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A new proof of the global existence theorem of Klainerman for quasi-linear wave equations

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Abstract

We give a new proof of the global existence theorem of Klainerman for the Cauchy problem of quasi-linear wave equations in space dimensions $n \geq 4$. In addition to the Klainerman-Sideris inequality, a space-time L^2-estimate plays a key role in the proof. We answer a question raised by Metcalfe.

1 Introduction

In this paper we consider the Cauchy problem for a system of quasi-linear wave equations

$$\Box u = F(\partial u, \partial^2 u) \text{ in } \mathbb{R}^{1+n}_+ \tag{1.1}$$

subject to the smooth, compactly supported initial data

$$u(0) = f,\ \partial_t u(0) = g. \tag{1.2}$$

Here and in the rest of this paper we mean by ∂u (resp. $\partial^2 u$) the set of all the first (resp. second) derivatives of components of vector-valued function $u: \mathbb{R}^{1+n}_+ \to \mathbb{R}^m$, $m \geq 1$. We define the d'Alembertian $\Box$ as

$$\Box = \text{diag}(\Box_1, \ldots, \Box_m),\ \Box_k = \frac{\partial^2}{\partial t^2} - c_k^2 \Delta, \tag{1.3}$$

which acts on vector-valued functions u. Since higher-order terms have no influence over our concern of large-time existence of small-amplitude smooth solutions, we suppose that the nonlinear term F is quadratic in $(\partial u, \partial^2 u)$ and linear in $\partial^2 u$. We therefore assume the k-th component of the vector function F to be of the form $F^k(\partial u, \partial^2 u) = G^k(u,u) + H^k(u,u)$, where

$$G^k(u,v) = \sum_{i,j=1}^{m} \sum_{\alpha,\beta,\gamma=0}^{n} G_{ij}^{k,\alpha\beta\gamma} (\partial_\alpha u^i)(\partial_\beta \partial_\gamma v^j),\ H^k(u,v) = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=0}^{n} H_{ij}^{k,\alpha\beta} (\partial_\alpha u^i)(\partial_\beta v^j) \tag{1.4}$$

$(\partial_0 = \partial_t)$ for real constants $G_{ij}^{k,\alpha\beta\gamma}$, $H_{ij}^{k,\alpha\beta}$. Since our proof is based on the energy integral method, we naturally assume the symmetry condition

$$G_{ij}^{k,\alpha\beta\gamma} = G_{ij}^{k,\alpha\gamma\beta} = G_{ik}^{j,\alpha\beta\gamma}. \tag{1.5}$$

The commuting vector fields method of John and Klainerman has brought a remarkable progress in the theory of large-time existence of small solutions to the Cauchy problem of nonlinear wave equations [3]. The theorem of Klainerman is the most fundamental in this research and it states global existence for $n \geq 4$ and almost global existence for $n = 3$ of small solutions to quadratic, quasi-linear wave equations [6]. The heart of the method of Klainerman is the use of the Killing vector fields and the radial vector

field $S = t\partial_t + x \cdot \nabla$ to prove the global Sobolev inequality in the Minkowski space $\mathbb{R}^{n+1}$, known by the name of the Klainerman inequality [7]. Thanks to good commutation relations between these vector fields and the d'Alembertian, the standard energy integral argument together with the Klainerman inequality efficiently works for the proof of the fundamental result.

Recently, one of the present authors has given another proof to the theorem of Klainerman [1]. His proof relies on an effective use of the Klainerman-Sideris inequality (see (3.1) below), by which we get some weighted $L^2(\mathbb{R}^n)$-estimates of the second and higher-order derivatives of local solutions. The weight involved is the type of $\langle ct-r\rangle$ for a wave-propagation speed c, and therefore we obtain time decay estimates of the $L^2(D_t)$-norms ($D_t := \{x \in \mathbb{R}^n : |x| < ct/2\}$ for each $t > 0$) of the second and higher-order derivatives. It is safe to say that weak decay estimates of the $L^\infty(D_t)$-norms have been compensated for by the time decay estimates of the $L^2(D_t)$-norms. In this way the use of Lorentz boosts $L_j = x_j\partial_t + t\partial_j$ has been completely avoided in [1], by which the validity of the theorem of Klainerman has been extended to systems of quasi-linear equations with multiple speeds.

Interestingly enough, from the motive for studying the initial-boundary value problem of semi-linear wave equations in an exterior domain, Metcalfe [9] has recently devised a method by expanding the enterprise of Keel, Smith and Sogge [4]. The feature of the analysis of Keel, Smith, Sogge and Metcalfe lies in a revival and an efficient use of the integrability in time over the interval $(0,\infty)$ of spatially local energy, the fact which was already in the literature (see, e.g., Morawetz [14] and Strauss [19]). Metcalfe has shown global existence of small solutions to the initial-boundary value problem as well as the Cauchy problem of quadratic, semi-linear equations in space dimensions $n \geq 4$, and moreover Metcalfe has raised a question how to handle quasi-linear equations via similar techniques of [9]. The purpose of the present paper is to explain that the integrability estimate of the local energy such as (2.5) below actually plays a prominent role in giving a new proof of global existence of small solutions to systems of quasi-linear wave equations with multiple speeds when we consider the Cauchy problem.

Technical differences between this paper and previous one [1] should be described clearly. For a clear explanation of some technical points it is suitable to give notation used in this paper. Let n denote the space dimensions. We consider systems of m quasi-linear equations. Points in $\mathbb{R}^{1+n}_+$ are denoted by $(x^0, x^1,...,x^n) = (t, x)$. In addition to the usual partial differential operators $\partial_\alpha = \partial/\partial x^\alpha$ $(\alpha = 0,...,n)$ with the abbreviation $\partial = (\partial_0, \partial_1,..., \partial_n) = (\partial_t, \nabla)$, we use the generators of Euclid rotations $\Omega = (\Omega_{12},..., \Omega_{1n}, \Omega_{23},..., \Omega_{n-1n})$ with $\Omega_{jk} = x^j\partial_k - x^k\partial_j$ $(1 \leq j < k \leq n)$, and of space-time scaling $S = t\partial_t + x \cdot \nabla$. The set of these $\nu = (n^2+n+4)/2$ vector fields are denoted by $\Gamma = (\Gamma_0, \Gamma_1,..., \Gamma_{\nu-1}) = (\partial, \Omega, S)$, and we also denote $\Gamma \setminus \{S\}$ by $\overline{\Gamma} = (\Gamma_0, \Gamma_1,..., \Gamma_{\nu-2}) = (\partial, \Omega)$. For multi-indices $a = (a_0,..., a_{\nu-1})$ and $b = (b_0,..., b_{\nu-2})$, we denote

$$\Gamma^a = \Gamma_0^{a_0} \cdots \Gamma_{\nu-1}^{a_{\nu-1}}, \quad \overline{\Gamma}^b = \Gamma_0^{b_0} \cdots \Gamma_{\nu-2}^{b_{\nu-2}}. \tag{1.6}$$

