \cap \{\text{odd numbers}\}$, then $C_1 \supset f(I_1)$ but C_1 proves to be a cycle, hence C_1 is the image of I_1:

$$C_1 = f(I_1) = \{9 \to 81 \to 63 \to 27 \to 45 \to (9)\}$$

The graph G_2 has two cycles C_0 and C_1 drawn as follows.



and the subgraph $\langle I_1 \rangle$ generated by I_1 is $\mathcal{F}(C_1) \cap f(G)$ and is drawn as follows.



§2..3 Case of $k = 3$

In the case of $k = 3$, $V = \mathbb{N}_3$ and the map $f = f_3$: $V \to V$ is given as

$$f(x) = |(100b + 10c + d) - (100d + 10c + b)| = 99|b - d|$$

for $x = 100b + 10c + d \in V$. This case is quite similar as the case $k = 2$.

V is decomposed as a sum of invariant subsets:

$$V = \tilde{I}_0 \cup \tilde{I}_1,$$

where

$$\tilde{I}_0 = \{100b + 10c + d \mid b = d\} \text{ and } \tilde{I}_1 = \{100b + 10c + d \mid b \neq d\}.$$

The subgraph $f(G)$ consists of 10 verteces, and is obtained from $G = G_3$ by dropping out 990 verteces of life 1.

Denote by I_i the f-image $f(\tilde{I}_i)$ $(i = 0, 1)$, then

$$I_0 = C_0 = \{0\} \text{ and } I_1 = 99Z^{\times} = \{99, 198, 297, 396, 495, 594, 693, 792, 891\}.$$

Since elements $100b + 10c + d$ of I_1 satisfy $b + d = 9$, $c = 9$, hence their images are odd as in the case $k = 2$. Note I_1 is obtained from $I_1(G_2)$ by inserting "9" in the middle of the numbers in $I_1(G_2)$:

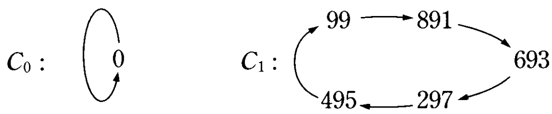
$$I_1(G_3) = \{100c + 90 + d \mid 10c + d \in I_1(G_2)\},$$

and this procedure is equivariant under f_2 and f_3, that is, if $10a + b \to 10c + d$ in G_2, then $100a + 10z + b \to 100c + 90 + d$ in G_3 for any $z \in Z$.

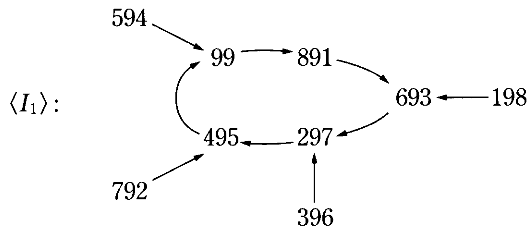
Let $C_1 = I_1 \cap \{\text{odd numbers}\}$, then $C_1 \supset f(I_1)$ but C_1 proves to be a cycle, hence C_1 is the image of I_1:

$$C_1 = f(I_1) = \{99 \to 891 \to 693 \to 297 \to 495 \to (99)\}$$

The graph G has two cycles C_0 and C_1 drawn as follows.



and the subgraph $\langle I_1 \rangle$ generated by I_1 is $\mathcal{F}(C_1) \cap f(G)$ and is drawn as follows.



§3. Case of k = 4

Now consider the graph $G = G_4$, that is, let $V = \mathbb{N}_4$ and the map $f = f_4$: $V \to V$ be given as

$$\begin{aligned} f(10^3a + 10^2b + 10c + d) &= |(10^3a + 10^2b + 10c + d) - (10^3d + 10^2c + 10b + a)| \\ &= |999(a-d) + 90(b-c)|. \end{aligned}$$

