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Classification of Dynamical Graphs with Vertex Number $\leqq 10$

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Introduction

I proposed several materials for Clinical Mathematics Education in [1] such as dynamical graphs (representing cause and effect), strategy games (equivalence relations generated by simple basic relations), and various inverse problems in arithmetics (techniques, skills, arts and structures in the world of numbers). I also developped in [2] the theory of dynamical graphs in the case of reduced divisor sums, and in [4] gave a brief review of a theory of dynamical graphs, and a detailed account in the case of Reversed Difference as an example.

In this note, I will give again a brief review of a theory of dynamical graphs (see [7] for details) which contains a few notions different from ones in [4], and propose the fundamental problems and give an answer for classification of dynamical graphs with vertex number $k \leqq 10$, in §8. This is equivalent with the classification problem for mappings of I_k to itself, where I_k is a k-point set.

§1. Definition of dynamical graphs and fundamental problems

A dynamical graph $G = (V, E)$ is an at most countable oriented graph whose every vertex v has only one outgoing arrow from v.

Proposition 1. *The set $\mathscr{D}(V)$ of dynamical graphs on V is bijective to the set $Map(V, V)$ of the maps of V to itself. The correspondence is given as follows.*

Given $f \in Map(V, V)$, take the set $E = \{(v, f(v)) \mid v \in V\}$ of pairs as the graph of f, then $G(f) = (V, E(f))$ is a dynamical graph.

Conversely, given a dynamical graph $G = (V, E)$, for any $v \in V$ we have only one vertex $w \in V$ with $(v, w) \in E$. So let $f(v) = w$. Denoting f by $f(G)$, we get that $G = G(f(G))$ and $f = f(G(f))$.

Hence a dynamical graph $G(f) = (V, E(f))$ is corresponding to a discrete dynamical system f on the set V.

Two mappings $f, g : V \to V$ are called *isomorphic*, if there exists a bijection $\varphi : V \to V$ (called an *isomorphism*) satisfying the equality

$$\varphi \circ f = g \circ \varphi \Leftrightarrow f = \varphi^{-1} \circ g \circ \varphi.$$

Isomorphic mappings are denoted by $f \cong g$, and the dynamical graphs $G(f)$ and $G(g)$ corresponding to isomorphic mappings $f, g : V \to V$ are called *isomorphic* and denoted by $G(f) \cong G(g)$.

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If f is bijective, the inverse mapping f^{-1} defines the dynagraph $G(f^{-1})$ called the inverse graph of $G = G(f)$, and G is called invertible.

Denote by $\mathscr{D}(V)$ the set of all dynamical graphs on V, and by $\mathscr{D}'(V)$ the set of all invertible dynamical graphs on V. The cardinality of V is called of *size* of $G = (V, E)$, denoted by $s = s(G)$, wchi coincides with the number $\#E$ of edges of G.

Denote by $\mathfrak{D}(V)$ and $\mathfrak{D}'(V)$ the set of isomorphism classes of $\mathscr{D}(V)$ and $\mathscr{D}'(V)$ respectively.

For explicit realization of graphs, fix the size k, and take the k-skelton of $\mathbb{N}$:

$$I_k = \begin{cases} \{i \in \mathbb{N} \mid 0 \leqq i < k\} = \{0, 1, 2, \ldots, k-1\} & (k : \textit{finite}) \\ \mathbb{N} & (k = \infty) \end{cases}$$

as a set of verteces. Denote $\mathscr{D}(I_k)$ and $\mathscr{D}'(I_k)$ by $\mathscr{D}_k$ and $\mathscr{D}'_k$ respectively. And $\mathfrak{D}_k$ and $\mathfrak{D}'_k$ by $\mathfrak{D}(I_k)$ and $\mathfrak{D}'(I_k)$ respectively. We know easily that $\#\mathscr{D}_k = k^k$, $\#\mathscr{D}'_k = k!$ and $\#\mathfrak{D}'_k = p(k)$, where $p(k)$ is the number of partitions of k.

Fundamental problem

1. **Isomorphism problem.**
   (a) Classify dynamical graphs of size k, that is, determine the set $\mathfrak{D}_k$.
   (b) Determine at least the number $\delta_k = \#\mathfrak{D}_k$.
   (c) Detemine invariants necessary for the classification.
2. **Normal form problem.**
   (a) Determine a (canonical) system of representatives of isomorphism classes $\mathfrak{D}_k$ of dynamical graphs on I_k.
   (b) Establish the correspondence between the values of invariants and the system of representatives.

§ 2. Basic notions of dynamical graphs

Here, we summarize basic notions of dynamical graphs. Let $G = (V, E) = G(f)$ be a dynamical graph.

1. *Future of a vertex.* For a vertex $v \in V$, the set of all 'descendants' of v,

   $$V^+(v) = \{w \in V \mid w = f^a(v) \text{ for some } a \geq 0\},$$

   is called the *future* of v. For a subset $U \subset V$, $V^+(U) = \bigcup_{v \in U} V^+(v) \supset U$ is called the future of U.
2. *Past of a vertex.* For a vertex $v \in V$, the set of all 'ancesterts' of v,

   $$V^-(v) = \{w \in V \mid v = f^a(w) \text{ for some } a \geq 0\},$$

   is called the *past* of v. For a subset $U \subset V$, $V^-(U) = \bigcup_{v \in U} V^-(v) \supset U$ is called the past of U.
3. *Subgraph.* Let $G = (V, E)$ and $G' = (V', E')$ be dynamical graphs. G' is called a *dynamical subgraph* (or simply *subgraph*) of G, if $V' \subset V$, $E' \subset E$ and every edge in E' has verteces in V'.

4. For a set $U \subset V$, the dynamical graph $G' = (V', E')$ such that V' is the future $V^+(U)$ of U, is called *dynamical subgraph generated by* U and is denoted by $\langle U \rangle$. For $U = \{v\}$, $\langle U \rangle$ is sometimes denoted by $\langle v \rangle$.
5. *future graph, derived graph.* For any $n \geqq 0$, the set $f^n(V)$ is f-invariant, the subgraph $G(f|_{f^n(V)})$ generated by $f^n(V)$ is called the *n-th future graph*, and is denoted by $G^{(n)}$. And the first future graph $G^{(1)}$ is also denoted by G', and is called *derived graph* of G.
6. *connectedness.* If $V^+(v) \cap V^+(w) \neq \emptyset$ for any two vertices $v, w \in V$, the graph G is called *connected.* For example, the subgraph $\langle v \rangle$ generated by a single vertex v is connected.
7. *connected component.* A maximal connected dynamical subgraph $\mathscr{F}$ of G is called a *connected component.* The number $c = c(G)$ of connected components in G is called *connectivity* of G. $c = 1$ means that G is connected.

   For a vertex v or a connected subgraph G', the connected component $\mathscr{F}$ containing v or G' is called the connected component of v or G', denoted by $\mathscr{F}(v)$ or $\mathscr{F}(G')$ respectively.
8. *cycle.* If a subset $Z = \{v_1, \dots, v_p\}$ of (mutually different) verteces satisfies

$$f(v_i) = \begin{cases} v_{i+1} & (i < p) \\ v_1 & (i = p), \end{cases}$$

   then the subgraph $\langle Z \rangle$ is called a *cycle.* Sometimes the set Z itself is also called cycle. The number $p = p(C)$ is called the *period* of the cycle C. A cycle with a period 1 consists of a single vertex, and is also called a *fixed point.*

   Regarding a cycle Z as a dynamical graph, $V^-(v) = V^+(v) = Z$ for any vertex v of Z. Denote by Z_p the isomorphism class of a cycle of period p.
9. *limit cycle* and *gate.* A cycle Z of G is called a *limit cycle* of G, if its connected component is actually larger than Z itself, that is $V^-(Z) \supsetneqq Z$. For any vertex v of a limit cycle Z, its past $V^-(v)$ coincides with $V^-(Z) = \mathscr{F}(v)$.

   Let $W = \{w \in V \backslash Z \mid w \to v\}$, then its past $V^-(W)$ is called the *outer past* of v, and is denoted by $O^-(v) \supset W$. The vertex v is called the *gate* for $O^-(v)$, and the number $w(v) = \# W$ is called the *width* of the gate v. We write $W = \{w_1, \dots, w_{w(v)}\}$, then $\# O^-(v) = \sum_{i=1}^{w(v)} \# V^-(w_i)$ is called the *weight* of the gate v, and is denoted by $\mathrm{wt}(v)$. For a vertex $z \in O^-(v)$ we say that *z belongs to the gate v.*

   Sometimes a vertex v of Z is called a gate of weight 0, if $O^-(v) = \emptyset$.
10. A connected subgraph G' of a dynamical graph G is called *regular*, if it contains actually one cycle Z. G is called *regular*, if any connected components are regular.

   Then the set of connected components $\{\mathscr{F}_1, \dots, \mathscr{F}_c\}$ corresponds to the set of cycles $\{Z_1, \dots, Z_c\}$ such that Z_i corresponds to $\mathscr{F}_i = \mathscr{F}_i(Z_i)$ containing Z_i. For a cycle Z, we say that any subsets or verteces of $\mathscr{F}(Z)$ *belong to the cycle* Z or *the Z-family.*

11. A vertex sequence $\{v_0, \dots, v_k\}$ is called a *path* from v_0 to v_k, if $v_i = f(v_{i-1})$, *i.e.* $v_i \leftarrow v_{i-1}$ for $i = 1, \dots, k$. If any verteces are differrent, then this sequence is called *simple*, and a simple path is determined by the end verteces v_0 and v_k, so is denoted by $[v_0, v_k]$. The simple path $[v_0, v_k]$ are not a dynamical sugraph, but an oriented graph with k edges. We call k a *length* of the simple path $[v_0, v_k]$ denoted by $\mathrm{len}([v_0, v_k])$. If $w \in V^+(v)$, there exists a unique simple path $[v, w]$.

§ 2..1 Characteristic values for verteces

1. *life* of a vertex. We say that the *life* $\ell(v)$ of a vertex $v \in V$ is n, if there exists a natural number n such that $v \in f^a(V)$ $(0 \le a \le n-1)$ and $v \notin f^n(V)$. If such number n does not exist, then such vertex has an *infinite* life. Denote by $\mathscr{L}_n(G)$ the set of all veteces of life n, that is, $\mathscr{L}_n(G) = \{v \in V \mid \ell(v) = n\}$, then
$$\mathscr{L}_n(G) = f^{n-1}(V) \setminus f^n(V) \qquad (n \geqq 1).$$
2. *degree* of a vertex. For a vertex $v \in V$, the number of arrows whose target is v is caleld the *degree* of v, and is denoted by $\deg(v)$. That is, $\deg(v)$ is the number of the preimage of v by $f = f(G)$:
$$\deg(v) = \# f^{-1}(v) = \#\{w \in V \mid w \to v\}.$$
Remark. In an ordinary graph theory, this notion of degree is called the *indegree*. The reason why we choose this definition, outdegree of every vertex is 1 (constant) in our theory.
3. *height* of a vertex. Denote by $\mathscr{F}(Z) = \mathscr{F}(Z; G)$ the connected component of a cycle Z of G. For a vertex $v \in \mathscr{F}(Z)$, put
$$\mathrm{ht}(v) = \mathrm{ht}_Z(v) = \min\{n \geqq 0 \mid f^n(v) \in Z\},$$
and call it the *height* of v. Write the set of verteces of height k as $\mathscr{F}_k(Z) = \{v \in V \mid \mathrm{ht}_Z(v) = k\}$, then
$$\mathscr{F}(Z) = \bigcup_{k \geqq 0} \mathscr{F}_k(Z), \qquad \mathscr{F}_0(Z) = Z.$$
Points and subsets of $\mathscr{F}(Z)$ are called of *points and subsets of Z-family*, and we say that they belong to the cycle Z, and sometimes to the period $p = p(Z)$.
4. *distance* between verteces. Let v and w be verteces. We define the $d(v, w)$ as follows. Put $d(v, v) = 0$, and $d(v, w) = \infty$ if v and w belong to different components.

   Assume that v and $w (\neq v)$ belong to a same cycle.

   If $v \in V^+(w) \cup V^-(w)$, then $d(v, w) = \min(\mathrm{len}([v, w]), \mathrm{len}([w, v])$.

   If '$v \in V^-(w)$ and $v \notin V^+(w)$' or '$v \in V^+(w)$ and $v \notin V^-(w)$', then $d(v, w) = \mathrm{len}([v, w])$ or $= \mathrm{len}([w, v])$ respectively.

   If $v \notin V^+(w) \cup V^-(w)$ and v and w belong a same gate u, then there is a branch point u' such that $[u', u]$ is the intersection of $[v, u]$ and $[w, u]$, and then $d(v, w) = \mathrm{len}([v, u']) + \mathrm{len}([w, u'])$.

   If $v \notin V^+(w) \cup V^-(w)$ and v belongs to a gate u_v and w belongs to a gate $u_w (\neq u_v)$, then $d(v, w) = \mathrm{len}([v, u_v]) + \mathrm{len}([w, u_w]) + d(u_v, u_w)$.

§ 2..2 Some properties

We get the following three propositions easily.

Proposition 2. (i) *Any finite dynamical graphs are regular.*

(ii) *For a finite dynamical graph $G = (V, E)$, there exists a number N such that the N-th future graph $G^{(N)}$ is of cycle class.*

(iii) $s(G) = \sum_{v \in V} \deg v$.

Proposition 3. *Assume that $G = G(f)$ is a regular graph.*

(i) *Any vertex v of infite life belongs to some cycle, and the subgraph $\mathscr{L}_\infty(G)$ is of cycle class.*

(ii) *The followings are equivalent with each other.*

1. *f is bijective.*
2. *G is invertible.*
3. $\deg v = 1$ *for every vertex v, that is, there are no branch points.*
4. *G is of cycle class.*
5. *The size characteristic of G coincides with the period characteristic of $G : \mathbb{P}(G) = \mathbb{S}(G)$ (the definitions will be given in § 2..5).*

(iii) *If G is connected, then G has only one cycle.*

(iv) *If the degree of every vertex is 1, then G itself is a union of cycles. Such graph is called* of cycle class.