Associated with the d'Alembertian $\Box$ given in (1.3), the energy is defined as

$$E_1(u(t)) = \frac{1}{2}\sum_{k=1}^m \int_{\mathbb{R}^n} \left(|\partial_t u^k(t, x)|^2 + c_k^2 |\nabla u^k(t,x)|^2 \right) dx. \tag{1.7}$$

We also introduce two types of generalized energy as

$$E_l(u(t)) = \sum_{\substack{|a|\le l-1 \\ a_{\nu-1}\le 1}} E_1(\Gamma^a u(t)) \tag{1.8}$$

$$\overline{E}_l(u(t)) = \sum_{|a|\le l-1} E_1(\overline{\Gamma}^a u(t)) \tag{1.9}$$

for $l = 2,3,\ldots$. Note that $\overline{E}_l(u(t)) \le E_l(u(t))$.

The auxiliary norm

$$M_l(u(t)) = \sum_{k=1}^{m} \sum_{|a|=2} \sum_{|b|\le l-2} \left\| \langle c_k t - r\rangle \partial^a \overline{\Gamma}^b u^k(t) \right\|_{L^2(\mathbb{R}^n)} \tag{1.10}$$

will play an intermediate role in the energy integral argument below. Here, and in what follows, we use the notation $r = |x|$ and $\langle A\rangle = \sqrt{1+|A|^2}$ for a scalar or vector A. For simplicity we often denote the $L^p(\mathbb{R}^n)$-norm by $\|\cdot\|_{L^p}$. The main theorem is stated as follows.

Theorem 1.1. *Let $n \ge 4$ and assume (1.4)–(1.5). Let δ be small so that $0 < \delta \le 1/2$, and let l be large so that*

$$\left[\frac{l}{2}\right] + \left[\frac{n}{2}\right] + 2 \le l - \left[\frac{n}{2}\right] - 2. \tag{1.11}$$

There exists a positive constant ε with the following property: If the initial data satisfy $E_l^{1/2}(u(0)) < \varepsilon$, then there exists a unique, smooth global (in time) solution to (1.1)–(1.2). It satisfies

$$E_{l-[n/2]-2}^{1/2}(u(t)) < 2\varepsilon,\ E_l^{1/2}(u(t)) \le 2E_l^{1/2}(u(0))(1+t)^{C\varepsilon},\ 0 < t < \infty \tag{1.12}$$

for a constant $C > 0$ independent of ε. Moreover, the solution also satisfies

$$\sum_{\alpha=0}^{n} \sum_{\substack{|a|\le l-[n/2]-2 \\ a_{\nu-1}\le 1}} \left\| \langle r\rangle^{-(1/2)-\delta} \partial_\alpha \Gamma^a u \right\|_{L^2((0,t)\times\mathbb{R}^n)} \le C\varepsilon(1+t)^{C\varepsilon},\ 0 < t < \infty. \tag{1.13}$$

Remark. The quantity $E_l^{1/2}(u(0))$ depends on the size of the initial data (f, g). Indeed, for sufficiently small data (f, g) at $t = 0$ we can calculate the derivatives of the solution u at $t = 0$ up to the l-th order by using the equation (1.1). In this way we can explicitly determine $E_l^{1/2}(u(0))$.

It was a key point in [1] that weak decay estimates of the $L^\infty(D_t)$-norms are compensated for by those of the $L^2(D_t)$-norms. On the other hand, we find in Section 6 that space-time L^2-estimates, which follow from the integrability estimate of the local energy, play a role as an alternative to the time decay estimates of the $L^2(D_t)$-norm. As a remarkable result, the number of vector fields S can be limited to at most one in the definition (1.8) of the generalized energy $E_l(u(t))$ which is employed in the a priori estimate of local solutions. This is in accordance with the thought in a recent paper of Keel, Smith and Sogge [5]. In that paper the method of vector fields is shown efficient in the proof of *almost global* existence of small solutions to initial-boundary value problems for quasi-linear wave equations in a three space dimensional domain exterior to a star-shaped obstacle with a compact, smooth boundary if the number of vector fields S is limited to at most one. In this paper we consider the Cauchy problem for quasi-linear wave equations in space dimensions $n \ge 4$ and we get the same result of global existence as the theorem of Klainerman. The main point of this paper is that the operator involved in our proof is somewhat restricted. Namely, we mainly use the generators of translations and spatial rotations (∂, Ω) and we use the generator of dilation S with only a single power. The authors have the hope that our present analysis will offer some insight into the study to prove global existence of small solutions to quasi-linear wave equations in an exterior domain of space dimensions $n \ge 4$.

We organize this paper as follows. In the next section some Sobolev-type inequalities and space-time L^2-estimates are presented. Section 3 is devoted to the weighted L^2-estimate of local solutions. In Section 4 we carry out the energy integral argument for the higher-order energy, and space-time L^2-estimates of local solutions are given in Section 5. In the final section we obtain a temporally uniform estimate of the lower-order energy and hence complete the proof of the main theorem.

2 Preliminaries

In addition to the well-known facts

$$[\partial_\alpha, \Box_k] = 0, \quad [\Omega_{ij}, \Box_k] = 0, \quad [S, \Box_k] = -2\Box_k, \tag{2.1}$$

we shall need the following Sobolev-type inequalities.