Decompose V in a formal way as

$$V = \tilde{I}_0 \cup \tilde{I}_1 \cup \tilde{I}_2 \cup \tilde{I}_3 \cup \tilde{I}_4,$$

where

$$\begin{aligned} \tilde{I}_0 &= \{10^3a + 10^2b + 10c + d \mid a = d,\ b = c\}, \\ \tilde{I}_1 &= \{10^3a + 10^2b + 10c + d \mid a = d,\ b \neq c\}, \\ \tilde{I}_2 &= \{10^3a + 10^2b + 10c + d \mid a \neq d,\ b = c\}, \\ \tilde{I}_3 &= \{10^3a + 10^2b + 10c + d \mid (a-d)\,(b-c) > 0\}, \\ \tilde{I}_4 &= \{10^3a + 10^2b + 10c + d \mid (a-d)\,(b-c) < 0\}. \end{aligned}$$

It is easily seen that $\tilde{I}_0$, $\tilde{I}_1$ and $\tilde{I}_2$ are f-invariant, but $\tilde{I}_3$ and $\tilde{I}_4$ are not so. But there are invariant subsets $\tilde{I}_i^0$ of $\tilde{I}_i$ (i = 3, 4):

$$\tilde{I}_3^0 = \{x \in \tilde{I}_3 \mid a-d = b-c \neq 0, \pm 5\},$$
$$\tilde{I}_4^0 = \{x \in \tilde{I}_4 \mid a-d = c-b \neq 0\}.$$

Each of these invariant subsets contains a cycle C_i $(0 \leqq i \leqq 4)$, and there are no other cycles:

$C_0 = \{0\}$, $\quad C_1: 90 \to 810 \to 630 \to 270 \to 450 \to 90$, $\quad C_3: 6534 \rightleftarrows 2178$

$C_2: 999 \to 8991 \to 6993 \to 2997 \to 4995 \to 999$, $\quad C_4: 0909 \to 8181 \to 6363 \to 2727 \to 4545 \to 0909$

Denote $f(\tilde{I}_i)$ by I_i and $f(\tilde{I}_i^0)$ by I_i^0, then $I_0 = C_0$,

$$I_1 = 90Z^\times = 10I_1(G_2) \supset C_1,$$
$$I_2 = 999Z^\times = 111I_1(G_2) \supset C_2,$$
$$I_3^0 = 1089(Z^\times \setminus \{5\}) \supset C_3,$$
$$I_4^0 = 909Z^\times = 101I_1(G_2) \supset C_4.$$

Thus the subgraphs $\langle I_1 \rangle$, $\langle I_2 \rangle$, $\langle I_4^0 \rangle$ are obtained from the subgraph $\langle I_1(G_2) \rangle$ of G_2, by adding "0" from the left and right sides of, inserting "99" in the middle of, and repeating twice the numbers in $I_1(G_2)$ respectively. These procedures are equivariant with f_2 and f_4.

Since there are many verteces of life 1, it is convenient to deal with the image graph $f(G_4)$:

$$V(f(G_4)) = I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_4,$$

whose members sum up to 181, since

$$|I_0| = 1, \quad |I_1| = |I_2| = 9, \quad |I_3| = |I_4| = 9 \times 9 = 81.$$

I_i $(i \leqq 2)$ are f-invariant, but I_3 and I_4 are not. So let us look at $\tilde{I}_3$ and $\tilde{I}_4$ more closely. For $i = 3, 4$, decompose $\tilde{I}_i$ as

$$\tilde{I}_i = \bigcup_{0 \leqq j \leqq 4} \tilde{I}_{j,i},$$

where

$$\tilde{I}_{j,i} = \{x \in \tilde{I}_i \mid f(x) \in \tilde{I}_j\}.$$

Let $I_{j,i} = f(\tilde{I}_{j,i}) \subset \tilde{I}_j$, then I_i's are decomposed as

$$I_3 = \bigcup_{0 \leqq j \leqq 4} I_{j,3} \text{ and } I_4 = \bigcup_{3 \leqq j \leqq 4} I_{j,4}.$$

Note that f is written as

$$f(x)=\begin{cases}999\,|a-d|+90\,|b-c| & (x\in\tilde{I}_3)\\ 999\,|a-d|-90\,|b-c| & (x\in\tilde{I}_4)\end{cases}$$