Remark. An at most countable (unoriented) graph $G = (V, E)$ is called *dynamicalizable*, if there exists a suitable assignment of the directions of edges which makes G dynamical. The resulting dynamical graph $\bar{G} = (V, \bar{E})$ is called a *dynamicalization* of the graph G.

Proposition 4. *Let $G = (V, E)$ be a finite unoriented graph. Then G is dynamicalizable, if and only if each connected component of G has only one cycle.*

A dynamicalization of G is determined by the assignment of directions of cycles of connected components, hence there are 2^c non-isomorphic dynamicalization of G, where c is the connectivity of G.

§ 2..3 Leaf, branch point and route

In the following, we assume that G is regular, otherwise stated.

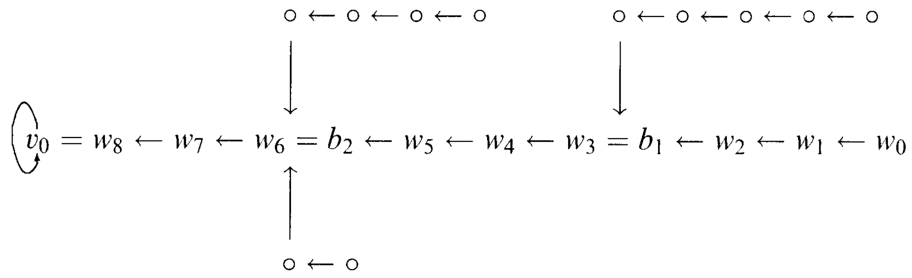
We say that a vertex v is a *branch point* if $\deg(v) > 1$, and v is a *leaf* if $\deg(v) = 0$, that is, $\ell(v) = 1$. Then we get easily the following, by computing the both sides of $s(G) = \#E$ separately w.r.t. degrees of verteces.

Proposition 5. *If G is finite, then the number of leaves equals with $\sum_b(\deg b - 1)$, where b runs over the set of branch points.*

Gates are branch points on cycles. Let Z be a cycle of G, v_0 be a gate of C, w_0 be a leaf belonging to this gate and be of height $h = \mathrm{ht}_Z(w_0)$.

Write the path $[w_0, v_0]$ from the leaf w_0 to the gate v_0 as $\{w_0, w_1, \ldots, w_h(= v_0)\}$, then it may have branch points $\{b_1, \ldots, b_k(= v_0)\}$ such that the path $[b_j, b_{j+1}]$ has no branch points other than the two end points. The sequence $\{w_0, b_1, \ldots, b_k(= v_0)\}$ is called a *route* from the leaf w_0 to the gate v_0.

For example, in the graph



$Z = \{v_0\}$ is a gate, w_0 is a leaf of height 8, $\{w_0, b_1, b_2, v_0\}$ is a route.

§ 2..4 Pseudotree and pseduoforest

A connected dynamical graph T is called *pseudotree*, if its cycle is a fixed point. A dynamical graph F is called *pseudoforest*, if any connected components are pseudotrees.

In a pseudotree T, the cycle consists of a single vertex v, this unique gate v is called a *root* of T. The weight $\mathrm{wt}(v)$ of this gate is called the *weight* of the pseudotree T denoted by $\mathrm{wt}(G)$. Then note $s(T) = \mathrm{wt}(T) + 1$.

Write connected components of a pseudoforest F as $T_1, \ldots, T_c$, then define the *weight* of the pseudoforest F as $\mathrm{wt}(F) = \sum_{i=1}^{c} \mathrm{wt}(T_i)$, where $c = c(F)$. Then note $\mathrm{wt}(F) = s(F) - c$.

For an integer $w \geqq 0$, denote by $\mathscr{T}_w$ and $\mathscr{F}_w$ the set of all pseudotrees and pseudoforests of weight w, and by $\mathfrak{T}_w$ and $\mathfrak{F}_w$ the sets of their isomorphism classes respectively. For integers $w, c \geqq 0$, denote by $\mathscr{F}_w^c$ the set of all pseudoforests of weight w and connectivity c, and by $\mathfrak{F}_w^c$ the set of its isomorphism classes. Pseudoforests which contain no cycles of weight 0 are called *regular pseudoforest* or *bonsai*. Denote by $\mathscr{F}_w'$ the set of all bonsai of weight w, and by $\mathfrak{F}_w'$ the set of its isomorphism classes.

Put $\tau_w = \#\mathfrak{T}_w$, $\phi_w' = \#\mathfrak{F}_w'$, $\phi_w'^{\,c} = \#\mathfrak{F}_w'^{\,c}$, then $\#\mathfrak{F}_w = \infty$, and $\phi_w'^{\,c} = 0$ if $c > w$.

Remark. In a regular dynamical graph, the subgraph $\langle v \rangle$ generated by a vertex $v \in V$ has no branch points outside its limit cycle. A pseudotree T is called *linear*, if the fixed point is the only one branch point.

§ 2..5 Invariants of dynamical graphs

For a connected graph G, we already know some invariants as follows.

1. The *size* $s(G) = \#V(G)$ is the number of verteces.
2. The *period* $p(G) = p(Z)$ is the period of the unique cycle Z in G.
3. Denote by $D_i = \#\mathscr{D}_i$ the number of verteces of degree i, where $\mathscr{D}_i = \{v \in V \mid \deg(v) = i\}$.

   The *maximal degree* $d(G)$ is the maximum of degrees of verteces, that is, $d(G) = \max\{i \mid D_G(i) \neq 0\}$.

   The D_0 is the number of leaves and $b(G) = \sum_{i \geqq 2} D_i$ is the number of branch points.

   There holds the degree equation $s(G) = \sum_{i \geqq 0} D_i = \sum_{i \geqq 0} iD_i$, from which Proposition 5 is easily obtained.

And let $\mathbb{D}_G = (D_0, D_1, D_2, D_3, \ldots) = \prod_{i \geq 0} D_i$. If $d(G)$ is finite, let $\mathbb{D}_G = (D_0, D_1, \ldots, D_{d(g)})$.

4. Denote by $L_i(G) = \#\mathscr{L}_i(G)$ the number of verteces of life i, where $\mathscr{L}_i = \{v \in V \mid \ell(v) = i\}$. Note that $\mathscr{L}_\infty(G) = Z$ and $L_\infty(G) = p(Z)$.
   The *finite maximal life* $\ell(G)$ is the maximum of finite lives of verteces, that is, $\ell(G) = \max\{i \in \mathbb{N} \mid L_i(G) \neq 0\}$. And let $\mathbb{L}_G = (L_0, L_1, L_2, \ldots; L_\infty)$ or $\mathbb{L}_G = (L_0, L_1, \ldots, L_{\ell(G)}; L_\infty)$ if $\ell(G)$ is finite.
5. The *gate number* $g = g(Z) = g(G)$ is the number of gates on the cycle. Denote by $\mathscr{G}(Z) = \mathscr{G}(G)$ the set of gates.
6. The *weight* $\mathrm{wt}(v)$ of a gate $v \in \mathscr{G}(Z)$ is $\#O^-(v)$, $b(v)$ is the number of branch points in $O^-(v)$, and $e(v)$ is the number of leaves in $O^-(v)$.
7. Denote by $H_i(G) = \#\mathscr{H}_i(G)$ the number of verteces of height i, where $\mathscr{H}_i = \{v \in V \mid \mathrm{ht}(v) = i\}$. Note that $\mathscr{H}_0(G) = Z$ and $H_0(G) = p(Z)$.
   The *maximal height* $\mathrm{ht}(G)$ is the maximum of heights of verteces, that is, $\mathrm{ht}(G) = \max\{i \in \mathbb{N} \mid H_i(G) \neq 0\}$. And let $\mathbb{H}_G = (H_0, H_1, H_2, \ldots)$, or $\mathbb{H}_G = (H_0, H_1, \ldots, H_{\mathrm{ht}(G)})$ if $\mathrm{ht}(G)$ is finite.

For a disconnected graph G, the connectivity $c = c(G)$ is essential. Assume that c is finite, and write connected components of G as $\{G^1, \ldots, G^c\}$. The corresponding invariants are denoted as c-vectors. For example,

1. the *size characteristic*: $\mathbb{S}_G = (s^1, \ldots, s^c)$, where $s^k = s(G^k)$. There holds that $s(G) = s^1 + \cdots + s^c$.
2. *period characteristic*: $\mathbb{P}_G = (p^1, \ldots, p^c)$, where $p^k = p(G^k)$.
   G is a pseudotree, if and only if $\mathbb{P}_G = (1, 1, \ldots, 1)$ denoted also by $1^{c(G)}$. G is of cycle class, if and only if $\mathbb{P}_G = \mathbb{S}_G$.
3. *maximal degree characteristic.* $M\mathbb{D}_G = (d^1, \ldots, d^c)$, where $d^k = d(G^k)$.
4. *maximal life characteristic.* $M\mathbb{L}_G = (\ell^1, \ldots, \ell^c)$, where $\ell^k = \ell(G^k)$.
5. *maximal height characteristic.* $M\mathbb{H}_G = (\mathrm{ht}^1, \ldots, \mathrm{ht}^c)$, where $\mathrm{ht}^k = \mathrm{ht}(G^k)$.

§ 2..6 Operations and deformations in dynamical graphs

First we fix a vertex set V and consider operations on $\mathscr{D}(V)$.

1. *Product.*
   The product of $F = G(f)$ and $G = G(g) \in \mathscr{D}(V)$ is defined as

$$FG = G(f)G(g) = G(fg).$$

   Denote $G(f)$ by $E = E(V)$, where f is given as $f(v) = v$ $(v \in V)$, then $\mathscr{D}(V)$ is a semigroup with the unit E, *i.e.*

$$G(f) = EG(f) = G(f)E.$$

   (Denote also $E_k = E(I_k)$.)

2. *Pointwise Sum and Pointwise Product.*
   On $V = I_k$, we can define operations by using operations in Z_k.
   For $F = G(f)$, $G = G(g) \in \mathscr{D}_k$, define the *pointwise sum* as

$$F \oplus G = G(f+g), \qquad (f+g)(i) = f(i) + g(i) \pmod{k}$$

and *poinwise product* as

$$F \otimes G = G(f \times g), \qquad (f \times g)(i) = f(i)g(i) \pmod{k}.$$

Next we define dynamical systems on different vertex sets.

1. *cup product*
Let V, V' a set of verteces, where verteces in different sets are considered different. Define the cup product $D \cup D'$ of $D \in \mathscr{D}(V)$ and $D' \in \mathscr{D}(V')$ as

$$\cup : \mathscr{D}(V) \times \mathscr{D}(V') \ni (D(f), D(f')) \to D(g) \in \mathscr{D}(V \cup V'),$$

$$g(v) = \begin{cases} f(v) & v \in V \\ f'(v) & v \in V'. \end{cases}$$

If graphs are realized on I_k, then the cup product is given as

$$\cup : \mathscr{D}_k \times \mathscr{D}_{k'} \ni (D(f), D(f')) \to D(g) \in \mathscr{D}_{k+k'}$$

$$g(i) = \begin{cases} f(i) & (0 \leqq i < k) \\ f'(i-k) + k & (k \leqq i < k + k'). \end{cases}$$

The notion of cup product can be factored to isomorphism classes, and be restricted to invertible graphs. Then we get easily

Proposition 6. *The cup product implies the isomorphism of the set of partitions of k to the isomorphism of invertible graphs of size k:*

$$\mathbb{P}(k) \ni (k_1, \dots, k_c) \leftrightarrow Z_{k_1} \cup \cdots \cup Z_{k_c} \in \mathfrak{D}_k,$$

where $k_1 + \cdots + k_c = k$, $(k_1 \leqq \cdots \leqq k_c)$.

Denote by mG the cup prduct of m copies of a graph G (or its isomorphism class), then for example

$$E_1 \cup E_1 = E_2 = 2E_1, \qquad E_1 \cup E_k = E_{k+1} = E_k \cup E_1 = (k+1)E_1.$$

2. *attachment.* Given a graph $G = G(f) \in \mathscr{D}(V)$, a vertex $v \in V$, a pseudotree $T = G(t) = (U, F) \in \mathscr{T}$ with the root $u \in U$, we define the dynamical graph $G \vee_v T = H = G(h) \in \mathscr{D}(V')$ by

$$h(w) = \begin{cases} f(w) & (w \in V) \\ t(w) & (w \in U \text{ and } t(w) \neq u) \\ v & (w \in U \text{ and } t(w) = u), \end{cases}$$

where V' is the disjoint sum of V and $U \backslash \{u\}$. We say that H is obtained from G attached by T at v. Then

$$s(H) = s(G) + s(T) - 1 = s(G) + \mathrm{wt}(T), \qquad c(H) = c(G), \qquad \mathbb{P}(H) = \mathbb{P}(G).$$

Any connected dynamical graphs can be expressed as a cycle Z with pseudotrees T_i attached at gates v_i $(i = 1, \dots, g)$: $G = Z \vee_{v_1} T_1 \cdots \vee_{v_g} T_g$. Then the size of G is given as

$$s(G) = p + \sum_{i=1}^{g} \mathrm{wt}(T_i),$$

where p is the period of the cycle.

Linear pseudotrees of weight w are isomorphic with each others, so denote their isomorphism class by L_w.

Any pseudotree T can be expressed as a linear pseudotree L_{w_0} with linear pseudotrees L_{w_i} attached at branch points v_i $(i = 1, \dots, b(T))$: $T = L_{w_0} \vee_{v_1} L_{w_1} \vee_{v_2} \cdots \vee_{v_b} L_{w_b}$. Then $s(T) = 1 + \sum_{i=0}^{b} w_i$.