Lemma 2.1. *Let $\alpha = 0,1,\ldots n$ and $j = 1,2,\ldots,m$.*

(1) *Let $n \geq 3$. The inequality*

$$\langle r\rangle^{(n/2)-1}\langle c_j t - r\rangle|\partial_\alpha u^j(t,x)| \leq C\overline{E}^{1/2}_{[n/2]+1}(u(t)) + CM_{[n/2]+2}(u(t)) \tag{2.2}$$

holds.

(2) *Let $n = 4$ and $0 \leq d < 1/2$. The inequality*

$$\langle r\rangle^{1+d}|\partial_\alpha u^j(t,x)| \leq C\overline{E}^{1/2}_4(u(t)) \tag{2.3}$$

holds.

(3) *Suppose $n \geq 5$. The inequality*

$$\langle r\rangle^{(n/2)-1}|\partial_\alpha u^j(t,x)| \leq C\overline{E}^{1/2}_{[n/2]+2}(u(t)) \tag{2.4}$$

holds.

Proof. This lemma has been proved in Hidano [1]. □

Remark. After completing this work, the authors knew that Metcalfe, Nakamura and Sogge proved an exterior-domain analogue of (2.2) and they used it as one of their key tools in the proof of global existence of small solutions to the initial-boundary value problem in a domain exterior to an obstacle. See [10] [11], and [12]. See also the review article of Sogge [18].

As is mentioned in Introduction, the following space-time L^2-estimate also plays an intermediate role in our energy integral argument.

Lemma 2.2. *Let $n \geq 1$, $\delta > 0$ and let $(f, g) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. Suppose that v solves the Cauchy problem $\Box v = G, v(0) = f, \partial_t v(0) = g$. Then the estimate*

$$\sum_{\alpha=0}^{n} \left\| \langle r\rangle^{-(1/2)-\delta}\partial_\alpha v \right\|_{L^2((0,T)\times\mathbb{R}^n)} \leq C(\|\nabla f\|_{L^2} + \|g\|_{L^2}) + C\|G\|_{L^1((0,T);L^2(\mathbb{R}^n))} \tag{2.5}$$

holds.

Proof. We draw attention to the fact that, essentially in line with § 27 of Mochizuki [13], one can prove by the multiplier method that the solution $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ of $\Box u = 0$ with data (f, g) at $t = 0$ satisfies

$$\sum_{\alpha=0}^{n}\left\|\langle r\rangle^{-(1/2)-\delta}\partial_\alpha u\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\le C(\|\nabla f\|_{L^2}+\|g\|_{L^2}) \tag{2.6}$$

if $n=1$ or $n\ge 3$. It is also possible by the Duhamel principle to show that the solution $v:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ of $\Box v=G$ with zero data at $t=0$ satisfies

$$\sum_{\alpha=0}^{n}\left\|\langle r\rangle^{-(1/2)-\delta}\partial_\alpha v\right\|_{L^2((0,T)\times\mathbb{R}^n)}\le C\|G\|_{L^1((0,T);L^2(\mathbb{R}^n))} \tag{2.7}$$

if $n=1$ or $n\ge 3$.

In Proposition 2.8 of [9] Metcalfe has discussed global (in time) integrability of the local energy by making use of the space-time Fourier transform, and he has proved

$$\sum_{\alpha=0}^{n}\left\|\langle r\rangle^{-(n-1)/4}\partial_\alpha u\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\le C(\|\nabla f\|_{L^2}+\|g\|_{L^2})$$

for $n\ge 4$. The present authors have verified in the Appendix of [2] that, with slight modifications, the argument of Metcalfe is actually valid for the proof of a variant of (2.6) for all $n\ge 1$ (see (8.1) of [2]). The argument of [2] obviously remains true for the proof of (2.6) in any space dimension without any essential modification, and therefore the estimate (2.5) also holds for all $n\ge 1$ by the Duhamel principle. For the details, we refer to the Appendix of [2]. □

3 Weighted $L^2(\mathbb{R}^n)$-estimates

Since weighted L^2-norms $M_l(u(t))$ appear on the right-hand side of the Sobolev-type inequality presented in the previous section, it is necessary to bound $M_l(u(t))$ by $E_l^{1/2}(u(t))$ for the completion of the energy integral argument. The next crucial inequality, which is due to Klainerman and Sideris, is the starting point of our proof.

Lemma 3.1 (Klainerman-Sideris inequality). *Assume $\sigma\ge 2$ and $n\ge 2$. The inequality*

$$M_\sigma(u(t))\le CE_\sigma^{1/2}(u(t))+C\sum_{k=1}^{m}\sum_{|a|\le\sigma-2}\left\|(t+r)\Box_k\bar\Gamma^a u^k(t)\right\|_{L^2} \tag{3.1}$$

holds for any smooth function $u:\mathbb{R}_+^{1+n}\to\mathbb{R}^m$ if the right-hand side is finite.

Proof. See Lemma 3.1 of Klainerman and Sideris [8] and Lemma 7.1 of Sideris and Tu [16]. Note that their proof is obviously valid for all $n\ge 2$. □

Following Sideris [15] and Hidano [1], we prove a couple of lemmas.

Lemma 3.2. *Let u be a smooth solution of (1.1)−(1.2). Set $\sigma'=[(\sigma-1)/2]+[n/2]+2$ for $\sigma\ge 2$. Then, for all $|a|\le\sigma-2$*

$$\sum_{k=1}^{m}\left\|(t+r)\Box_k\bar\Gamma^a u^k(t)\right\|_{L^2}\le CE_{\sigma'}^{1/2}(u(t))\,E_\sigma^{1/2}(u(t))+CM_{\sigma'}(u(t))\,E_\sigma^{1/2}(u(t)). \tag{3.2}$$

Proof. We may focus on the estimate of the L^2-norm of $t\Box_k\bar\Gamma^a u^k$ because we can treat that of $r\Box_k\bar\Gamma^a u^k$ in a similar way. Set $p=[(\sigma-1)/2]$. By (1.4) it is necessary to estimate the contribution from the quasi-linear parts

$$\sum_{\alpha,\beta,\gamma=0}^{n}\left\|t\partial_\alpha\bar\Gamma^b u^i(t)\cdot\partial_\beta\partial_\gamma\bar\Gamma^c u^j(t)\right\|_{L^2},\quad |b|+|c|\le\sigma-2 \tag{3.3}$$

as well as the contribution from the semi-linear parts

$$\sum_{\alpha,\beta=0}^{n} t\left\|\partial_\alpha \overline{\Gamma}^b u^i(t)\cdot\partial_\beta \overline{\Gamma}^c u^j(t)\right\|_{L^2},\ |b|+|c|\le \sigma-2. \tag{3.4}$$