Explicitly, the sets above are expressed as

$$\tilde{I}_{0,3}=\{x=10^3a+10^2b+10c+d\in\tilde{I}_3 \mid a-d=b-c=\pm 5\}.$$
$$\tilde{I}_{1,3}=\{x\in\tilde{I}_3 \mid |a-d|=5, |b-c|\neq 5\},$$
$$\tilde{I}_{2,3}=\{x\in\tilde{I}_3 \mid |a-d|\neq 5, |b-c|=5\},$$
$$\tilde{I}_{3,3}=\{x\in\tilde{I}_3 \mid |a-d|, |b-c|\leqq 4, \text{or } |a-d|, |b-c|\geqq 6\}\supset\tilde{I}_3^0,$$
$$\tilde{I}_{4,3}=\{x\in\tilde{I}_3 \mid |a-d|\leqq 4, |b-c|\geqq 6, \text{or } |a-d|\geqq 6, |b-c|\leqq 4\}.$$

$\tilde{I}_{4,3}$ contains the set

$$\tilde{I}_{4,3}^0=\{x\in\tilde{I}_3 \mid |a-d|+|b-c|=10, |a-d|, |b-c|\neq 5\}=\tilde{I}_3\cap f^{-1}(I_4^0).$$

The map f is explicitly given as

$$f(x)=1089\,|a-d|,\quad (x\in\tilde{I}_{0,3}\cup\tilde{I}_3^0)$$
$$f(x)=4995+90\,|b-c|,\quad (x\in\tilde{I}_{1,3})$$
$$f(x)=450+999\,|a-d|,\quad (x\in\tilde{I}_{2,3})$$
$$f(x)=900+909\,|a-d|,\quad (x\in\tilde{I}_{4,3}^0)$$

hence

$$|I_{0,3}|=1,\ |I_{1,3}|=|I_{2,3}|=8,\ |I_{3,3}|=|I_{4,3}|=32.$$

More explicitly,

$I_{0,3}$ = {5445} belongs to the C_0-family of height 1,

$I_{1,3}$ = {5085, 5175, 5265, 5355, 5535, 5625, 5715, 5805} belongs to the C_1-family of height 2,

$I_{2,3}$ = {1449, 2448, 3447, 4446, 6444, 7443, 8442, 9441} belongs to the C_2-family of height 2,

$I_{3,3}=I_3^0\cup I_{4,3,3}^0\cup\bar{I}_{3,3}$, $|I_{3,3}|=32=8+8+16$,

$I_{4,3,3}^0=I_3\cap f^{-1}(I_{4,3}^0)=\{1359, 2268, 3177, 4086, 6804, 7713, 8622, 9531\}$ belongs to the C_4-family of height 3,

$\bar{I}_{3,3}=I_{3,3}\setminus(I_3^0\cup I_{4,3,3}^0)$,

$=\{1179, 1269, 2088, 2358, 3087, 3357, 4176, 4266\}$

$\cup\{9711, 9621, 8802, 8532, 7803, 7533, 6714, 6624\}$,

$I_{4,3}=I_{4,3}^0\cup\bar{I}_{4,3}$, $|I_{4,3}|=32=8+24$,

$I_{4,3}^0(=I_{4,3}\cap\tilde{I}_4^0)$= {1089, 2718, 3627, 4536, 6354, 7263, 8172, 9081} belongs to the C_4-family of height 2.

$\bar{I}_{4,3}=I_{4,3}\setminus I_{4,3}^0$

$=\{1359, 1629, 1719, 2538, 2628, 2808, 3537, 3717, 3807, 4626, 4716, 4806\}$

$\cup\{9531, 9261, 9171, 8352, 8262, 8082, 7353, 7173, 7083, 6264, 6174, 6084\}$.

The image of $\tilde{I}_3^0$ is also contained in itself, so will be denoted by I_3^0. Then

$$I_3^0 = \{1089, 2178, 3267, 4356, 6534, 7623, 8712, 9801\},$$
$$f(I_3^0) = \{2178, 4356, 6534, 8712\}, \quad f^2(I_3^0) = C_3 = \{2178, 6534\}.$$

As for $I_4 = I_{3,4} \cup I_{4,4}$, decompose $I_{3,4}$ and $I_{4,4}$ as follows.