In particular, $L_0 = K_1^0 = Z_1$, and attaching L_0 does not change any graph: $G \vee_v L_0 = G$ for any $v \in V$.

If v is a leaf of a linear pseudotree T_w, then $L_w \vee_v L_{w'} = L_{w+w'}$.

§3. Examples

In this section, fix a size k $(1 \leqq k \leqq \infty)$, and consider the set $\mathscr{D}_k$ of all dynamical graphs on I_k. Denote by $\mathscr{C}_k$ the set of all connected dynamical graphs on I_k, and by $\mathfrak{C}_k$ the set of its isomorphism classes.

§3..1 Elementary dynamical graphs

Let $P \in \mathbb{Z}[x]$ be a polynomial with integral coefficients, then define a mapping $P_k : I_k \to I_k$ as

$$P_k(i) = P(i) \qquad (\mathrm{mod}\ k),$$

and the corresponding dynamical graph $G(P_k)$ is also denoted by $G_k(P)$. Such dynamical graphs are called *elementary*. Denote by $\mathscr{E}_k$ the set of all elementary dynamical graphs on I_k.

We use the convention $P_\infty = P$. Note that $P_k = Q_k$ may happen even if $P \neq Q \in \mathbb{Z}[x]$.

Remark. $\mathfrak{C}_k$ plays an important role in the isomorphism problem. As for the normal form problem, we seek a representative of an isomorphism class in the region $\mathscr{E}_k$.

Here we list elementary graphs. Let a be a natural number.

1. The *Constant Graph* K_k^a stands for $G_k(P)$, where $P(x) = a$. K_k^a is a pseudotree, a is the root of degree k, and the other $k-1$ points are leaves.
2. The *Addition Graph* A_k^a stands for $G_k(P)$, where $P(x) = x + a$. Obviously, if k is finite, $A_k^{a+k} = A_k^a$ and every A_k^a is of cycle class, hence $A_k^a \in \mathscr{D}_k'$. Choose A_p^1 as the representative for the cycle Z_p of period p.

   Write A_∞^a as A^a, then its connectivity c is a, and

$$\mathscr{L}_n(A^a) = \{i \mid (n-1)a \leqq i < na\}, \qquad \mathscr{L}_\infty(A^a) = \varnothing, \qquad \mathbb{D}_{A^a} = (a, \infty),$$

   and $\ell(i) = [i/a] + 1$ for a vertex i of the addition graph A^a. A^a has no branch points.

$$A_k^1 \cup \cdots \cup A_k^1 (= mA_k^1) \cong A_{mk}^m$$

3. The *Multiplication Graph* M_k^a stands for $G_k(P)$, where $P(x) = ax$. Obviously, if k is finite, $M_k^{a+k} = M_k^a$. M_k^a is of cycle class, if and only if a and k are coprime, that is, $(a, k) = 1$.
4. The *Power Graph* P_k^a stands for $G_k(P)$, where $P(x) = x^a$.
5. The general *Polynomial Graph* $P_k^a(f)$ stands for $G_k(P)$, where $P(x) = f(x)$ is a polynomial in x of degree a.

General polynomial graphs can be represented as a finite products of addition graphs and multiplication graphs.

$A_k^0 = M_k^1 (= E_k)$ is the identity graph w.r.t. the pointwise product in $\mathscr{D}_k$.

Examples. $A_k^a A_k^b = A_k^b A_k^a = A_k^{a+b}$, $M_k^a M_k^b = M_k^b M_k^a = M_k^{ab}$, but in general, $M_k^a A_k^b \neq A_k^b M_k^a$, and they are no more addition graphs nor multiplication graphs. For example, both of

$$M_5^2 A_5^3 = P_5(2x+1) \neq A_5^3 M_5^2 = P_5(2x+3)$$

is isomorphic to M_5^2.

$$A_k^a \oplus A_k^b = P_k(2x+a+b), \qquad M_k^a \otimes M_k^b = P_k(abx^2).$$

$$K_k^a \oplus K_k^b = K_k^{a+b}, \qquad K_k^a \otimes K_k^b = K_k^{ab}.$$

$$G(f) = K_k^0 \oplus G(f), \qquad G(f) = K_k^1 \otimes G(f).$$

Unfortunately, threre are not sufficiently many elementary dynamical graphs for classification for large k. We can compute all dynamical graphs for small k. In fact,

Proposition 7. *As polynomial functions on* I_k $(k \leqq 10)$,

$$x^2 \equiv x \pmod 2, \qquad x^4 \equiv x^2 \pmod 4, \qquad x^7 \equiv x \pmod 7, \qquad x^5 \equiv x^3 \pmod 8,$$

$$x^3 \equiv x \ (\mathrm{mod}\ 3, \text{ or } 6), \qquad x^5 \equiv x \ (\mathrm{mod}\ 5, \text{ or } 10), \qquad x^8 \equiv x^2 \pmod 9.$$

Hence for example, there are at most 4^4 elementary graphs on I_4, since any polynomial functions on I_4 can be written as $f(x) = a_0 x^3 + a_1 x^2 + a_3 x + a_4$, $(a_i \in I_4)$.

§ 3..2 Realization of pseudotrees and pseudoforests

For an integer $a > 0$, let f be a mapping of I_k defined by $f(x) = \max\{x - a, 0\}$. The corresponding graph $G(f)$ is called a *subtraction graph*, denoted by D_k^a. Then D_k^1 is a linear pseudotree L_{k-1} of weight $k-1$.

For $a > 1$, D_∞^a is a pseudotree and is L_0 with a linear pseudotrees L_∞ attached at the root 0, *i.e.*

$$D^a = L_0 \vee_0 aL_\infty = L_\infty \vee_0 (a-1)L_\infty,$$

where the weight of 0 is a. (Note that the left hand side D^a is a dynamical graph, and the right hand side is its isomorphism class, but readers will not be confused.)

For finite k, D_k^a is a pseudotree and is L_0 with i linear pseudotrees L_{b+1} and $(a-i)L_b$ attached at the root 0, *i.e.*

$$D_k^a = L_0 \vee_0 (iL_{b+1} \cup (a-i)L_b) = L_{b+1} \vee_0 ((i-1)L_{b+1} \cup (a-i)L_b),$$

where $b = \left\lfloor \dfrac{k-1}{a} \right\rfloor$, $i = k-1-ab$, and the weight of 0 is a.

§ 3..3 p-ary pseudotree

Fix (p,ℓ) $(p > 1, \ell > 0)$, we define *p-ary pseudotrees* B_p^ℓ of length ℓ inductively on ℓ as follows: At first let $B_p^0 = L_0$, then put

$$B_p^{\ell+1} = L_0 \vee_0 p(L_1 \vee_1 B_p^\ell),$$

then

$$s(B_p^\ell) = \sum_{i=1}^{\ell} p^i = \frac{p^{\ell+1}-1}{p-1}, \qquad \mathrm{wt}(B_p^\ell) = p\frac{p^\ell - 1}{p-1}, \qquad D_0(B_p^\ell) = p^\ell.$$

In fact,

$$\mathrm{wt}(B_p^{\ell+1}) = p\left(1 + \frac{p^{\ell+1}-1}{p-1} - 1\right) = p\frac{p^{\ell+1}-1}{p-1}.$$

B_1^k is a linear pseudotree L_k of weight k, and B_2^k is called a binary pseudotree of length k.

The multiplication graph $M_{2^k}^2$ is expressed as $L_1 \vee_1 B_2^{k-1}$, and in general the multiplication graph $M_{p^k}^p$ is expressed as $L_0 \vee_0 (p-1)(L_1 \vee_1 B_p^{k-1})$. Its size can be computed as

$$s(M_{p^k}^p) = 1 + \mathrm{wt}(M_{p^k}^p) = 1 + (p-1)\left(1 + \frac{p^k - 1}{p-1} - 1\right) = p^k.$$

§ 3..4 Deformation of future graphs and derived graphs

Let $h, n > 0$ be integers. Assume that the size of the n-th future graph $G^{(n)}$ of $G = G(f) \in \mathscr{D}_h$ is positive, namely k. Denote by J_k the vertex set $f^n(I_h)$ of $G^{(n)}$, then J_k is f-invariant. Consider a different dynamical graph $G(g) \in \mathscr{D}_h$, then J_k is also $f^n g$-invariant.

Even if $G(g)$ is elementray, the dynamical graph on J_k corresponding to fg is not isomorphic to the original $G(f)$ in general.

There is a graph $H \in \mathscr{D}_k$ which is isomorphic to $G^{(n)}$, and an bijection $\varphi : I_k \to J_k$ such that $H = G(\varphi^{-1} f^n g \varphi)$. So we can get many explicit examples in $\mathscr{D}_k$.

In [6], we gave various examples on ten verteces, by using deformations of this type. There, we considered the reversed difference graph $R_2 = G(d)$ on 2 place numbers whose derived graph (the first future graph) R_2' is of size 10. So taking as above g additions and multiplications on I_{100}, we get various graphs $G(dg|_{R_2'})$ on $V(R_2') \cong I_{10}$. Even the connectivity of $G(dg|_{R_2'})$ is different from the one of R_2' in general.

Example. Let k a positive integer. A number x in $I = I_{k^2} = \{i \in \mathbb{N} \mid 0 \leqq i < k^2\}$ is uniquely expressed as $x = ak + b$, $(0 \leqq a, b < k)$. Define a mapping f of I to itself by $f(x) = |(ak+b) - (bk+a)| = (k-1)|a-b|$, then the subset $J = (k-1)I_k \subset I$ is f-invariant, and $G(f|_J)$ is the derived graph whose vertex set J is isomorphic with I_k. Hence, many examples in $\mathscr{D}_k$ are given from $G(f|_J)$ by deformations of this type. In this article, we use notations R_2^k for $G(f)$ on I_{k^2} and $R_k'(g)$ for $G(fg|_J)$ on $J(\cong I_k) \subset I$ with $g \in Map(I, I)$.

§4. $\mathfrak{D}_k$ and their representatives for $k \leqq 4$

Since $\mathscr{D}_k \cong Map(I_k, I_k)$ and the invertible $\mathscr{D}_k'$ corresponds to the set of bijections of I_k to itself, $\#\mathscr{D}_k = k^k$ and $\#\mathfrak{D}_k \geqq k^k/k!$. Hence $\delta_{10} = \#\mathfrak{D}_{10} \geqq 2756$. It is too large for listing all members of $\mathfrak{D}_{10}$.

However we will try it for small k, namely $k \leqq 4$.

It is obvious that $\mathfrak{D}_1 = \mathscr{D}_1 = \{Z_1 = L_0\}$ and $\delta_1 = 1$.

For $k = 2$, $\mathfrak{D}_2 = \{Z_2, L_1; 2Z_1\}$ and $\delta_2 = 3$. We can give their representatives by elementary graphs, such as

$$A_2^1 \in Z_2, \qquad M_2^0 = K_2^0 \in L_1; \qquad A_2^0 = M_2^1 \in 2Z_1,$$

where the semicolon seperates w.r.t. the connectivity.

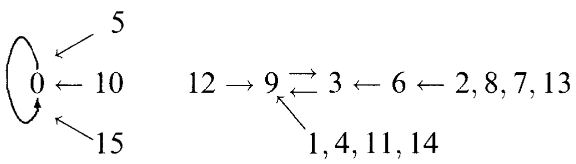
For $k = 3$, $\mathfrak{D}_3 = \{Z_3, L_2, Z_2 \vee_0 L_1, L_1 \vee_0 L_1 = Z_0 \vee_0 2L_1; Z_1 \cup Z_2, Z_1 \cup L_1; 3Z_1\}$ and $\delta_3 = 7$. We can give their representatives by elementary graphs, such as

$$A_3^1 \in Z_3, \qquad P_3(x^2+1) \in L_2, \qquad P_3(x^2+2) \in Z_2 \vee_0 L_1, \qquad M_3^0 \in L_1 \vee_0 L_1;$$
$$M_3^2 \in Z_1 \cup Z_2, \qquad P_3(x^2) \in Z_1 \cup L_1; \qquad A_3^0 \in 3Z_1.$$

For $k = 4$, $\mathfrak{D}_4 = \{Z_4, Z_3 \vee_0 L_1, Z_2 \vee_0 L_2, Z_2 \vee_0 2L_1, Z_2 \vee_0 L_1 \vee_1 L_1, L_3, L_2 \vee_0 L_1, L_2 \vee_1 L_1, Z_1 \vee_0 3L_1; Z_3 \cup Z_1, 2Z_2, Z_2 \cup L_1, 2L_1, Z_1 \cup (Z_2 \vee_0 L_1), Z_2 \cup L_1, Z_1 \cup L_2; 2Z_1 \cup L_1, Z_2 \cup 2Z_1; 4Z_1\}$ and $\delta_4 = 18$. In this case, we cannot give all their representatives by elementary graphs. In fact,

$A_4^1 \in Z_4$, $R_4'(x+4) \in Z_3 \vee_0 L_1$, $P_4(x^3+1) \in Z_2 \vee_0 L_2$, $R_4'(x^2+4) \in Z_2 \vee_0 2L_1$,
$P_4(x^2+1) \in Z_2 \vee_0 L_1 \vee_1 L_1$, $R_4'(x+6) \in L_3$, $P_4(3x^2+2) \in L_2 \vee_0 L_1$,
$M_4^2 \in L_2 \vee_1 L_1 = L_1 \vee_1 B_2^1$, $M_4^0 \in Z_1 \vee_0 3L_1 = B_3^1$;
$R_4'(x+10) \in Z_3 \cup Z_1$, $A_4^2 \in 2Z_2$, $P_4(x^3+2) \in Z_2 \cup L_1$,
$R_4' \in Z_1 \cup (Z_2 \vee_0 L_1)$, $P_4(x^2) \in 2L_1$, $R_4'(x+1) \in Z_1 \cup L_2$,
$M_4^3 \in Z_2 \cup 2Z_1$, $P_4(x^5) \in 2Z_1 \cup L_1$; $A_4^0 = M_4^1 \in 4L_1$.