We shall start with the estimate of (3.4). Let us assume $|b|\le p$ without loss of generality. It follows from (2.2) that

$$\begin{aligned} &t\left\|\partial_\alpha \overline{\Gamma}^b u^i(t)\cdot\partial_\beta \overline{\Gamma}^c u^j(t)\right\|_{L^2} \\ &\le C\left\|\langle r\rangle\langle c_i t-r\rangle\partial_\alpha \overline{\Gamma}^b u^i(t)\right\|_{L^\infty}\left\|\partial_\beta \overline{\Gamma}^c u^j(t)\right\|_{L^2} \\ &\le C\left(\overline{E}^{1/2}_{|b|+[n/2]+1}(u(t))+M_{|b|+[n/2]+2}(u(t))\right)\overline{E}^{1/2}_{|c|+1}(u(t)) \\ &\le C\left(\overline{E}^{1/2}_{p+[n/2]+1}(u(t))+M_{p+[n/2]+2}(u(t))\right)\overline{E}^{1/2}_{\sigma-1}(u(t)) \\ &\le C\left(\overline{E}^{1/2}_{\sigma'}(u(t))+M_{\sigma'}(u(t))\right)\overline{E}^{1/2}_{\sigma-1}(u(t)). \end{aligned} \tag{3.5}$$

For the estimate of (3.3) we separate two cases: $|b|\le p$ or $|c|\le p-1$. For the former case the estimate is carried out as

$$\begin{aligned} &t\left\|\partial_\alpha \overline{\Gamma}^b u^i(t)\cdot\partial_\beta\partial_\gamma \overline{\Gamma}^c u^j(t)\right\|_{L^2} \\ &\le C\left\|\langle r\rangle\langle c_i t-r\rangle\partial_\alpha \overline{\Gamma}^b u^i(t)\right\|_{L^\infty}\left\|\partial_\beta\partial_\gamma \overline{\Gamma}^c u^j(t)\right\|_{L^2} \\ &\le C\left(\overline{E}^{1/2}_{|b|+[n/2]+1}(u(t))+M_{|b|+[n/2]+2}(u(t))\right)\overline{E}^{1/2}_{|c|+2}(u(t)) \\ &\le C\left(E^{1/2}_{\sigma'}(u(t))+M_{\sigma'}(u(t))\right)\overline{E}^{1/2}_{\sigma}(u(t)). \end{aligned} \tag{3.6}$$

On the other hand, for $|c|\le p-1$, we have only to exchange the roles which $\partial_\alpha \overline{\Gamma}^b u^i(t)$ and $\partial_\beta\partial_\gamma \overline{\Gamma}^c u^j(t)$ have played in (3.6). The proof of Lemma 3.2 has been completed. □

Lemma 3.3. *Let $n\ge 4$ and let l be large so that*

$$\left[\frac{l}{2}\right]+\left[\frac{n}{2}\right]+2\le l-\left[\frac{n}{2}\right]-2. \tag{3.7}$$

Set $\mu=l-[n/2]-2$. There exists a small, positive constant ε_0 with the following property: Suppose that, for a local smooth solution u of (1.1)—(1.2), the supremum of $E^{1/2}_\mu(u(t))$ on an interval $(0,T)$ is sufficiently small so that

$$\sup_{0<t<T} E^{1/2}_\mu(u(t))\le\varepsilon_0. \tag{3.8}$$

Then the inequalities

$$M_\mu(u(t))\le CE^{1/2}_\mu(u(t)),\ 0<t<T \tag{3.9}$$

and

$$M_l(u(t))\le CE^{1/2}_l(u(t)),\ 0<t<T \tag{3.10}$$

hold with a constant C independent of T.

Remark. This lemma is actually valid for any integer l satisfying

$$\left[\frac{l-1}{2}\right]+\left[\frac{n}{2}\right]+2\le l-\left[\frac{n}{2}\right]-2. \tag{3.11}$$

We have assumed (3.7) for the later use.

Proof. Set

$$\mu' \equiv \left[\frac{\mu-1}{2}\right]+\left[\frac{n}{2}\right]+2,\ l' \equiv \left[\frac{l-1}{2}\right]+\left[\frac{n}{2}\right]+2.$$

Employing Lemmas 3.1 and 3.2 with $\sigma \equiv \mu$, we see

$$M_\mu(u(t)) \le CE_\mu^{1/2}(u(t)) + C\sum_{k=1}^{m}\sum_{|a|\le\mu-2} \left\|(t+r)\Box_k \bar{\Gamma}^a u^k(t)\right\|_{L^2} \tag{3.12}$$

$$\le CE_\mu^{1/2}(u(t)) + CE_{\mu'}^{1/2}(u(t))\, E_\mu^{1/2}(u(t)) + CM_{\mu'}(u(t))\, E_\mu^{1/2}(u(t))$$

$$\le CE_\mu^{1/2}(u(t)) + C\varepsilon_0 E_\mu^{1/2}(u(t)) + C\varepsilon_0 M_{\mu'}(u(t)),$$

which yields (3.9). Taking account of a simple but crucial inequality $\mu' \le l' \le \mu \le l$, we see that $E_{l'}(u(t)) \le E_\mu(u(t))$, $M_{l'}(u(t)) \le M_\mu(u(t)) \le CE_\mu^{1/2}(u(t))$, $0<t<T$ and therefore

$$M_l(u(t)) \le CE_l^{1/2}(u(t)) + C\sum_{k=1}^{m}\sum_{|a|\le l-2} \left\|(t+r)\Box_k \bar{\Gamma}^a u^k(t)\right\|_{L^2} \tag{3.13}$$

$$\le CE_l^{1/2}(u(t)) + CE_{l'}^{1/2}(u(t))\, E_l^{1/2}(u(t)) + CM_{l'}(u(t))\, E_l^{1/2}(u(t))$$

$$\le CE_l^{1/2}(u(t)) + C\varepsilon_0 E_l^{1/2}(u(t)),$$

which leads us to (3.10). □

4 Energy estimates I. Higher-order energy

The main result of this section is the following proposition.