$I_{3,4} = I_{0,3,4} \cup I_{3,4}^0 \cup I_{4,3,4}^0 \cup \bar{I}_{3,4}$, $\quad |I_{3,4}| = 2+6+6+26 = 40$

$I_{0,3,4} = I_4 \cap f^{-1}(I_{0,3}) = \{2277, 7722\}$,

$I_{3,4}^0 = I_4 \cap f^{-1}(I_3^0) = \{1188, 3366, 4455, 5544, 6633, 8811\}$,

$I_{4,3,4}^0 = I_4 \cap f^{-1}(I_{4,3}^0)$ = {459, 5904, 1368, 3186, 6813, 8631} belongs to the C_4-family of height 3.

$\bar{I}_{3,4} = I_{3,4} \setminus (I_{0,3,4} \cup I_{3,4}^0 \cup I_{4,3,4}^0)$

$= \{369, 279, 189, 6903, 7902, 8901\}$

$\cup \{1458, 1278, 2457, 2367, 2187, 3456, 3276, 4365, 4275, 4685\}$

$\cup \{8541, 8721, 7542, 7632, 7812, 6543, 6723, 5634, 5724, 5864\}$.

$I_{4,4} = I_4^0 \cup \bar{I}_{4,4}$, $\quad |I_{4,4}| = 9+32 = 41$,

$\bar{I}_{4,4} = I_{4,4} \setminus I_4^0$

$= \{819, 729, 639, 549, 1908, 2907, 3906, 4905\}$

$\cup \{1728, 1638, 1548, 2817, 2637, 2547, 3816, 3726, 3546, 4815, 4725, 4635\}$

$\cup \{8271, 8361, 8451, 7182, 7362, 7452, 6183, 6273, 6453, 5184, 5274, 5364\}$.

§4. Drawings of the Dynamical Graph G_4

Let $\tilde{\mathcal{F}}(C_i)$ and $\mathcal{F}(C_i)$ be the set of verteces belonging to the cycle C_i in $V = \mathbb{N}_4$ and $f(V)$ respectively. And let $\tilde{\mathcal{F}}_k(C_i)$ and $\mathcal{F}_k(C_i)$ be their subsets consisiting of verteces of height k, *i.e.*

$$\mathbb{N}_4 = \bigcup_{0 \leqq i \leqq 4} \tilde{\mathcal{F}}(C_i), \qquad \tilde{\mathcal{F}}(C_i) = \bigcup_{k=0}^{\infty} \tilde{\mathcal{F}}_k(C_i)$$

$$f(\mathbb{N}_4) = \bigcup_{0 \leqq i \leqq 4} \mathcal{F}(C_i), \qquad \mathcal{F}(C_i) = \bigcup_{k=0}^{\infty} \mathcal{F}_k(C_i)$$

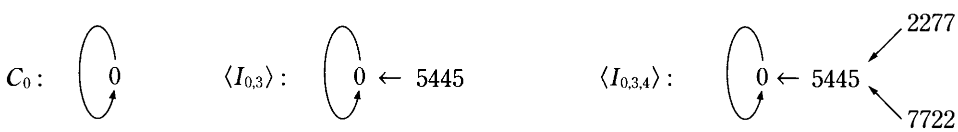
$$\tilde{\mathcal{F}}_k(C_i) = \{x \in \mathbb{N}_4 \mid f^k(x) \in C_i, \ f^{k-1}(x) \notin C_i\},$$

$$\mathcal{F}_k(C_i) = f(\mathbb{N}_4) \cap \tilde{\mathcal{F}}_k(C_i) = f(\tilde{\mathcal{F}}_{k+1}(C_i)),$$

where the unions on k are actually finite unioins.

The graph G_4 exhibits an interesting phenomena: "gate".

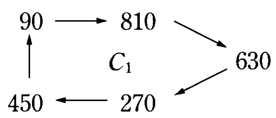
Any vertex $x \in f(\mathbb{N}_4)$ belongs to some $\mathcal{F}_k(C_i)$. For a subset X of $\mathcal{F}(C_i)$, x is called a *gate* for X, if X is contained in the past of x. In the case where $X = \mathcal{F}_h(C_i)$ with some $h > k$, x is a gate for X, if $f^{h-k}(X) = x$.