For calculations of $R_4'(g)$, here we draw the graph R_4^2 in an abbreviated form:



We can show that other elementary graphs on I_4 are isomorphic with some graph drawn above as elementary graphs.

Thus explicit listing of isomorphism classes ($\mathfrak{D}_k$) for higher k seems very cumbersome and tedious, so in the following we will compute only the class number δ_k for $k \leqq 10$.

§ 5. Plan for computation

Now we start the computation of δ_k $(k \leqq 10)$.
Here we summarize subfamilies of $\mathscr{D}_k$ defined before:

$$\mathscr{D}'_k = \{D \in \mathscr{D}_k \mid D : \text{invertible}\} \subset \mathscr{D}_k$$

$$\mathscr{C}_k = \{D \in \mathscr{D}_k \mid D : \text{connected}\} \subset \mathscr{D}_k$$

$$\bigcup$$

$$\mathscr{T}_{k-1} = \{D \in \mathscr{C}_k \mid p(D) = 1\}.$$

They are compatible with the equivalence relation given by isomorphisms, so

$$\mathfrak{D}_k \supset \mathfrak{D}'_k, \qquad \mathfrak{D}_k \supset \mathfrak{C}_k \supset \mathfrak{T}_{k-1}.$$

In the preceding section, we already know the class numbers $\tau_w = \#\mathfrak{T}_w$ $(w \leqq 3)$, $\gamma_k = \#\mathfrak{C}_k$ and $\delta_k = \#\mathfrak{D}_k$ $(k \leqq 4)$ as

$$\delta_1 = \gamma_1 = \tau_0 = 1, \qquad \delta_2 = 3 > \gamma_2 = 2 > \tau_1 = 1,$$

$$\delta_3 = 7 > \gamma_3 = 4 > \tau_2 = 2, \qquad \delta_4 = 18 > \gamma_4 = 9 > \tau_3 = 4.$$

In the following, we will determine τ_w $(w \leqq 9)$, in § 6., γ_k in § 7. and δ_k $(k \leqq 10)$ in § 8. inductively.

§ 5..1 Partitions

Here we give the class number $\delta'_k = \#\mathfrak{D}'_k$ explicitly. By Proposition 8, $\delta'_k = \#\mathbb{P}(k)$, where $\mathbb{P}(k)$ is the set of all partitions of k. They are well-known as

Proposition 8.

k	1	2	3	4	5	6	7	8	9	10
$p(k)$	1	2	3	5	7	11	15	22	30	42

Since we use the facts on the partitions in the computations of class numbers, we give its brief review. A partition of k is given as $\Bbbk = (k_1, \dots, k_r)$, $k_1 \leqq \cdots \leqq k_r$, $1 \leqq r \leqq k$. Denote $p(k,r) = \#\mathbb{P}(k,r)$, where $\mathbb{P}(k,r)$ is the set of partitions of k to r numbers, then

$$p(k) = \sum_{r=1}^{k} p(k,r), \qquad p(k,k) = p(k,1) = 1,$$

and the recursion formula

$$p(k,r) = p(k-r,r) + p(k-1,r-1)$$

holds under the convention $p(k,r) = 0$ for $r > k$.

Then we get the table of $p(k,r)$:

$k\backslash r$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0
4	1	2	1	1	0	0	0	0	0	0
5	1	2	2	1	1	0	0	0	0	0
6	1	3	3	2	1	1	0	0	0	0
7	1	3	4	3	2	1	1	0	0	0
8	1	4	5	5	3	2	1	1	0	0
9	1	4	7	6	5	3	2	1	1	0
10	1	5	8	9	7	5	3	2	1	1

For example, $p(10) = 1+5+8+9+7+5+3+2+1+1 = 42$.

In the computations, we alwalys consult this table and use the other expression of a partition $\Bbbk$ as $\Bbbk = (1^{i_1}, 2^{i_2}, \ldots, k^{i_k})$, $0 \leqq i_j \leqq k$, $\sum_{j=1}^{k} j i_j = k$.

§6. Pseudotrees

In this section, we will show the following by induction on k.

Theorem 1.

k	0	1	2	3	4	5	6	7	8	9
τ_k	1	1	2	4	9	20	48	115	286	719

Let v_0 be the unique gate of a pseudotree T, $w = w(v_0)$ be the width of v_0 $(1 \leqq w \leqq k = \mathrm{wt}(v_0))$, and $\Bbbk$ be a partition of k to w numbers, that is,

$$\Bbbk = (1^{i_1}, 2^{i_2}, \ldots, k^{i_k}), \qquad k = \sum_{j=1}^{k} j i_j, \qquad w = \sum_{j=1}^{k} i_j, \quad 0 \leqq i_j \leqq k.$$

Denote by $\tau(\Bbbk)$ the class number of pseudotrees obtained from the fixed point $Z_1 = L_0$ attached at v_0, by the cup product of w pseudotrees among which there are i_j pseudotrees of weight j. Then

$$\tau_k = \sum_{\Bbbk \in \mathbb{P}(k)} \tau(\Bbbk) \qquad \text{and} \qquad \tau(\Bbbk) = \prod_{j=1}^{k} {}_{\tau_{j-1}}H_{i_j},$$

where ${}_sH_r = {}_{s+r-1}C_r$ is the number of ways of choosing r elements allowing repetition from a set of s elements.

Note. ${}_1H_r = 1$, ${}_tH_0 = 1$, ${}_tH_1 = t$. $\tau_0 = \tau_1 = 1$ is obvious.

For $k = 2$, we get

w	$\mathbb{k}$	$\tau(\mathbb{k})$
2	(1^2)	${}_{\tau_0}H_2 = 1$
1	(2)	$\tau_{2-1} = 1$

where $\mathbb{k} \in \mathbb{P}(2, w)$, and $\tau_2 = 1 + 1 = 2$.

For $k = 3$, we get

w	$\mathbb{k}$	$\tau(\mathbb{k})$
3	(1^3)	${}_{\tau_0}H_3 = 1$
2	$(1, 2)$	$\tau_{1-1}\tau_{2-1} = 1 \cdot 1 = 1$
1	(3)	$\tau_{3-1} = \tau_2 = 2$

where $\mathbb{k} \in \mathbb{P}(3, w)$, and $\tau_3 = 1 + 1 + 2 = 4$.

Remark. These results coincide with the ones in §5..

For $k = 4$, we get

w	$\mathbb{k}$	$\tau(\mathbb{k})$
4	(1^4)	${}_{\tau_0}H_4 = 1$
3	$(1^2, 2)$	${}_{\tau_0}H_2\tau_{2-1} = 1 \cdot 2 = 1$
2	$(1, 3)$	$\tau_{1-1}\tau_{3-1} = 1 \cdot 2 = 2$
2	(2^2)	${}_{\tau_{1-1}}H_2 = {}_1H_2 = {}_2C_2 = 1$
1	(4)	$\tau_{4-1} = \tau_3 = 4$

where $\mathbb{k} \in \mathbb{P}(4, w)$, and $\tau_4 = 1 + 1 + 2 + 1 + 4 = 9$.

For $k = 5$, we get

w	$\mathbb{k}$	$\tau(\mathbb{k})$
5	(1^5)	${}_{\tau_0}H_5 = 1$
4	$(1^3, 2)$	${}_{\tau_0}H_3\tau_{2-1} = 1 \cdot 1 = 1$
3	$(1^2, 3)$	${}_{\tau_0}H_2\tau_{3-1} = 1 \cdot 2 = 2$
3	$(1, 2^2)$	$\tau_{1-1}{}_{\tau_{2-1}}H_2 = 1 \cdot 1 = 1$
2	$(1, 4)$	$\tau_{1-1}\tau_{4-1} = 1 \cdot 4 = 4$
2	$(2, 3)$	$\tau_{2-1}\tau_{3-1} = 1 \cdot 2 = 2$
1	(5)	$\tau_{5-1} = \tau_4 = 9$

where $\mathbb{k} \in \mathbb{P}(5, w)$, and $\tau_5 = 1 + 1 + 2 + 1 + 4 + 2 + 9 = 20$.

For $k = 6$, we get

w	$\Bbbk$	$\tau(\Bbbk)$
6	(1^6)	${}_{\tau_0}H_6 = 1$
5	$(1^4, 2)$	${}_{\tau_0}H_4\tau_1 = 1 \cdot 1 = 1$
4	$(1^3, 3)$	${}_{\tau_0}H_3\tau_2 = 1 \cdot 2 = 2$
4	$(1^2, 2^2)$	${}_{\tau_0}H_2 \cdot {}_{\tau_1}H_2 = 1 \cdot 1 = 1$
3	$(1^2, 4)$	${}_{\tau_0}H_2\tau_3 = 1 \cdot 4 = 4$
3	$(1, 2, 3)$	$\tau_0\tau_1\tau_2 = 1 \cdot 1 \cdot 2 = 2$
3	(2^3)	${}_{\tau_1}H_3 = 1$
2	$(1, 5)$	$\tau_0\tau_4 = 1 \cdot 9 = 9$
2	$(2, 4)$	$\tau_1\tau_3 = 1 \cdot 4 = 4$
2	(3^2)	${}_{\tau_2}H_2 = {}_2H_2 = {}_3C_2 = 3$
1	(6)	$\tau_{6-1} = \tau_5 = 20$

where $\Bbbk \in \mathbb{P}(6, w)$, and $\tau_6 = 1 + 1 + 2 + 1 + 4 + 2 + 1 + 9 + 4 + 3 + 20 = 48$.

For $k = 7$, we get

w	$\Bbbk$	$\tau(\Bbbk)$
7	(1^7)	${}_{\tau_0}H_7 = 1$
6	$(1^5, 2)$	${}_{\tau_0}H_5\tau_1 = 1 \cdot 1 = 1$
5	$(1^4, 3)$	${}_{\tau_0}H_4\tau_2 = 1 \cdot 2 = 2$
5	$(1^3, 2^2)$	${}_{\tau_0}H_3 \cdot {}_{\tau_1}H_2 = 1 \cdot 1 = 1$
4	$(1^3, 4)$	${}_{\tau_0}H_3\tau_3 = 1 \cdot 4 = 4$
4	$(1^2, 2, 3)$	${}_{\tau_0}H_2 \cdot \tau_1\tau_2 = 1 \cdot 1 \cdot 2 = 2$
4	$(1, 2^3)$	$\tau_0 {}_{\tau_1}H_3 = 1$
3	$(1, 2, 4)$	$\tau_0\tau_1\tau_3 = 1 \cdot 1 \cdot 4 = 4$
3	$(1, 3^2)$	$\tau_0 {}_{\tau_2}H_2 = {}_2H_2 = 3$
3	$(2^2, 3)$	${}_{\tau_1}H_2\tau_2 = 2$
3	$(1^2, 5)$	${}_{\tau_0}H_2\tau_4 = 9$
2	$(1, 6)$	$\tau_0\tau_5 = 1 \cdot 20 = 20$
2	$(2, 5)$	$\tau_1\tau_4 = 1 \cdot 9 = 9$
2	$(3, 4)$	$\tau_2\tau_3 = 2 \cdot 4 = 8$
1	(7)	$\tau_6 = 48$

where $\Bbbk \in \mathbb{P}(7, w)$, and $\tau_7 = 1 + 1 + 2 + 1 + 4 + 2 + 1 + 4 + 3 + 2 + 9 + 20 + 9 + 8 + 48 = 115$.

For $k = 8$, we get

w	$\Bbbk$	$\tau(\Bbbk)$
8	(1^8)	${}_{\tau_0}H_8 = 1$
7	$(1^6, 2)$	${}_{\tau_0}H_7\tau_1 = 1$
6	$(1^5, 3)$	${}_{\tau_0}H_5\tau_2 = 2$
6	$(1^4, 2^2)$	${}_{\tau_0}H_{2\tau_1}H_2 = 1$
5	$(1^4, 4)$	${}_{\tau_0}H_4\tau_3 = 4$
5	$(1^3, 2, 3)$	${}_{\tau_0}H_3\tau_1\tau_2 = 2$
5	$(1^2, 2^3)$	${}_{\tau_0}H_{2\tau_1}H_3 = 1$
4	$(1^3, 5)$	${}_{\tau_0}H_3\tau_4 = 9$
4	$(1^2, 2, 4)$	${}_{\tau_0}H_2\tau_1\tau_3 = \tau_3 = 4$
4	$(1^2, 3^2)$	${}_{\tau_0}H_{2\tau_2}H_2 = {}_2H_2 = 3$
4	$(1, 2^2, 3)$	$\tau_{0\tau_1}H_2\tau_2 = 2$
4	(2^4)	${}_{\tau_1}H_4 = 1$
3	$(1^2, 6)$	${}_{\tau_0}H_2\tau_5 = 20$
3	$(1, 2, 5)$	$\tau_0\tau_1\tau_4 = 9$
3	$(1, 3, 4)$	$\tau_0\tau_2\tau_3 = 2 \cdot 4 = 8$
3	$(2^2, 4)$	${}_{\tau_1}H_2\tau_3 = 4$
3	$(2, 3^2)$	$\tau_{1\tau_2}H_2 = {}_2H_2 = 3$
2	$(1, 7)$	$\tau_0\tau_6 = 48$
2	$(2, 6)$	$\tau_1\tau_5 = 20$
2	$(3, 5)$	$\tau_2\tau_4 = 2 \cdot 9 = 18$
2	(4^2)	${}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
1	(8)	$\tau_7 = 115$

where $\Bbbk \in \mathbb{P}(8, w)$, and $\tau_8 = 5 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 8 + 2 \cdot 9 + 10 + 18 + 2 \cdot 20 + 48 + 115 = 286$.