Proposition 4.1. *Let $n \ge 4$ and suppose that l is large so that (3.7) holds. Set $\mu \equiv l-[n/2]-2$. Suppose that initial data of a local solution to (1.1)−(1.2) satisfy $E_l^{1/2}(u(0)) < \varepsilon$ for a sufficiently small ε such that $2\varepsilon \le \varepsilon_0$ (see (3.8) for ε_0). Let T_0 be the supremum of all $T>0$ for which the unique local solution satisfies*

$$E_\mu^{1/2}(u(t)) < 2\varepsilon,\ 0<t<T. \tag{4.1}$$

Then the solution has the bound

$$E_l^{1/2}(u(t)) \le 2E_l^{1/2}(u(0))(1+t)^{C\varepsilon},\ 0<t<T_0. \tag{4.2}$$

Remark. Suppose $T_0 < \infty$. By the continuity of $E_\mu^{1/2}(u(t))$ on $[0, T_0]$ as well as the definition of T_0 we see that the maximum of $E_\mu^{1/2}(u(t))$ on the closed interval $[0, T_0]$ is 2ε. In the last section it will be shown that $E_\mu^{1/2}(u(t)) < 2\varepsilon$ on the interval $0 \le t \le T_0$, which is the contradiction and hence means that the local solution actually exists for any length of time.

Proof. The proof is in line with the previous works [16], [1]. Note that the following calculations are valid on the interval $0 \le t \le T_0$. Introducing the modified energy

$$\tilde{E}_\sigma(u(t)) \tag{4.3}$$

$$\equiv E_\sigma(u(t)) - \frac{1}{2}\sum_{\substack{|a|=\sigma-1\\ a_{\nu-1}\le l}}\sum_{i,j,k=1}^{m}\sum_{\alpha,\beta,\gamma,\delta=0}^{n} G_{ij}^{k,\alpha\beta\gamma}\eta_\gamma^\delta \int_{\mathbb{R}^n} \partial_\alpha u^i \cdot \partial_\beta \Gamma^a u^j \cdot \partial_\delta \Gamma^a u^k dx$$

($\eta_\gamma^\delta = \mathrm{diag}\,(1,-1,\ldots,-1)$), we get

$$\tilde{E}_l'(u(t)) \le C\sum_{i,j,k}\sum_{\alpha,\beta,\gamma,\delta}\sum_{\substack{|a|\le l-1\\ a_{\nu-1}\le 1}}\sum_{\substack{|b|+|c|\le|a|,c\ne a\\ b_{\nu-1}+c_{\nu-1}\le 1}}\left\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2}\left\|\partial_\delta\Gamma^a u^k\right\|_{L^2} \tag{4.4}$$
$$+C\sum_{i,j,k}\sum_{\alpha,\beta,\gamma}\sum_{\substack{|a|\le l-1\\ a_{\nu-1}\le 1}}\sum_{\substack{|b|+|c|\le|a|\\ b_{\nu-1}+c_{\nu-1}\le 1}}\left\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\Gamma^c u^j\right\|_{L^2}\left\|\partial_\gamma\Gamma^a u^k\right\|_{L^2}$$

(see Sideris and Tu [16] on page 484). Since it easily follows from the Sobolev embedding that

$$\frac{1}{2}E_\sigma(u(t)) \le \tilde{E}_\sigma(u(t)) \le 2E_\sigma(u(t)),\ 0<t<T_0\ \ (\sigma=\mu,\dots,l) \tag{4.5}$$

for the small solution u satisfying (4.1), we may be free to replace the norm $E_\sigma^{1/2}(u(t))$ with $\tilde{E}_\sigma^{1/2}(u(t))$ in the estimates below. Set $q=[l/2]$. We start with the estimate of the first term on the right-hand side of (4.4) which is the contribution from the quasi-linear part.

Quasi-linear part. We separate two cases: $|b|\le q$ or $|c|\le q-1$.

Case $|b|\le q$. If Γ^b contains the operator S, then we have by (2.3)—(2.4)

$$\begin{aligned}
&\left\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2} \\
&\le C\langle t\rangle^{-1}\left(\left\|\partial_\alpha\bar{\Gamma}^b u^i\cdot\langle c_j t-r\rangle\partial_\beta\partial_\gamma\bar{\Gamma}^c u^j\right\|_{L^2}+\left\|\langle r\rangle\partial_\alpha\bar{\Gamma}^b u^i\cdot\partial_\beta\partial_\gamma\bar{\Gamma}^c u^j\right\|_{L^2}\right) \\
&\le C\langle t\rangle^{-1}\left(\left\|\partial_\alpha\Gamma^b u^i\right\|_{L^\infty}\left\|\langle c_j t-r\rangle\partial_\beta\partial_\gamma\bar{\Gamma}^c u^j\right\|_{L^2}+\left\|\langle r\rangle\partial_\alpha\Gamma^b u^i\right\|_{L^\infty}\left\|\partial_\beta\partial_\gamma\bar{\Gamma}^c u^j\right\|_{L^2}\right) \\
&\le C\langle t\rangle^{-1}\left(E^{1/2}_{|b|+[n/2]+2}(u(t))\,M_{|c|+2}(u(t))+E^{1/2}_{|b|+[n/2]+2}(u(t))\,\bar{E}^{1/2}_{|c|+2}(u(t))\right) \\
&\le C\langle t\rangle^{-1}E^{1/2}_{q+[n/2]+2}(u(t))\left(\bar{E}^{1/2}_l(u(t))+M_l(u(t))\right) \\
&\le C\langle t\rangle^{-1}E^{1/2}_\mu(u(t))\,E^{1/2}_l(u(t)).
\end{aligned} \tag{4.6}$$