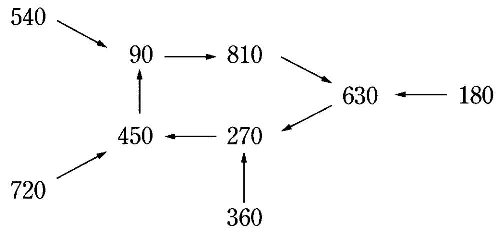
So, 5445 is a gate for $\mathcal{F}_2(C_0)$, and

$$\mathcal{F}_0(C_0) = C_0,\ \mathcal{F}_1(C_0) = I_{0,3} = \{5445\},\ \mathcal{F}_2(C_0) = I_{0,3,4} = \{2277, 7722\}.$$

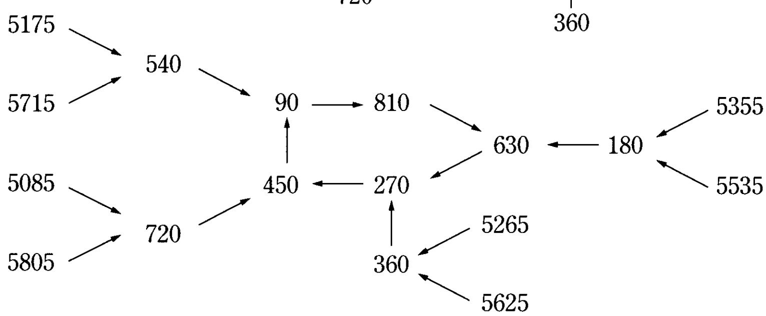
$$\circlearrowleft C_0 \leftarrow I_{0,3} \leftarrow I_{0,3,4} \qquad |\mathcal{F}(C_0)| = 4$$



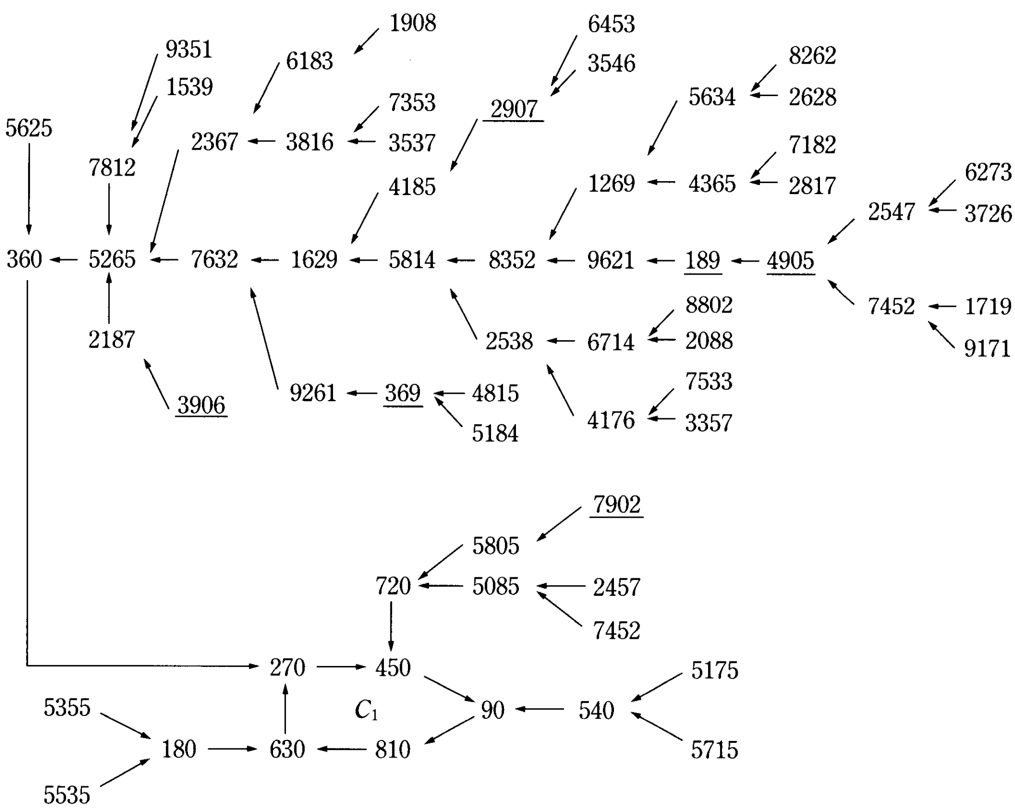
$\langle I_1 \rangle$:



⟨I1,3⟩



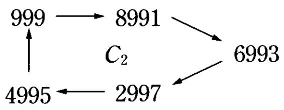
$\mathcal{F}(C_1)$



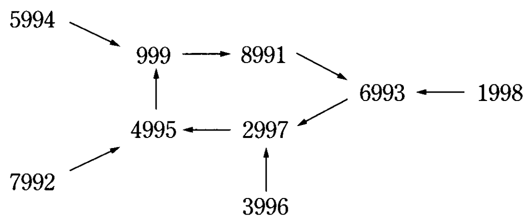
So,

$$\mathcal{F}_0(C_1) = C_1,\ \mathcal{F}_1(C_1) = I_1 \setminus \mathcal{F}_2(C_1) = I_{1,3}.$$

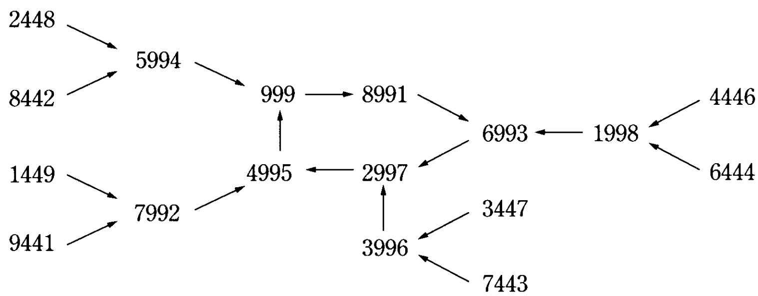
$5265 \in I_{1,3} = \mathcal{F}_2(C_1)$ is a gate for $\mathcal{F}_k(C_1)$ for $k > 3$, and $|\bigcup_{k>2} \mathcal{F}_k(C_1)| = 66 - 17 = 49$. The connected component $\mathcal{F}(C_1)$ of C_1 has 66 verteces.