For $k=9$, we get

w	$\Bbbk$	$\tau(\Bbbk)$
9	(1^9)	${}_{\tau_0}H_9 = 1$
8	$(1^7, 2)$	${}_{\tau_0}H_7 = 1$
7	$(1^6, 3)$	${}_{\tau_0}H_6\tau_2 = 2$
7	$(1^5, 2^2)$	${}_{\tau_0}H_5\,{}_{\tau_1}H_2 = 1$
6	$(1^5, 4)$	${}_{\tau_0}H_5\tau_3 = 4$
6	$(1^4, 2, 3)$	${}_{\tau_0}H_4\tau_1\tau_2 = 2$
6	$(1^3, 2^3)$	${}_{\tau_0}H_3\,{}_{\tau_1}H_3 = 1$
5	$(1^4, 5)$	${}_{\tau_0}H_4\tau_4 = 9$
5	$(1^3, 2, 4)$	${}_{\tau_0}H_3\tau_1\tau_3 = 4$
5	$(1^3, 3^2)$	${}_{\tau_0}H_3\,{}_{\tau_2}H_2 = {}_2H_2 = 3$
5	$(1^2, 2^2, 3)$	${}_{\tau_0}H_2\,{}_{\tau_1}H_2\tau_2 = 2$
5	$(1, 2^4)$	$\tau_0\,{}_{\tau_1}H_4 = 1$
4	$(1^3, 6)$	${}_{\tau_0}H_3\tau_5 = 20$
4	$(1^2, 2, 5)$	${}_{\tau_0}H_2\tau_1\tau_4 = 9$
4	$(1^2, 3, 4)$	${}_{\tau_0}H_2\tau_2\tau_3 = 2\cdot 4 = 8$
4	$(1, 2^2, 4)$	$\tau_0\,{}_{\tau_1}H_2\tau_3 = 4$
4	$(1, 2, 3^2)$	$\tau_0\tau_1\,{}_{\tau_2}H_2 = {}_2H_2 = 3$
4	$(2^3, 3)$	${}_{\tau_1}H_3\tau_2 = 2$
3	$(1^2, 7)$	${}_{\tau_0}H_2\tau_6 = 48$
3	$(1, 2, 6)$	$\tau_0\tau_1\tau_5 = 20$
3	$(1, 3, 5)$	$\tau_0\tau_2\tau_4 = 2\cdot 9 = 18$
3	$(1, 4^2)$	$\tau_0\,{}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
3	$(2, 3, 4)$	$\tau_1\tau_2\tau_3 = 2\cdot 4 = 8$
3	$(2^2, 5)$	${}_{\tau_1}H_2\tau_4 = 9$
3	(3^3)	${}_{\tau_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
2	$(1, 8)$	$\tau_0\tau_7 = 115$
2	$(2, 7)$	$\tau_1\tau_6 = 48$
2	$(3, 6)$	$\tau_2\tau_5 = 2\cdot 20 = 40$
2	$(4, 5)$	$\tau_3\tau_4 = 4\cdot 9 = 36$
1	(9)	$\tau_8 = 286$

where $\Bbbk \in \mathbb{P}(9, w)$, and $\tau_9 = 5 \cdot 1 + 4 \cdot 2 + 2 \cdot 3 + 4 \cdot 4 + 2 \cdot 8 + 3 \cdot 9 + 10 + 18 + 2 \cdot 20 + 36 + 40 + 2 \cdot 48 + 115 + 286 = 719$.

§ 6..1 Bonsai

From Theorem 3, we can compute $\phi(s) = \#\mathfrak{F}(s)$ and $\phi_k'^{\,c} = \#\mathfrak{F}_k'^{\,c}$ similarly as above, where $\mathfrak{F}(s)$ is the set of isomorphism classes of pseudoforests of size s and $\mathfrak{F}_k'^{\,c}$ is the set of bonsai of weight k and connectivity c.

Let $\Bbbk \in \mathbb{P}(k, c)$ and denote by $\phi'(\Bbbk)$ the class number of bonsai with weight characteristic $\Bbbk$, then

$$\phi_k'^{\,c} = \sum_{\Bbbk \in \mathbb{P}(k,c)} \phi'(\Bbbk) \qquad \text{and} \qquad \phi'(\Bbbk) = \prod_{j=1}^{k} {}_{\tau_j}H_{i_j}.$$

Then we get the following table of $\phi_k'^{\,c}$ inductively on k.

Proposition 9.

$k \backslash c$	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0
3	4	2	1	0	0	0	0	0	0
4	9	7	2	1	0	0	0	0	0
5	20	17	7	2	1	0	0	0	0
6	48	48	21	7	2	1	0	0	0
7	115	124	60	21	7	2	1	0	0
8	286	336	181	65	21	7	2	1	0
9	719	892	336	197	65	21	7	2	1

Before the proof of the proposition, we remark the following.
Divide the set $\mathfrak{F}(s)$ by the number of trival cycles Z_0, then

$$\begin{aligned}\mathfrak{F}(s) &= \{sZ_0\} \cup \left(\bigcup_{k=1}^{s} \bigcup_{c=1}^{k-1} ((s-k)Z_0 \cup \mathfrak{F}_{k-c}'^{\,c}) \right) \\ &= \{sZ_0\} \cup \left(\bigcup_{k=1}^{s-1} \bigcup_{c=1}^{k-1} ((s-k)Z_0 \cup \mathfrak{F}_{k-c}'^{\,c}) \right) \cup \left(\bigcup_{c=1}^{s-1} \mathfrak{F}_{s-c}'^{\,c} \right) \\ &= \mathfrak{F}(s-1) \cup \left(\bigcup_{c=1}^{s-1} \mathfrak{F}_{s-c}'^{\,c} \right).\end{aligned}$$

Hence,

$$\phi(s) = \phi(s-1) + \sum_{c=1}^{s-1} \phi_{s-c}'^{\,c}, \qquad \phi(1) = 1,$$

so

s	1	2	3	4	5	6	7	8	9	10
$\phi(s)$	1	2	4	9	20	48	115	286	719	1842

These values are identical with τ_s. This fact has an intrinsic reasoning. Consider the mapping

$$\mathscr{F}(s) \to \mathscr{T}_s, \qquad F \mapsto Z_1 \vee_0 F$$

given by attaching F at the fixed point of the cycle Z_1, then this gives a bijection.

Now we start to compute $\phi_k'^{\,c}$ inductively.

Note. $\phi_k{}^c = 0$ for $c > k$ and $\phi_k'^{\,1} = \tau_k$. $\phi_1'^{\,1} = 1$ is obvious.

For $k = 2$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
2	(1^2)	${}_{\tau_1}H_2 = 1$
1	(2)	$\tau_2 = 2$

where $\Bbbk \in \mathbb{P}(2, c)$, and $\phi_2'^{\,2} = 1$, $\phi_2'^{\,1} = 2$.

For $k = 3$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
3	(1^3)	${}_{\tau_1}H_3 = 1$
2	$(1, 2)$	$\tau_1\tau_2 = 2$
1	(3)	$\tau_3 = 4$

where $\Bbbk \in \mathbb{P}(3, c)$, and $\phi_3'^{\,3} = 1$, $\phi_3'^{\,2} = 2$, $\phi_3'^{\,1} = 4$.

For $k = 4$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
4	(1^4)	${}_{\tau_1}H_4 = 1$
3	$(1^2, 2)$	${}_{\tau_1}H_2\tau_2 = 1 \cdot 2 = 2$
2	$(1, 3)$	$\tau_1\tau_3 = 1 \cdot 4 = 4$
2	(2^2)	${}_{\tau_2}H_2 = {}_2H_2 = 3$
1	(4)	$\tau_4 = 9$

where $\Bbbk \in \mathbb{P}(4, c)$, and $\phi_4'^{\,4} = 1$, $\phi_4'^{\,3} = 2$, $\phi_4'^{\,2} = 4 + 3 = 7$, $\phi_4'^{\,1} = 9$.

For $k = 5$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
5	(1^5)	${}_{\tau_1}H_5 = 1$
4	$(1^3, 2)$	${}_{\tau_1}H_3\tau_2 = 1 \cdot 2 = 2$
3	$(1^2, 3)$	${}_{\tau_1}H_2\tau_3 = 1 \cdot 4 = 4$
3	$(1, 2^2)$	${\tau_1}_{\tau_2}H_2 = 1 \cdot 3 = 3$

c	$\Bbbk$	$\phi'(\Bbbk)$
2	$(1, 4)$	$\tau_1\tau_4 = 1 \cdot 9 = 9$
2	$(2, 3)$	$\tau_2\tau_3 = 2 \cdot 4 = 8$
1	(5)	$\tau_5 = 20$

where $\Bbbk \in \mathbb{P}(5, c)$, and ${\phi'_5}^5 = 1$, ${\phi'_5}^4 = 2$, ${\phi'_5}^3 = 4 + 3 = 7$, ${\phi'_5}^2 = 9 + 8 = 17$, ${\phi'_5}^1 = 20$.

For $k = 6$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
6	(1^6)	${}_{\tau_1}H_6 = 1$
5	$(1^4, 2)$	${}_{\tau_1}H_4\tau_2 = 1 \cdot 2 = 2$
4	$(1^3, 3)$	${}_{\tau_1}H_3\tau_3 = 1 \cdot 4 = 4$
4	$(1^2, 2^2)$	${}_{\tau_1}H_2 \cdot {}_{\tau_2}H_2 = 1 \cdot 3 = 3$
3	$(1^2, 4)$	${}_{\tau_1}H_2\tau_4 = 1 \cdot 9 = 9$
3	$(1, 2, 3)$	$\tau_1\tau_2\tau_3 = 1 \cdot 2 \cdot 4 = 8$

c	$\Bbbk$	$\phi'(\Bbbk)$
3	(2^3)	${}_{\tau_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
2	$(1, 5)$	$\tau_1\tau_5 = 1 \cdot 20 = 20$
2	$(2, 4)$	$\tau_2\tau_4 = 2 \cdot 9 = 18$
2	(3^2)	${}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
1	(6)	$\tau_6 = 48$

where $\Bbbk \in \mathbb{P}(6, c)$, and ${\phi'_6}^6 = 1$, ${\phi'_6}^5 = 2$, ${\phi'_6}^4 = 4 + 3 = 7$, ${\phi'_6}^3 = 9 + 8 + 4 = 21$, ${\phi'_6}^2 = 20 + 18 + 10 = 48$, ${\phi'_6}^1 = 48$.

For $k = 7$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
7	(1^7)	${}_{\tau_1}H_7 = 1$
6	$(1^5, 2)$	${}_{\tau_1}H_5\tau_2 = 1 \cdot 2 = 2$
5	$(1^4, 3)$	${}_{\tau_1}H_4\tau_3 = 1 \cdot 4 = 4$
5	$(1^3, 2^2)$	${}_{\tau_1}H_3 \cdot {}_{\tau_2}H_2 = 1 \cdot 3 = 3$
4	$(1^3, 4)$	${}_{\tau_1}H_3\tau_4 = 1 \cdot 9 = 9$
4	$(1^2, 2, 3)$	${}_{\tau_1}H_2 \cdot \tau_2\tau_3 = 1 \cdot 2 \cdot 4 = 8$
4	$(1, 2^3)$	${\tau_1}_{\tau_2}H_3 = 1 \cdot {}_2H_3 = {}_4C_3 = 4$
3	$(1^2, 5)$	${}_{\tau_1}H_2\tau_5 = 20$

c	$\Bbbk$	$\phi'(\Bbbk)$
3	$(1, 2, 4)$	$\tau_1\tau_2\tau_4 = 1 \cdot 2 \cdot 9 = 18$
3	$(1, 3^2)$	${\tau_1}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
3	$(2^2, 3)$	${}_{\tau_2}H_2\tau_3 = 3 \cdot 4 = 12$
2	$(1, 6)$	$\tau_1\tau_6 = 1 \cdot 48 = 48$
2	$(2, 5)$	$\tau_2\tau_5 = 2 \cdot 20 = 40$
2	$(3, 4)$	$\tau_3\tau_4 = 4 \cdot 9 = 36$
1	(7)	$\tau_7 = 115$

where $\Bbbk \in \mathbb{P}(7, c)$, and ${\phi'_7}^7 = 1$, ${\phi'_7}^6 = 2$, ${\phi'_7}^5 = 4 + 3 = 7$, ${\phi'_7}^4 = 9 + 8 + 4 = 21$, ${\phi'_7}^3 = 20 + 18 + 10 + 12 = 60$, ${\phi'_7}^2 = 48 + 40 + 36 = 124$, ${\phi'_7}^1 = 115$.