If Γ^b does not contain S, then we obtain by (2.2)

$$\begin{aligned}
&\left\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2} \\
&\le C\langle t\rangle^{-1}\left\|\langle r\rangle\langle c_i t-r\rangle\partial_\alpha\bar{\Gamma}^b u^i\right\|_{L^\infty}\left\|\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2} \\
&\le C\langle t\rangle^{-1}\left(\bar{E}^{1/2}_{|b|+[n/2]+1}(u(t))+M_{|b|+[n/2]+2}(u(t))\right)E^{1/2}_{|c|+2}(u(t)) \\
&\le C\langle t\rangle^{-1}E^{1/2}_\mu(u(t))\,E^{1/2}_l(u(t)).
\end{aligned} \tag{4.7}$$

Case $|c|\le q-1$. If Γ^b contains S, then we easily have

$$\begin{aligned}
&\left\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2} \\
&\le C\langle t\rangle^{-1}\left\|\partial_\alpha\Gamma^b u^i\right\|_{L^2}\left\|\langle r\rangle\langle c_j t-r\rangle\partial_\beta\partial_\gamma\bar{\Gamma}^c u^j\right\|_{L^\infty} \\
&\le C\langle t\rangle^{-1}E^{1/2}_l(u(t))\left(\bar{E}^{1/2}_{q+[n/2]+1}(u(t))+M_{q+[n/2]+2}(u(t))\right) \\
&\le C\langle t\rangle^{-1}E^{1/2}_l(u(t))\,E^{1/2}_\mu(u(t)).
\end{aligned} \tag{4.8}$$

If Γ^b does not contain S, then we see, noting $|b|\le l-2$ in this case,

$$\begin{aligned}
&\left\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2} \\
&\le C\langle t\rangle^{-1}\left(\left\|\frac{1}{r}\langle c_i t-r\rangle\,\partial_\alpha\bar{\Gamma}^b u^i\cdot r\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2}+\left\|\partial_\alpha\bar{\Gamma}^b u^i\cdot\langle r\rangle\partial_\beta\partial_\gamma\Gamma^c u^j\right\|_{L^2}\right)
\end{aligned} \tag{4.9}$$

$$\le C\langle t\rangle^{-1}\Big(\Big\|\frac{1}{r}\langle c_i t-r\rangle\partial_\alpha\bar\Gamma^b u^i\Big\|_{L^2}+\big\|\partial_\alpha\bar\Gamma^b u^i\big\|_{L^2}\Big)\big\|\langle r\rangle\partial_\beta\partial_\gamma\Gamma^c u^j\big\|_{L^\infty}$$

$$\le C\langle t\rangle^{-1}\Big(\bar E^{1/2}_{|b|+1}(u(t))+M_{|b|+2}(u(t))\Big)E^{1/2}_{|c|+[n/2]+3}(u(t))$$

$$\le C\langle t\rangle^{-1}\Big(\bar E^{1/2}_{l-1}(u(t))+M_l(u(t))\Big)E^{1/2}_{q+[n/2]+2}(u(t))$$

$$\le C\langle t\rangle^{-1}E_l^{1/2}(u(t))\,E_\mu^{1/2}(u(t)),$$

where we have used the Hardy inequality at the third inequality.

Semi-linear part. When estimating the second term on the right-hand side of (4.4) we may assume $|b|\le q$ $(q=[l/2])$ without loss of generality. If Γ^b contains the operator S, then we have, noting $|c|\le l-2$ in this case,

(4.10)
$$\big\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\Gamma^c u^j\big\|_{L^2}$$

$$\le C\langle t\rangle^{-1}\Big(\big\|r\partial_\alpha\Gamma^b u^i\cdot\frac{1}{r}\langle c_j t-r\rangle\partial_\beta\bar\Gamma^c u^j\big\|_{L^2}+\big\|\langle r\rangle\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\bar\Gamma^c u^j\big\|_{L^2}\Big)$$

$$\le C\langle t\rangle^{-1}\big\|\langle r\rangle\partial_\alpha\Gamma^b u^i\big\|_{L^\infty}\Big(\Big\|\frac{1}{r}\langle c_j t-r\rangle\partial_\beta\bar\Gamma^c u^j\Big\|_{L^2}+\big\|\partial_\beta\bar\Gamma^c u^j\big\|_{L^2}\Big)$$

$$\le C\langle t\rangle^{-1}E^{1/2}_{|b|+[n/2]+2}(u(t))\Big(\bar E^{1/2}_{|c|+1}(u(t))+M_{|c|+2}(u(t))\Big)$$

$$\le C\langle t\rangle^{-1}E^{1/2}_{q+[n/2]+2}(u(t))\Big(\bar E^{1/2}_{l-1}(u(t))+M_l(u(t))\Big)$$

$$\le C\langle t\rangle^{-1}E_\mu^{1/2}(u(t))+E_l^{1/2}(u(t))$$

If Γ^b does not contain S, then we get, using (2.2),

(4.11)
$$\big\|\partial_\alpha\Gamma^b u^i\cdot\partial_\beta\Gamma^c u^j\big\|_{L^2}$$

$$\le C\langle t\rangle^{-1}\big\|\langle r\rangle\langle c_i t-r\rangle\partial_\alpha\bar\Gamma^b u^i\big\|_{L^\infty}\big\|\partial_\beta\Gamma^c u^j\big\|_{L^2}$$

$$\le C\langle t\rangle^{-1}\Big(\bar E^{1/2}_{|b|+[n/2]+1}(u(t))+M_{|b|+[n/2]+2}(u(t))\Big)E^{1/2}_{|c|+1}(u(t))$$

$$\le C\langle t\rangle^{-1}\bar E^{1/2}_{q+[n/2]+2}(u(t))\,E_l^{1/2}(u(t))\le C\langle t\rangle^{-1}E_\mu^{1/2}(u(t))\,E_l^{1/2}(u(t)).$$

Conclusion of the proof. Using the equivalence (4.5), we have from (4.6)—(4.11)

(4.12)
$$\tilde E_l'(u(t))\le C\varepsilon\langle t\rangle^{-1}\tilde E_l(u(t)),\quad 0\le t<T_0,$$

which yields

$$\frac{1}{2}E_l(u(t))\le\tilde E_l(u(t))\le\tilde E_l(u(0))(1+t)^{C\varepsilon}\le 2E_l(u(0))(1+t)^{C\varepsilon},$$

therefore,

(4.13)
$$E_l^{1/2}(u(t))\le 2E_l^{1/2}(u(0))(1+t)^{C\varepsilon}$$

for a suitable constant C. □

5 Space-time L^2-estimates

What makes a crucial difference between the two proofs of [1] and the present paper is a use of temporal

integrability estimates of the local solution. We shall employ Lemma 2.2 to prove the following proposition which plays a key role in the estimate of the lower-order energy in the final section.