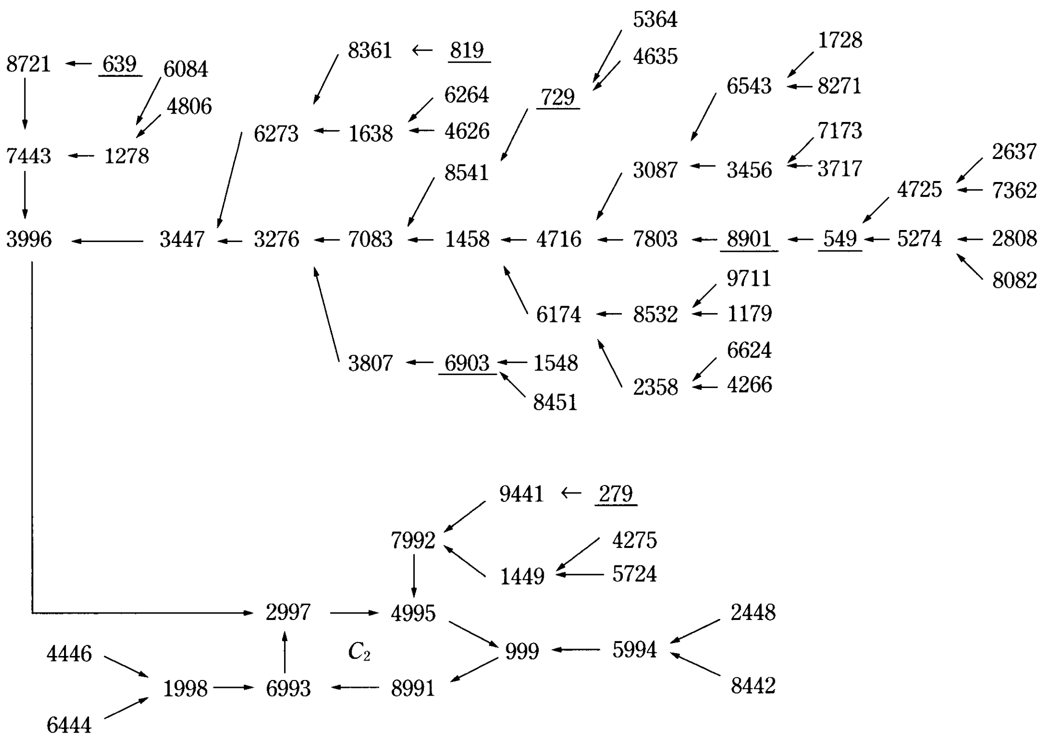
$\langle I_2 \rangle$



$\langle I_{2,3} \rangle$



$\mathcal{F}(C_2)$

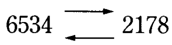


So,

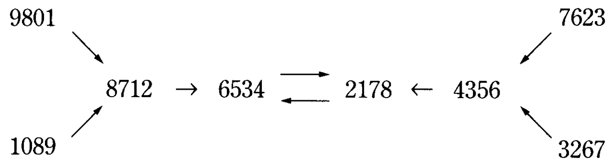
$$\mathcal{F}_0(C_2) = C_2,\ \mathcal{F}_1(C_2) = I_2 \setminus C_2,\ \ \mathcal{F}_2(C_2) = I_{2,3}.$$

$3996 \in \mathcal{F}_1(C_2)$ is a gate for $\mathcal{F}_k(C_2)$ for $k > 2$, and $|\bigcup_{k>1} \mathcal{F}_k(C_2)| = 66 - 9 = 57$. The connected component $\mathcal{F}(C_2)$ of C_2 has 66 verteces.

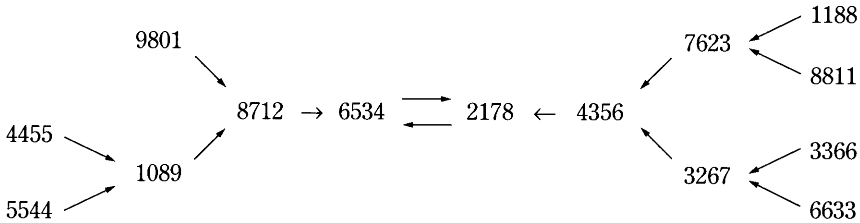
C_3 :



$\langle I_3^0 \rangle$



$\langle I_{3,4}^0 \rangle$:



So,

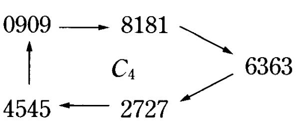
$$\mathcal{F}_0(C_3) = C_3,\ \mathcal{F}_1(C_3) = \{4356, 8712\},\ \mathcal{F}_2(C_3) = \{1089, 3267, 7623, 9801\},$$

and

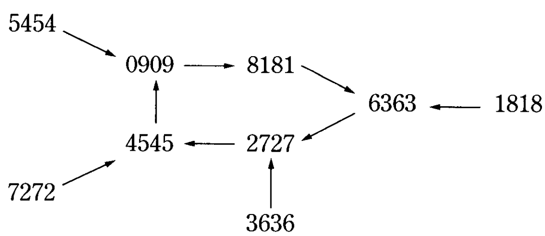
$$\mathcal{F}_3(C_3) = I_{4,3}^0,\ \ \mathcal{F}(C_3) = \bigcup_{0 \leqq k \leqq 3} \mathcal{F}_k(C_3).$$