For $k = 8$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
8	(1^8)	${}_{\tau_1}H_8 = 1$
7	$(1^6, 2)$	${}_{\tau_1}H_6\tau_2 = 1 \cdot 2 = 2$
6	$(1^5, 3)$	${}_{\tau_1}H_5\tau_3 = 4$
6	$(1^4, 2^2)$	${}_{\tau_1}H_4{}_{\tau_2}H_2 = 3$
5	$(1^4, 4)$	${}_{\tau_1}H_4\tau_4 = 9$
5	$(1^3, 2, 3)$	${}_{\tau_1}H_3\tau_2\tau_3 = 2 \cdot 4 = 8$
5	$(1^2, 2^3)$	${}_{\tau_1}H_2{}_{\tau_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
4	$(1^3, 5)$	${}_{\tau_1}H_3\tau_5 = 20$
4	$(1^2, 2, 4)$	${}_{\tau_1}H_2\tau_2\tau_4 = 2 \cdot 9 = 18$
4	$(1^2, 3^2)$	${}_{\tau_1}H_2{}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
4	$(1, 2^2, 3)$	$\tau_1{}_{\tau_2}H_2\tau_3 = 1 \cdot 3 \cdot 4 = 12$
4	(2^4)	${}_{\tau_2}H_4 = {}_2H_4 = {}_5C_4 = 5$
3	$(1^2, 6)$	${}_{\tau_1}H_2\tau_6 = 48$
3	$(1, 2, 5)$	$\tau_1\tau_2\tau_5 = 2 \cdot 20 = 40$
3	$(1, 3, 4)$	$\tau_1\tau_3\tau_4 = 4 \cdot 9 = 36$
3	$(2^2, 4)$	${}_{\tau_2}H_2\tau_4 = 3 \cdot 9 = 27$
3	$(2, 3^2)$	$\tau_2{}_{\tau_3}H_2 = 2 \cdot {}_4H_2 = 2 \cdot {}_5C_2 = 2 \cdot 10 = 20$
2	$(1, 7)$	$\tau_1\tau_7 = 115$
2	$(2, 6)$	$\tau_2\tau_6 = 2 \cdot 48 = 96$
2	$(3, 5)$	$\tau_3\tau_5 = 4 \cdot 20 = 80$
2	(4^2)	${}_{\tau_4}H_2 = {}_9H_2 = {}_{10}C_2 = 45$
1	(8)	$\tau_8 = 286$

where $\Bbbk \in \mathbb{P}(8, c)$, and ${\phi_8'}^8 = 1$, ${\phi_8'}^7 = 2$, ${\phi_8'}^6 = 4 + 3 = 7$, ${\phi_8'}^5 = 9 + 8 + 4 = 21$, ${\phi_8'}^4 = 20 + 18 + 10 + 12 + 5 = 65$, ${\phi_8'}^3 = 48 + 40 + 36 + 27 + 20 = 181$, ${\phi_8'}^2 = 115 + 96 + 80 + 45 = 336$, ${\phi_8'}^1 = 286$.

For $k = 9$, we get

c	$\Bbbk$	$\phi'(\Bbbk)$
9	(1^9)	${}_{\tau_1}H_9 = 1$
8	$(1^7, 2)$	${}_{\tau_1}H_7\tau_2 = 1 \cdot 2 = 2$
7	$(1^6, 3)$	${}_{\tau_1}H_6\tau_3 = 4$
7	$(1^5, 2^2)$	${}_{\tau_1}H_5\ {}_{\tau_2}H_2 = 3$
6	$(1^5, 4)$	${}_{\tau_1}H_5\tau_4 = 9$
6	$(1^4, 2, 3)$	${}_{\tau_1}H_4\tau_2\tau_3 = 2 \cdot 4 = 8$
6	$(1^3, 2^3)$	${}_{\tau_1}H_3\,{}_{\tau_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
5	$(1^4, 5)$	${}_{\tau_1}H_4\tau_5 = 20$
5	$(1^3, 2, 4)$	${}_{\tau_1}H_3\tau_2\tau_4 = 2 \cdot 9 = 18$
5	$(1^3, 3^2)$	${}_{\tau_1}H_3\,{}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
5	$(1^2, 2^2, 3)$	${}_{\tau_1}H_2\,{}_{\tau_2}H_2\tau_3 = {}_2H_2 \cdot 4 = 12$
5	$(1, 2^4)$	$\tau_1\,{}_{\tau_2}H_4 = {}_2H_4 = {}_5C_4 = 5$
4	$(1^3, 6)$	${}_{\tau_1}H_3\tau_6 = 48$
4	$(1^2, 2, 5)$	${}_{\tau_1}H_2\tau_2\tau_5 = 2 \cdot 20 = 40$
4	$(1^2, 3, 4)$	${}_{\tau_1}H_2\tau_3\tau_4 = 4 \cdot 9 = 36$
4	$(1, 2^2, 4)$	$\tau_1\,{}_{\tau_2}H_2\tau_4 = {}_2H_2 \cdot 9 = 27$
4	$(1, 2, 3^2)$	$\tau_1\tau_2\,{}_{\tau_3}H_2 = 2 \cdot {}_4H_2 = 2 \cdot {}_5C_2 = 2 \cdot 10 = 20$
4	$(2^3, 3)$	${}_{\tau_2}H_3\tau_3 = {}_2H_3 \cdot 4 = {}_4C_3 \cdot 4 = 16$
3	$(1^2, 7)$	${}_{\tau_1}H_2\tau_7 = 115$
3	$(1, 2, 6)$	$\tau_1\tau_2\tau_6 = 2 \cdot 48 = 96$
3	$(1, 3, 5)$	$\tau_1\tau_3\tau_5 = 4 \cdot 20 = 80$
3	$(1, 4^2)$	$\tau_1\,{}_{\tau_4}H_2 = {}_9H_2 = {}_{10}C_2 = 45$
3	$(2, 3, 4)$	$\tau_2\tau_3\tau_4 = 2 \cdot 4 \cdot 9 = 72$
3	$(2^2, 5)$	${}_{\tau_2}H_2\tau_5 = 3 \cdot 20 = 60$
3	(3^3)	${}_{\tau_3}H_3 = {}_4H_3 = {}_6C_3 = 20$
2	$(1, 8)$	$\tau_1\tau_8 = 286$
2	$(2, 7)$	$\tau_2\tau_7 = 2 \cdot 115 = 230$
2	$(3, 6)$	$\tau_3\tau_6 = 4 \cdot 48 = 196$
2	$(4, 5)$	$\tau_4\tau_5 = 9 \cdot 20 = 180$
1	(9)	$\tau_9 = 719$

where $\Bbbk \in \mathbb{P}(9,c)$, and $\phi_9'^9 = 1$, $\phi_9'^8 = 2$, $\phi_9'^7 = 4+3 = 7$, $\phi_9'^6 = 9+8+4 = 21$, $\phi_9'^5 = 20+18+10+12+5 = 65$, $\phi_9'^4 = 48+40+36+27+20+16 = 197$, $\phi_9'^3 = 115+96+80+45+72+60+20 = 336$, $\phi_9'^2 = 286+230+196+180 = 892$, $\phi_9'^1 = 719$.

§ 7. Connected graphs

In this section, we will show the following by induction on k. This part is most complicated and important in all computation.

Theorem 2.

k	1	2	3	4	5	6	7	8	9	10
γ_k	1	2	4	9	20	51	125	329	862	2251

Let G be a conneted graph of size $k = s(G)$, $p = p(G)$ $(1 \leqq p \leqq k)$ be the period of the unique cycle $Z = Z(G)$, $g = g(G)$ be the number of gates of Z, then $1 \leqq g \leqq \min\{p, k-p\}$ if $k > p$.

Then $k - g$ is the sum of the weights of gates, so $k - g$ is called the weight of Z, denoted by $\mathrm{wt}(Z)$. The weights of gates $v_1, \dots, v_g$ of Z give a partition $\Bbbk$ of $\mathrm{wt}(Z) = \sum_{i=1}^{g} \mathrm{wt}(v_i)$, *i.e.* $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$. Denote by $\mathscr{C}_k(p,g;\Bbbk)$ the set of such connected graphs, and put $\gamma(p,g;\Bbbk) = \#\mathfrak{C}_k(p,g;\Bbbk)$, where $\mathfrak{C}_k(p,g;\Bbbk)$ is the set of isomorphism classes of graphs in $\mathscr{C}_k(p,g;\Bbbk)$.

If $p = k$, such graphs must be a cycle, so we use a convention $\mathfrak{C}_k(k,0;\mathbb{O}) = \{Z_k\}$ and $\gamma(k,0;\mathbb{O}) = 1$, then

$$\gamma_k = 1 + \sum_{p=1}^{k-1} \sum_{g=1}^{\min\{p,k-p\}} \sum_{\Bbbk \in \mathbb{P}(k-g,g)} \gamma(p,g;\Bbbk)$$

(Note that it is necessary to take into account distributions of gates on the cycle.)

For $k = 2$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
2	0	0		1
1	1	1	(1)	$\tau_1 = 1$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_2 = 1 + 1 = 2$.

For $k = 3$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
3	0	0		1
2	1	1	(1)	$\tau_1 = 1$
1	2	1	(2)	$\tau_2 = 2$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_3 = 1 + 1 + 2 = 4$.

For $k = 4$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
4	0	0		1
3	1	1	(1)	$\tau_1 = 1$
2	2	2	(1^2)	${}_{\tau_1}H_2 = 1$
2	2	1	(2)	$\tau_2 = 2$
1	3	1	(3)	$\tau_3 = 4$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_4 = 1 + 1 + 1 + 2 + 4 = 9$.

For $k = 5$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
5	0	0		1
4	1	1	(1)	$\tau_1 = 1$
3	2	2	(1^2)	1
3	2	1	(2)	$\tau_2 = 2$

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
2	3	2	(1, 2)	$\tau_1\tau_2 = 2$
2	3	1	(3)	$\tau_3 = 4$
1	4	1	(4)	$\tau_4 = 9$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_5 = 1 + 1 + 1 + 2 + 2 + 4 + 9 = 20$.

For $k = 6$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
6	0	0		1
5	1	1	(1)	$\tau_1 = 1$
4	2	2	(1^2)	2(*1)
4	2	1	(2)	$\tau_2 = 2$
3	3	3	(1^3)	1
3	3	2	(1, 2)	$2\tau_2 = 4$

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
3	3	1	(3)	$\tau_3 = 4$
2	4	2	(1, 3)	$\tau_1\tau_3 = 4$
2	4	2	(2^2)	${}_{\tau_2}H_2 = 3$
2	4	1	(4)	$\tau_4 = 9$
1	5	1	(5)	$\tau_5 = 20$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_6 = 1 + 1 + 2 + 2 + 1 + 4 + 4 + 4 + 3 + 9 + 20 = 51$.

(*1) Two configurations of gates (their distances are 1 and 2).

For $k = 7$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
7	0	0		1
6	1	1	(1)	$\tau_1 = 1$
5	2	2	(1^2)	$2(^*1)$
5	2	1	(2)	$\tau_2 = 2$
4	3	3	(1^3)	1
4	3	2	(1,2)	$3\tau_2 = 6$
4	3	1	(3)	$\tau_3 = 4$
3	4	3	$(1^2,2)$	2

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
3	4	2	(1,3)	$2\tau_3 = 8(^*2)$
3	4	2	(2^2)	$\tau_2^2 = 4(^*3)$
3	4	1	(4)	$\tau_4 = 9$
2	5	2	(1,4)	$\tau_1\tau_4 = 9$
2	5	2	(2,3)	$\tau_2\tau_3 = 2\cdot 4 = 8$
2	5	1	(5)	$\tau_5 = 20$
1	6	1	(6)	$\tau_6 = 48$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_7 = 1+1+2+2+1+6+4+2+8+4+9+9+8+20+48 = 125$.

(^{*1}) Distances of gates are of 2 types (1 and 2).

(^{*2}) Distances from the gate of weight 3 to the gate of weight 1 are of 2 types (1 and 2).

(^{*3}) The gate of weight 0 determines the order of the other two gates.

For $k = 8$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
8	0	0		1
7	1	1	(1)	$\tau_1 = 1$
6	2	2	(1^2)	$3(^*1)$
6	2	1	(2)	$\tau_2 = 2$
5	3	3	(1^3)	$2(^*2)$
5	3	2	(1,2)	$4\tau_2 = 8(^*3)$
5	3	1	(3)	$\tau_3 = 4$
4	4	4	(1^4)	1
4	4	3	$(1^2,2)$	$3\tau_2 = 6(^*4)$
4	4	2	(1,3)	$3\tau_3 = 12(^*5)$
4	4	2	(2^2)	${}_{\tau_2}H_2 + \tau_2^2 = 3+4 = 7(^*6)$

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
4	4	1	(4)	$\tau_4 = 9$
3	5	3	$(1^2,3)$	$\tau_3 = 4$
3	5	3	$(1,2^2)$	$\tau_2^2 = 4$
3	5	2	(1,4)	$2\tau_4 = 18(^*7)$
3	5	2	(2,3)	$2\tau_2\tau_3 = 16(^*8)$
3	5	1	(5)	$\tau_5 = 20$
2	6	2	(1,5)	$\tau_1\tau_5 = 20$
2	6	2	(2,4)	$\tau_2\tau_4 = 2\cdot 9 = 18$
2	6	2	(3^2)	${}_{\tau_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
2	6	1	(6)	$\tau_6 = 48$
1	7	1	(7)	$\tau_7 = 115$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_8 = 1+1+3+2+2+8+4+1+6+12+7+9+4+4+18+16+20+20+18+10+48+115 = 329$.

(^{*1}) Distances between 2 gates are of 3 types (1,2,3).

(^{*2}) Runs of gates are of 2 types (2,3).

(^{*3}) Distances from the gate of weight 2 to the gate of weight 1 are of 4 types (1,2,3,4).

(*4) Fix a gate of weight 2, then the locations of the gate of weight 0 are of 3 types.

(*5) Fix a gate of weight 3, then the locations of the gate of weight 1 are of 3 types.

(*6) ${}_{\tau_2}H_2$ for neighbouring gates and τ_2^2 for gates in opposite veteces.

(*7) Distances from the gate of weight 4 to the gate of weight 1 are of 2 types $(1,2)$.

(*8) Distances from the gate of weight 3 to the gate of weight 2 are of 2 types $(1,2)$.