Proposition 5.1. *Let $\delta > 0$. Under the same assumptions as in Proposition 4.1 the local solution satisfies*

$$\sum_{\alpha=0}^{n} \sum_{\substack{|a|\le\mu \\ a_{\nu-1}\le 1}} \left\| \langle r\rangle^{-(1/2)-\delta} \partial_\alpha \Gamma^a u \right\|_{L^2((0,t)\times\mathbb{R}^n)} \le C E_{\mu+1}^{1/2}(0) + C\varepsilon \langle t\rangle^{C\varepsilon}, \quad 0 < t < T_0 \tag{5.1}$$

for constants C independent of ε.

Proof. By Lemma 2.2 we may focus our effort on the estimate of

$$\sum_{\alpha,\beta,\gamma=0}^{n} \sum_{\substack{|b|+|c|\le\mu \\ b_{\nu-1}+c_{\nu-1}\le 1}} \int_0^t \left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} d\tau. \tag{5.2}$$

If Γ^b contains S, we see by noting $|c| \le \mu - 1$

$$\begin{aligned} &\left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \overline{\Gamma}^c u^j(\tau) \right\|_{L^2} \\ &\le C\langle\tau\rangle^{-1} \left\| \partial_\alpha \Gamma^b u^i(\tau) \right\|_{L^2} \left\| \langle r\rangle \langle c_j\tau - r\rangle \partial_\beta \partial_\gamma \overline{\Gamma}^c u^j(\tau) \right\|_{L^\infty} \\ &\le C\langle\tau\rangle^{-1} E_{|b|+1}^{1/2}(u(\tau)) \left(E_{|c|+[n/2]+2}^{1/2}(u(\tau)) + M_{|c|+[n/2]+3}(u(\tau)) \right) \\ &\le C\langle\tau\rangle^{-1} E_l(u(\tau)) \le C\langle\tau\rangle^{-1+C\varepsilon} E_l(u(0)). \end{aligned} \tag{5.3}$$

Here we have made use of Lemma 2.1 at the second, Lemma 3.3 at the third, and Proposition 4.1 at the last inequality.

On the other hand, if Γ^b does not contain S, it is easy to obtain

$$\begin{aligned} &\left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} \\ &\le C\langle\tau\rangle^{-1} \left\| \langle r\rangle \langle c_i\tau - r\rangle \partial_\alpha \overline{\Gamma}^b u^i(\tau) \right\|_{L^\infty} \left\| \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} \\ &\le C\langle\tau\rangle^{-1} \left(E_{|b|+[n/2]+1}^{1/2}(u(\tau)) + M_{|b|+[n/2]+2}(u(\tau)) \right) E_{|c|+2}^{1/2}(u(\tau)) \\ &\le C\langle\tau\rangle^{-1} E_l(u(\tau)) \le C\langle\tau\rangle^{-1+C\varepsilon} E_l(u(0)). \end{aligned} \tag{5.4}$$

Combining (5.3)—(5.4) with (5.2), we find that the estimate of (5.2) is continued as

$$\cdots \le C\varepsilon \langle t\rangle^{C\varepsilon}, \quad t < T_0. \tag{5.5}$$

The estimate of

$$\sum_{\alpha,\beta=0}^{n} \sum_{\substack{|b|+|c|\le\mu \\ b_{\nu-1}+c_{\nu-1}\le 1}} \int_0^t \left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \Gamma^c u^j(\tau) \right\|_{L^2} d\tau \tag{5.6}$$

remains to be done. But it is obvious that the arguments in (5.3)—(5.4) are still valid for the estimate of (5.6). We have therefore finished the proof of Proposition 5.1. □

6 Energy estimates II. Lower-order energy

Let T_0 be the time defined in (4.1). Suppose $T_0 < \infty$. The last section is devoted to the proof of the first inequality of (1.12) for $0 \le t \le T_0$, which completes the proof of Theorem 1.1 (see Remark below

Proposition 4.1). Our consideration starts with the standard inequality

$$E_\mu^{1/2}(u(t)) \le E_\mu^{1/2}(u(0)) + C \sum_{\substack{|a|\le \mu-1 \\ a_{\nu-1}\le 1}} \left\| \Gamma^a F(\partial u, \partial^2 u) \right\|_{L^1(0,t;L^2(\mathbb{R}^n))} \tag{6.1}$$

for $\mu \equiv l - [n/2] - 2$ as before. It is necessary to estimate the contributions from quasi-linear terms

$$\sum_{i,j=1}^{m} \sum_{\alpha,\beta,\gamma=0}^{n} \sum_{\substack{|b|+|c|\le \mu-1 \\ b_{\nu-1}+c_{\nu-1}\le 1}} \int_0^t \left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} d\tau \tag{6.2}$$

and from semi-linear terms

$$\sum_{i,j=1}^{m} \sum_{\alpha,\beta=0}^{n} \sum_{\substack{|b|+|c|\le \mu-1 \\ b_{\nu-1}+c_{\nu-1}\le 1}} \int_0^t \left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \Gamma^c u^j(\tau) \right\|_{L^2} d\tau. \tag{6.3}$$