Note

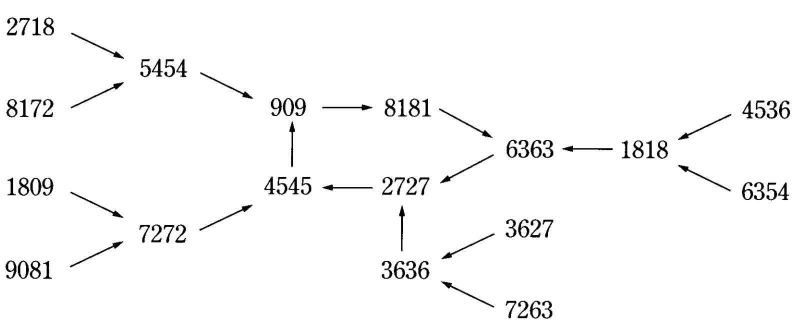
$$I_3^0 = C_3 \cup \mathcal{F}_1(C_3) \cup \mathcal{F}_2(C_3) \text{ and } |\ \mathcal{F}(C_3)| = 2 + 2 + 4 + 6 = 14.$$



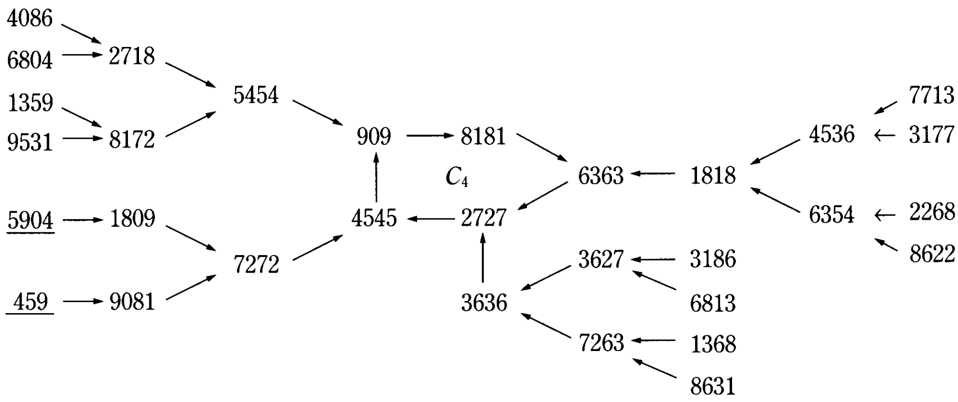
$\langle I_4^0 \rangle$:



$\langle I_{4,3}^0 \rangle$



$\mathcal{F}(C_4)$



So,

$$\mathcal{F}_0(C_4) = C_4,\ \mathcal{F}_1(C_4) = I_4^0 \setminus C_4,\ \mathcal{F}_2(C_4) = I_{4,3}^0,$$
$$|\mathcal{F}(C_4)| = 5 + 4 + 8 + 14 = 31.$$

The maximum height in $f(\mathbb{N}_4)$ is 11, and $|\mathcal{F}_{11}(C_1)| = |\mathcal{F}_{11}(C_2)| = 4$, and the sizes of connected components $\mathcal{F}(C_i)$ of $f(\mathbb{N}_4)$ are

$$|\mathcal{F}(C_0)| = 4,\quad |\mathcal{F}(C_1)| = 66,\quad |\mathcal{F}(C_2)| = 66,\quad |\mathcal{F}(C_3)| = 14,\quad |\mathcal{F}(C_4)| = 31,$$

and sum up to $4 + 66 + 66 + 14 + 31 = 181 = |f(\mathbb{N}_4)|$.

In the above graphs, there are numbers with underlines. The order reversions of those numbers are not in $f(\mathbb{N}_4)$, so the image graph $f(\mathbb{N}_4)$ is not invariant under the reversion.

References

[1] Yuhihiro Kaine, *Towards Clinical Mathematics Education,* Bull. of the Fac. of Education, Mie University (Educational Science), 52(2001), 101–105 (in Japanese).

[2] ——, *Games of Number Structures I,* Bull. of the Fac. of Education, Mie University (Educational Science), 52(2001), 107–118 (in Japanese).

[3] ——, *Dynamical Graphs and Strategy Games — Roles of Materials in Clinical Mathematics Education,* Bull. of the Fac. of Education, Mie University (Educational Science), 53(2002), 73–83 (in Japanese).

[4} ——, *A Mathematical Theory of Graphical Illustration to Arithmetics,* in preparation.