For $k=9$, we get

p	wt	g	$\Bbbk$	$\gamma(p,g;\Bbbk)$
9	0	0		1
8	1	1	(1)	$\tau_1 = 1$
7	2	2	(1^2)	$3(^*1)$
7	2	1	(2)	$\tau_2 = 2$
6	3	3	(1^3)	$1+2+1=4(^*2)$
6	3	2	$(1,2)$	$5\tau_2 = 10(^*3)$
6	3	1	(3)	$\tau_3 = 4$
5	4	4	(1^4)	1
5	4	3	$(1^2,2)$	${}_4C_2\tau_2 = 12(^*4)$
5	4	2	$(1,3)$	$4\tau_3 = 16(^*5)$
5	4	2	(2^2)	$2\cdot\tau_2^2 = 8(^*6)$
5	4	1	(4)	$\tau_4 = 9$
4	5	4	$(1^3,2)$	$\tau_2 = 2$
4	5	3	$(1^2,3)$	$3\tau_3 = 12(^*7)$
4	5	3	$(1,2^2)$	$3\tau_2^2 = 12(^*8)$
4	5	2	$(1,4)$	$3\tau_4 = 27(^*9)$
4	5	2	$(2,3)$	$3\tau_2\tau_3 = 24(^*9)$
4	5	1	(5)	$\tau_5 = 20$
3	6	3	$(1^2,4)$	$\tau_4 = 9$
3	6	3	$(1,2,3)$	$2\tau_1\tau_2\tau_3 = 16(^*10)$
3	6	3	(2^3)	$4(^*11)$
3	6	2	$(1,5)$	$2\tau_5 = 40(^*12)$
3	6	2	$(2,4)$	$2\tau_2\tau_4 = 36(^*12)$
3	6	2	(3^2)	$\tau_3^2 = 16(^*13)$
3	6	1	(6)	$\tau_6 = 48$
2	7	2	$(1,6)$	$\tau_1\tau_6 = 48$
2	7	2	$(2,5)$	$\tau_2\tau_5 = 2\cdot 20 = 40$
2	7	2	$(3,4)$	$\tau_3\tau_4 = 36$
2	7	1	(7)	$\tau_7 = 115$
1	8	1	(8)	$\tau_8 = 286$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_9 = 1+1+3+2+4+10+4+1+12+16+8+9+2+12+12+27+24+20+9+16+4+40+36+16+48+48+40+36+115+286 = 862$.

(*1) Distances between gates are of 3 types $(1,2,3)$.

(*2) 1 type for 3 neighbouring gates, 2 types for distances from 2 neighbouring gates to the single gate, and 1 type for non-neibouring gates.

(*3) Distances from the gate of weight 2 to the gate of weight 1 are of 5 types $(1,2,3,4,5)$.

(*4) Fix a gate of weight 2, then the distributions of two gates of weight 1 are of ${}_4C_2 = 6$ types.

(*5) Fix a gate of weight 3, then the locations of the gate of weight 1 are of 4 types.

(*6) τ_2^2 for neighbouring gates and τ_2^2 for non-neighbouring gates.

(*7) Fix a gate of weight 3, then the distributions of two gates of weight 1 are of ${}_3C_2 = 3$ types.

(*8) Fix a gate of weight 1, then the distributions of two gates of weight 2 are of ${}_3C_2 = 3$ types.

(*9) Distances from the gate of weight 4 (or 3) to the gate of weight 1 (or 2) are of 3 types $(1, 2, 3)$.

(*10) Circular permutations of 3 elements are of 2 types.

(*11) The numbers of ways of choosing 3 elements from a 2 element set on a circle is 4.

(*12) Distances from the gate of weight 5 (or 4) to the gate of weight 1 (or 2) are of 2 types $(1, 2)$.

(*13) The gate of weight 0 determines orders of 2 gates of weight 3.

For $k = 10$, we get

p	wt	g	$\mathbb{k}$	$\gamma(p, g; \mathbb{k})$
10	0	0		1
9	1	1	(1)	$\tau_1 = 1$
8	2	2	(1^2)	4(*1)
8	2	1	(2)	$\tau_2 = 2$
7	3	3	(1^3)	$1 + 3 + 1 = 5$(*2)
7	3	2	$(1, 2)$	$6\tau_2 = 12$(*3)
7	3	1	(3)	$\tau_3 = 4$
6	4	4	(1^4)	3(*4)
6	4	3	$(1^2, 2)$	${}_5C_2\tau_2 = 20$(*5)
6	4	2	$(1, 3)$	$5\tau_3 = 20$(*6)
6	4	2	(2^2)	$3 + 8 = 11$(*7)
6	4	1	(4)	$\tau_4 = 9$
5	5	5	(1^5)	${}_1H_1 = 1$
5	5	4	$(1^3, 2)$	$4\tau_2 = 4 \cdot 2 = 8$(*8)
5	5	3	$(1^2, 3)$	$6\tau_3 = 24$(*9)
5	5	3	$(1, 2^2)$	$6\tau_2^2 = 24$(*10)
5	5	2	$(1, 4)$	$4\tau_4 = 36$(*11)
5	5	2	$(2, 3)$	$4\tau_2\tau_3 = 32$(*11)
5	5	1	(5)	$\tau_5 = 20$
4	6	4	$(1^3, 3)$	$\tau_3 = 4$
4	6	4	$(1^2, 2^2)$	$4 + 3 = 7$(*12)
4	6	3	$(1, 2, 3)$	$6 \cdot \tau_2\tau_3 = 48$(*13)
4	6	3	(2^3)	$\tau_2^3 = 8$(*14)
4	6	2	$(1, 5)$	$3\tau_5 = 27$(*15)
4	6	2	$(2, 4)$	$3\tau_2\tau_4 = 54$(*15)
4	6	2	(3^2)	$\tau_3^2 + {}_{\tau_3}H_2 = 26$(*16)
4	6	1	(6)	$\tau_6 = 48$
3	7	3	$(1, 2, 4)$	$2\tau_2\tau_4 = 36$
3	7	3	$(1, 3^2)$	$\tau_1\tau_3^2 = 16$
3	7	3	$(2^2, 3)$	$\tau_2^2\tau_3 = 16$
3	7	3	$(1^2, 5)$	$\tau_1^2\tau_5 = 20$
3	7	2	$(1, 6)$	$2\tau_6 = 96$
3	7	2	$(2, 5)$	$2\tau_2\tau_5 = 80$
3	7	2	$(3, 4)$	$2\tau_3\tau_4 = 72$
3	7	1	(7)	$\tau_7 = 115$
2	8	2	$(1, 7)$	$\tau_1\tau_7 = 115$
2	8	2	$(2, 6)$	$\tau_2\tau_6 = 2 \cdot 48 = 96$
2	8	2	$(3, 5)$	$\tau_3\tau_5 = 4 \cdot 20 = 80$
2	8	2	(4^2)	${}_{\tau_4}H_2 = {}_9H_2 = {}_{10}C_2 = 45$
2	8	1	(8)	$\tau_8 = 286$
1	9	1	(9)	$\tau_9 = 719$

where $\Bbbk \in \mathbb{P}(\mathrm{wt}, g)$, and $\gamma_{10} = 1+1+4+2+5+12+4+3+20+20+11+9+1+8+24+24+36+32+20+4+7+48+8+27+54+26+48+36+16+16+20+96+80+72+115+115+96+80+45+286+719 = 2251$.

(*1) Distances between 2 gates are of 4 types $(1,2,3,4)$.

(*2) 1 type for 3 neighbouring gates, 3 types for distances from 2 neighbouring gates to the single gate, and 1 type for non-neibouring gates.

(*3) Distances from the gate of weight 2 to the gate of weight 1 are of 6 types $(1 \sim 6)$.

(*4) 1 type for 4 neighbouring gates, 1 type for 3 neighbouring gates and a single gate, 1 types for two 2 neighbouring gates.

(*5) Fix a gate of weight 2, then the distributions of two gates of weight 1 are of ${}_5C_2 = 10$ types.

(*6) Fix a gate of weight 3, then the locations of the gate of weight 1 are of 5 types.

(*7) ${}_{\tau_2}H_2 = 3$ for gates in opposite positions, and $2 \cdot \tau_2^2 = 8$ for 2 non-symmetric locations of gates.

(*8) Fix a gate of weight 2, then the locations of the gate of weight 0 are of 4 types.

(*9) Fix a gate of weight 3, then the distributions of two gates of weight 1 are of ${}_4C_2 = 6$ types.

(*10) Fix a gate of weight 1, then the distributions of two gates of weight 2 are of ${}_4C_2 = 6$ types.

(*11) Distances from the gate of weight 4 (or 3) to the gate of weight 1 (or 2) are of 4 types $(1,2,3,4)$.

(*12) $\tau_2^2 = 4$ types for the unique distribution where 2 gates of same weight are neighbouring, and ${}_{\tau_2}H_2 = 3$ types for the unique distribution where 2 gates of same weight are not neighbouring.

(*13) Fix a gate of weight 3, then the distributions of the gate of weight 1 and the gate of weight 2 are of $3 \cdot 2 = 6$ types.

(*14) The gate of weight 0 determines orders of 3 gates of weight 2.

(*15) Distances from the gate of weight 5 (or 4) to the gate of weight 1 (or 2) are of 3 types $(1,2,3)$.

(*16) ${}_{\tau_3}H_2 = 10$ for gates in opposite positions, and $\tau_3^2 = 16$ for the non-symmetric distribution of gates.

§8. Computation of δ_k

In this section, we will show the following by induction on k.

Theorem 3.

k	1	2	3	4	5	6	7	8	9	10
δ_k	1	3	7	18	46	130	343	951	2615	7207

Let G be a dynamical graph of size $k = s(G)$, $c = c(G)$ $(1 \leqq c \leqq k)$ be the connectivity of G, and $\Bbbk$ be a partition of k to c numbers, that is, $\Bbbk \in \mathbb{P}(k, c)$.

Denote by $\delta(\Bbbk)$ the class number of dynamical graphs G with size characteristic $\mathbb{S}_G = \Bbbk$. Then

$$\delta_k = \sum_{\Bbbk \in \mathbb{P}(k,c)} \delta(\Bbbk) \qquad \text{and} \qquad \delta(\Bbbk) = \prod_{j=1}^{k} {}_{\gamma_j}H_{i_j},$$

where γ_j is the class number of connected graphs of size j.

For $k = 2$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
2	(1^2)	${}_{\gamma_1}H_2 = 1$
1	(2)	$\gamma_2 = 2$

where $\Bbbk \in \mathbb{P}(2,c)$, and $\delta_2 = 1 + 2 = 3$.

For $k = 3$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
3	(1^3)	${}_{\gamma_1}H_3 = 1$
2	$(1,2)$	$\gamma_1\gamma_2 = 1 \cdot 2 = 2$
1	(3)	$\gamma_3 = 4$

where $\Bbbk \in \mathbb{P}(3,c)$, and $\delta_3 = 1 + 2 + 4 = 7$.

For $k = 4$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
4	(1^4)	${}_{\gamma_1}H_4 = 1$
3	$(1^2,2)$	${}_{\gamma_1}H_2\gamma_2 = 1 \cdot 2 = 2$
2	$(1,3)$	$\gamma_1\gamma_3 = 1 \cdot 4 = 4$
2	(2^2)	${}_{\gamma_2}H_2 = {}_2H_2 = {}_3C_2 = 3$
1	(4)	$\gamma_4 = 9$

where $\Bbbk \in \mathbb{P}(4,c)$, and $\delta_4 = 1 + 2 + 4 + 3 + 9 = 19$.

For $k = 5$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
5	(1^5)	${}_{\gamma_1}H_5 = 1$
4	$(1^3,2)$	${}_{\gamma_1}H_3\gamma_2 = 1 \cdot 2 = 2$
3	$(1^2,3)$	${}_{\gamma_1}H_2\gamma_3 = 1 \cdot 4 = 4$
3	$(1,2^2)$	$\gamma_1 {}_{\gamma_2}H_2 = 1 \cdot 3 = 3$

c	$\Bbbk$	$\delta(\Bbbk)$
2	$(1,4)$	$\gamma_1\gamma_4 = 1 \cdot 9 = 9$
2	$(2,3)$	$\gamma_2\gamma_3 = 2 \cdot 4 = 8$
1	(5)	$\gamma_5 = 20$

where $\Bbbk \in \mathbb{P}(5,c)$, and $\delta_5 = 1 + 2 + 4 + 3 + 9 + 8 + 20 = 47$.

For $k = 6$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
6	(1^6)	${}_{\gamma_1}H_6 = 1$
5	$(1^4, 2)$	${}_{\gamma_1}H_4\gamma_2 = 1 \cdot 2 = 2$
4	$(1^3, 3)$	${}_{\gamma_1}H_3\gamma_3 = 1 \cdot 4 = 4$
4	$(1^2, 2^2)$	${}_{\gamma_1}H_2 \cdot {}_{\gamma_2}H_2 = 1 \cdot 3 = 3$
3	$(1^2, 4)$	${}_{\gamma_1}H_2\gamma_4 = 1 \cdot 9 = 9$
3	$(1, 2, 3)$	$\gamma_1\gamma_2\gamma_3 = 1 \cdot 2 \cdot 4 = 8$
3	(2^3)	${}_{\gamma_2}H_3 = {}_4C_3 = 4$
2	$(1, 5)$	$\gamma_1\gamma_5 = 1 \cdot 20 = 20$
2	$(2, 4)$	$\gamma_2\gamma_4 = 2 \cdot 9 = 18$
2	(3^2)	${}_{\gamma_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
1	(6)	$\gamma_6 = 51$

where $\Bbbk \in \mathbb{P}(6, c)$, and $\delta_6 = 1 + 2 + 4 + 3 + 9 + 8 + 4 + 20 + 18 + 10 + 51 = 130$.