Set $c_0 \equiv \min\{c_j : j = 1, 2, \ldots, m\}$ for the propagation speeds. Dividing (6.2) into three pieces

$$\begin{aligned} &\sum \int_0^1 \left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} d\tau \\ &+ \sum \int_1^t \| \cdots \|_{L^2(r<c_0\tau/2)} d\tau + \sum \int_1^t \| \cdots \|_{L^2(r>c_0\tau/2)} d\tau \\ &\equiv I_1 + I_2 + I_3, \end{aligned} \tag{6.4}$$

we begin with the estimate of I_2. If Γ^b contains S, we see for $0 < \delta \le 1/2$

$$\begin{aligned} &\left\| \partial_\alpha \Gamma^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \overline{\Gamma}^c u^j(\tau) \right\|_{L^2(r<c_0\tau/2)} \\ &\le \langle \tau \rangle^{-1} \left\| \langle \tau \rangle^{-(1/2)-\delta} \partial_\alpha \Gamma^b u^i(\tau) \right\|_{L^2} \left\| \langle \tau \rangle^{(1/2)+\delta} \langle c_j \tau - r \rangle \partial_\beta \partial_\gamma \overline{\Gamma}^c u^j(\tau) \right\|_{L^\infty} \\ &\le C \langle \tau \rangle^{-1+C\varepsilon} E_l^{1/2}(u(0)) \left\| \langle r \rangle^{-(1/2)-\delta} \partial_\alpha \Gamma^b u^i(\tau) \right\|_{L^2} \end{aligned} \tag{6.5}$$

by modifying the computation in (5.3). On the other hand, if Γ^b does not contain S, we have

$$\begin{aligned} &\left\| \partial_\alpha \overline{\Gamma}^b u^i(\tau) \cdot \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2(r<c_0\tau/2)} \\ &\le \langle \tau \rangle^{-1} \left\| \langle \tau \rangle^{(1/2)+\delta} \langle c_i \tau - r \rangle \partial_\alpha \overline{\Gamma}^b u^i(\tau) \right\|_{L^\infty} \left\| \langle r \rangle^{-(1/2)-\delta} \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} \\ &\le C \langle \tau \rangle^{-1+C\varepsilon} E_l^{1/2}(u(0)) \left\| \langle r \rangle^{-(1/2)-\delta} \partial_\beta \partial_\gamma \Gamma^c u^j(\tau) \right\|_{L^2} \end{aligned} \tag{6.6}$$

as in (5.4). With the help of a useful technique of the dyadic decomposition of an interval as in Sogge [17] (see [17] on page 363), we obtain

$$\begin{aligned} I_2 &\le C\varepsilon \sum_{i=1}^{m} \sum_{\alpha=0}^{n} \sum_{\substack{|a|\le \mu \\ a_{\nu-1}\le 1}} \sum_{j=0}^{N} \int_{2^j}^{2^{j+1}} \langle \tau \rangle^{-1+C\varepsilon} \left\| \langle r \rangle^{-(1/2)-\delta} \partial_\alpha \Gamma^a u^i(\tau) \right\|_{L^2} d\tau \\ &\le C\varepsilon \sum_{i=1}^{m} \sum_{\alpha=0}^{n} \sum_{\substack{|a|\le \mu \\ a_{\nu-1}\le 1}} \sum_{j=0}^{N} \left(\int_{2^j}^{2^{j+1}} \langle \tau \rangle^{-2+2C\varepsilon} d\tau \right)^{1/2} \left\| \langle r \rangle^{-(1/2)-\delta} \partial_\alpha \Gamma^a u^i \right\|_{L^2((0,2^{j+1})\times \mathbb{R}^n)} \\ &\le C\varepsilon^2 \sum_{j=0}^{\infty} (2^j)^{(-1/2)+C\varepsilon} \le C\varepsilon^2, \end{aligned} \tag{6.7}$$

where we have abused the notation to mean T_0 by 2^{N+1}. This is the place where Proposition 5.1 plays a crucial role.

For the estimate of I_3 we separate two cases: $|b| \le [\mu/2]$ or $|c| \le [\mu/2] - 1$. If $|b| \le [\mu/2]$, then we get

by (2.3)−(2.4), (4.1)−(4.2)

$$
\begin{aligned}
(6.8)\qquad & \left\|\partial_\alpha \Gamma^b u^i(\tau)\cdot \partial_\beta\partial_\gamma \Gamma^c u^j(\tau)\right\|_{L^2(r>c_0\tau/2)} \\
& \le C\langle\tau\rangle^{-1-\delta'} \left\|\langle r\rangle^{1+\delta'}\partial_\alpha \Gamma^b u^i(\tau)\right\|_{L^\infty} \left\|\partial_\beta\partial_\gamma \Gamma^c u^j(\tau)\right\|_{L^2} \\
& \le C\langle\tau\rangle^{-1-\delta'} E^{1/2}_{[\mu/2]+[n/2]+2}(u(\tau))\, E^{1/2}_{\mu+1}(u(\tau)) \\
& \le C\langle\tau\rangle^{-1-\delta'} E^{1/2}_{\mu}(u(\tau))\, E^{1/2}_{l}(u(\tau)) \le C\varepsilon^2\langle\tau\rangle^{-1-\delta'+C\varepsilon}
\end{aligned}
$$

for a suitable $\delta' > 0$. Otherwise, we have

$$
\begin{aligned}
(6.9)\qquad & \left\|\partial_\alpha \Gamma^b u^i(\tau)\cdot \partial_\beta\partial_\gamma \Gamma^c u^j(\tau)\right\|_{L^2(r>c_0\tau/2)} \\
& \le C\langle\tau\rangle^{-1-\delta'} \left\|\partial_\alpha \Gamma^b u^i(\tau)\right\|_{L^2} \left\|\langle r\rangle^{1+\delta'}\partial_\beta\partial_\gamma \Gamma^c u^j(\tau)\right\|_{L^\infty} \le C\varepsilon^2\langle\tau\rangle^{-1-\delta'+C\varepsilon}
\end{aligned}
$$

as in (6.8). Since ε is sufficiently small, I_3 is estimated as

$$
(6.10)\qquad I_3 \le C\varepsilon^2 \int_1^\infty \langle\tau\rangle^{-1-\delta'+C\varepsilon}\, d\tau \le C\varepsilon^2.
$$

Finally, it is easy to see $I_1 \le C\varepsilon^2$, and we have shown that the estimate of (6.3) is continued as

$$
(6.11)\qquad \cdots \le C\varepsilon^2
$$

for a constant C independent of ε. Obviously the computations in (6.4)−(6.11) are still valid for the estimate of (6.3), and we have therefore proved

$$
(6.12)\qquad E^{1/2}_{\mu}(u(t)) \le \varepsilon + C\varepsilon^2,\ 0 < t < T_0
$$

Since ε is sufficiently small, we can conclude $E^{1/2}_{\mu}(u(t)) \le 3\varepsilon/2$ on the interval $[0,\ T_0]$. The proof of the first inequality of (1.12) has been finished.

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