For $k = 7$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
7	(1^7)	${}_{\gamma_1}H_7 = 1$
6	$(1^5, 2)$	${}_{\gamma_1}H_5\gamma_2 = 1 \cdot 2 = 2$
5	$(1^4, 3)$	${}_{\gamma_1}H_4\gamma_3 = 1 \cdot 4 = 4$
5	$(1^3, 2^2)$	${}_{\gamma_1}H_3 \cdot {}_{\gamma_2}H_2 = 1 \cdot 3 = 3$
4	$(1^3, 4)$	${}_{\gamma_1}H_3\gamma_4 = 1 \cdot 9 = 9$
4	$(1^2, 2, 3)$	${}_{\gamma_1}H_2 \cdot \gamma_2\gamma_3 = 1 \cdot 2 \cdot 4 = 8$
4	$(1, 2^3)$	$\gamma_1 {}_{\gamma_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
3	$(1, 2, 4)$	$\gamma_1\gamma_2\gamma_4 = 1 \cdot 2 \cdot 9 = 18$
3	$(1, 3^2)$	$\gamma_1 {}_{\gamma_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
3	$(2^2, 3)$	${}_{\gamma_2}H_2\gamma_3 = 12$
3	$(1^2, 5)$	${}_{\gamma_1}H_2\gamma_5 = 20$
2	$(1, 6)$	$\gamma_1\gamma_6 = 1 \cdot 51 = 51$
2	$(2, 5)$	$\gamma_2\gamma_5 = 2 \cdot 20 = 40$
2	$(3, 4)$	$\gamma_3\gamma_4 = 4 \cdot 9 = 36$
1	(7)	$\gamma_7 = 125$

where $\Bbbk \in \mathbb{P}(7, c)$, and $\delta_7 = 1 + 2 + 4 + 3 + 9 + 8 + 4 + 18 + 10 + 12 + 20 + 51 + 40 + 36 + 125 = 326$.

For $k = 8$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
8	(1^8)	${}_{\gamma_1}H_8 = 1$
7	$(1^6, 2)$	${}_{\gamma_1}H_6\gamma_2 = 2$
6	$(1^5, 3)$	${}_{\gamma_1}H_5\gamma_3 = 4$
6	$(1^4, 2^2)$	${}_{\gamma_1}H_4{}_{\gamma_2}H_2 = 3$
5	$(1^4, 4)$	${}_{\gamma_1}H_4\gamma_4 = 9$
5	$(1^3, 2, 3)$	${}_{\gamma_1}H_3\gamma_2\gamma_3 = 8$
5	$(1^2, 2^3)$	${}_{\gamma_1}H_2{}_{\gamma_2}H_3 = {}_4C_3 = 4$
4	$(1^3, 5)$	$0_{\gamma_1}H_3\gamma_5 = 20$
4	$(1^2, 2, 4)$	${}_{\gamma_1}H_2\gamma_2\gamma_4 = 18$
4	$(1^2, 3^2)$	${}_{\gamma_1}H_2{}_{\gamma_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
4	$(1, 2^2, 3)$	$\gamma_1{}_{\gamma_2}H_2\gamma_3 = 3 \cdot 4 = 12$
4	(2^4)	${}_{\gamma_2}H_4 = {}_2H_4 = {}_5C_4 = 5$
3	$(1^2, 6)$	${}_{\gamma_1}H_2\gamma_6 = 51$
3	$(1, 2, 5)$	$\gamma_1\gamma_2\gamma_5 = 2 \cdot 20 = 40$
3	$(1, 3, 4)$	$\gamma_1\gamma_3\gamma_4 = 4 \cdot 9 = 36$
3	$(2^2, 4)$	${}_{\gamma_2}H_2\gamma_4 = 3 \cdot 9 = 27$
3	$(2, 3^2)$	$\gamma_2{}_{\gamma_3}H_2 = 2{}_4H_2 = 20$
2	$(1, 7)$	$\gamma_1\gamma_7 = 125$
2	$(2, 6)$	$\gamma_2\gamma_6 = 2 \cdot 51 = 102$
2	$(3, 5)$	$\gamma_3\gamma_5 = 4 \cdot 20 = 80$
2	(4^2)	${}_{\gamma_4}H_2 = {}_9H_2 = {}_{10}C_2 = 45$
1	(8)	$\gamma_8 = 329$

where $\Bbbk \in \mathbb{P}(8, c)$, and $\delta_8 = 1 + 2 + 4 + 3 + 9 + 8 + 4 + 20 + 18 + 10 + 12 + 5 + 51 + 40 + 36 + 27 + 20 + 125 + 102 + 80 + 45 + 329 = 951$.

For $k = 9$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
9	(1^9)	${}_{\gamma_1}H_9 = 1$
8	$(1^7, 2)$	${}_{\gamma_1}H_7\gamma_2 = 2$
7	$(1^6, 3)$	${}_{\gamma_1}H_6\gamma_3 = 4$
7	$(1^5, 2^2)$	${}_{\gamma_1}H_5{}_{\gamma_2}H_2 = {}_2H_2 = 3$
6	$(1^5, 4)$	${}_{\gamma_1}H_5\gamma_4 = 9$
6	$(1^4, 2, 3)$	${}_{\gamma_1}H_4\gamma_2\gamma_3 = 2 \cdot 4 = 8$
6	$(1^3, 2^3)$	${}_{\gamma_1}H_3{}_{\gamma_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
5	$(1^4, 5)$	${}_{\gamma_1}H_4\gamma_5 = 20$
5	$(1^3, 2, 4)$	${}_{\gamma_1}H_3\gamma_2\gamma_4 = 2 \cdot 9 = 18$
5	$(1^3, 3^2)$	${}_{\gamma_1}H_3{}_{\gamma_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
5	$(1^2, 2^2, 3)$	${}_{\gamma_1}H_2{}_{\gamma_2}H_2\gamma_3 = {}_2H_2 4 = 12$
5	$(1, 2^4)$	$\gamma_1{}_{\gamma_2}H_4 = {}_2H_4 = {}_5C_4 = 5$
4	$(1^3, 6)$	${}_{\gamma_1}H_3\gamma_6 = 51$
4	$(1^2, 2, 5)$	${}_{\gamma_1}H_2\gamma_2\gamma_5 = 2 \cdot 20 = 40$
4	$(1^2, 3, 4)$	${}_{\gamma_1}H_2\gamma_3\gamma_4 = 4 \cdot 9 = 36$
4	$(1, 2^2, 4)$	$\gamma_1{}_{\gamma_2}H_2\gamma_4 = 3 \cdot 9 = 27$
4	$(1, 2, 3^2)$	$\gamma_1\gamma_2{}_{\gamma_3}H_2 = 2{}_4H_2 = 2{}_5C_2 = 20$
4	$(2^3, 3)$	${}_{\gamma_2}H_3\gamma_3 = {}_2H_3 \cdot 4 = {}_4C_3 \cdot 4 = 16$
3	$(1^2, 7)$	${}_{\gamma_1}H_2\gamma_7 = 125$
3	$(1, 2, 6)$	$\gamma_1\gamma_2\gamma_6 = 2 \cdot 51 = 102$
3	$(1, 3, 5)$	$\gamma_1\gamma_3\gamma_5 = 4 \cdot 20 = 80$
3	$(1, 4^2)$	$\gamma_1{}_{\gamma_4}H_2 = {}_9H_2 = {}_{10}C_2 = 45$
3	$(2, 3, 4)$	$\gamma_2\gamma_3\gamma_4 = 2 \cdot 4 \cdot 9 = 72$
3	$(2^2, 5)$	${}_{\gamma_2}H_2\gamma_5 = 3 \cdot 20 = 60$
3	(3^3)	${}_{\gamma_3}H_3 = {}_4H_3 = {}_6C_3 = 20$
2	$(1, 8)$	$\gamma_1\gamma_8 = 329$
2	$(2, 7)$	$\gamma_2\gamma_7 = 2 \cdot 125 = 250$
2	$(3, 6)$	$\gamma_3\gamma_6 = 4 \cdot 51 = 204$
2	$(4, 5)$	$\gamma_4\gamma_5 = 9 \cdot 20 = 180$
1	(9)	$\gamma_9 = 862$

where $\Bbbk \in \mathbb{P}(9, c)$, and $\delta_9 = 1 + 2 + 4 + 3 + 9 + 8 + 4 + 20 + 18 + 10 + 12 + 5 + 51 + 40 + 36 + 27 + 20 + 16 + 125 + 102 + 80 + 45 + 72 + 60 + 20 + 329 + 250 + 204 + 180 + 862 = 2615$.

For $k = 10$, we get

c	$\Bbbk$	$\delta(\Bbbk)$
10	(1^{10})	${}_{\gamma_1}H_{10} = 1$
9	$(1^8, 2)$	${}_{\gamma_1}H_8\gamma_2 = 2$
8	$(1^7, 3)$	${}_{\gamma_1}H_7\gamma_3 = 4$
8	$(1^6, 2^2)$	${}_{\gamma_1}H_{6\gamma_2}H_2 = {}_2H_2 = 3$
7	$(1^6, 4)$	${}_{\gamma_1}H_6\gamma_4 = 9$
7	$(1^5, 2, 3)$	${}_{\gamma_1}H_5\gamma_2\gamma_3 = 2 \cdot 4 = 8$
7	$(1^4, 2^3)$	${}_{\gamma_1}H_{4\gamma_2}H_3 = {}_2H_3 = {}_4C_3 = 4$
6	$(1^5, 5)$	${}_{\gamma_1}H_5\gamma_5 = 20$
6	$(1^4, 2, 4)$	${}_{\gamma_1}H_4\gamma_2\gamma_4 = 2 \cdot 9 = 18$
6	$(1^4, 3^2)$	${}_{\gamma_1}H_{4\gamma_3}H_2 = {}_4H_2 = {}_5C_2 = 10$
6	$(1^3, 2^2, 3)$	${}_{\gamma_1}H_{3\gamma_2}H_2\gamma_3 = {}_2H_24 = 12$
6	$(1^2, 2^4)$	${}_{\gamma_1}H_{2\gamma_2}H_4 = {}_2H_4 = {}_5C_4 = 5$
5	$(1^4, 6)$	${}_{\gamma_1}H_4\gamma_6 = 51$
5	$(1^3, 2, 5)$	${}_{\gamma_1}H_3\gamma_2\gamma_5 = 2 \cdot 20 = 40$
5	$(1^3, 3, 4)$	${}_{\gamma_1}H_3\gamma_3\gamma_4 = 4 \cdot 9 = 36$
5	$(1^2, 2^2, 4)$	${}_{\gamma_1}H_{2\gamma_2}H_2\gamma_4 = 3 \cdot 9 = 27$
5	$(1^2, 2, 3^2)$	${}_{\gamma_1}H_2\gamma_{2\gamma_3}H_2 = 2{}_4H_2 = 2{}_5C_2 = 20$
5	$(1, 2^3, 3)$	$\gamma_{1\gamma_2}H_3\gamma_3 = {}_2H_3 \cdot 4 = {}_4C_3 \cdot 4 = 16$
5	(2^5)	${}_{\gamma_2}H_5 = {}_6C_5 = 6$
4	$(1^3, 7)$	${}_{\gamma_1}H_3\gamma_7 = 125$
4	$(1^2, 2, 6)$	${}_{\gamma_1}H_2\gamma_2\gamma_6 = 2 \cdot 51 = 102$

c	$\Bbbk$	$\delta(\Bbbk)$
4	$(1^2, 3, 5)$	${}_{\gamma_1}H_2\gamma_3\gamma_5 = 4 \cdot 20 = 80$
4	$(1^2, 4^2)$	${}_{\gamma_1}H_{2\gamma_4}H_2 = {}_9H_2 = {}_{10}C_2 = 45$
4	$(1, 2, 3, 4)$	$\gamma_1\gamma_2\gamma_3\gamma_4 = 2 \cdot 4 \cdot 9 = 72$
4	$(1, 2^2, 5)$	$\gamma_{1\gamma_2}H_2\gamma_5 = 3 \cdot 20 = 60$
4	$(1, 3^3)$	$\gamma_{1\gamma_3}H_3 = {}_4H_3 = {}_6C_3 = 20$
4	$(2^2, 3^2)$	${}_{\gamma_2}H_{2\gamma_3}H_2 = 3 \cdot {}_4H_2 = 3 \cdot {}_5C_2 = 30$
4	$(2^3, 4)$	${}_{\gamma_2}H_3\gamma_4 = {}_2H_3 \cdot 9 = {}_4C_3 \cdot 9 = 36$
3	$(1^2, 8)$	${}_{\gamma_1}H_2\gamma_8 = 329$
3	$(1, 2, 7)$	$\gamma_1\gamma_2\gamma_7 = 2 \cdot 125 = 250$
3	$(1, 3, 6)$	$\gamma_1\gamma_3\gamma_6 = 4 \cdot 51 = 204$
3	$(1, 4, 5)$	$\gamma_1\gamma_4\gamma_5 = 9 \cdot 20 = 180$
3	$(2^2, 6)$	${}_{\gamma_2}H_2\gamma_6 = 3 \cdot 51 = 153$
3	$(2, 3, 5)$	$\gamma_2\gamma_3\gamma_5 = 2 \cdot 4 \cdot 20 = 160$
3	$(2, 4^2)$	$\gamma_{2\gamma_4}H_2 = 2{}_9H_2 = 2{}_{10}C_2 = 90$
3	$(3^2, 4)$	${}_{\gamma_3}H_2\gamma_4 = {}_4H_2 \cdot 9 = {}_5C_2 \cdot 9 = 90$
2	$(1, 9)$	$\gamma_1\gamma_9 = 862$
2	$(2, 8)$	$\gamma_2\gamma_8 = 2 \cdot 329 = 658$
2	$(3, 7)$	$\gamma_3\gamma_7 = 4 \cdot 125 = 500$
2	$(4, 6)$	$\gamma_4\gamma_6 = 8 \cdot 51 = 408$
2	(5^2)	${}_{\gamma_5}H_2 = {}_{20}H_2 = {}_{21}C_2 = 210$
1	(10)	$\gamma_{10} = 2251$

where $\Bbbk \in \mathbb{P}(10, c)$, and $\delta_{10} = 1 + 2 + 4 + 3 + 9 + 8 + 4 + 20 + 18 + 10 + 12 + 5 + 51 + 40 + 36 + 27 + 20 + 16 + 6 + 125 + 102 + 80 + 45 + 72 + 60 + 20 + 30 + 36 + 329 + 250 + 204 + 180 + 153 + 160 + 90 + 90 + 862 + 658 + 500 + 408 + 210 + 2251 = 7207$.

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