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On absolute continuity of convex functions.

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凸関数の絶対連続性について

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Abstract

A concept of a convex function was introduced by Jensen. Convex functions play an important role in the information theory, statistic, engineering ... as well as lots of branches of mathematics. A lot of mathematicians work on analyses of the convex function, and many important properties, for example continuity, differentiability and monotonicity of it have been clarified. As is well known a convex function defined on an open interval I is absolutely continuous on every closed interval contained I. Unfortunately it is not true on I in general. The aim of this paper is to consider the problem of absolute continuity of a convex function on an open interval. We shall also discuss integrability of the derivative of a convex function on infinite interval. Moreover we study the same problem about a Q-convex function.

1 Introduction

Let I be an interval in the real line.

A real-valued function f defined on I is said to be *convex* (*or strictly convex*) on I if it satisfies

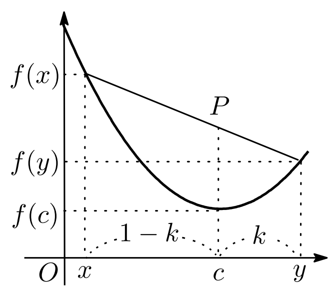
(★) $$f(kx + (1-k)y) \underset{(<)}{\leq} kf(x) + (1-k)f(y)$$

for *any* $0 < k < 1$ and for *any* $x, y \in I,\ x \neq y$.

Geometrically a convex function is defined as follows.

Let c be a point in the segment $xy \subseteq I$ and assume $0 < k = \dfrac{y-c}{y-x} < 1$. Then

$$c = kx + (1-k)y.$$



Moreover let $P(c, y')$ be a point on the straight line segment connecting $(x, f(x))$ and $(y, f(y))$. Then

$$y' = kf(x) + (1-k)f(y).$$

The point $(c,\ f(c))$ on the curve of a convex function f is always *below or on* the point P. Hence it satisfies

$$f(c) \leq kf(x) + (1-k)f(y).$$

This definition is equal to the analytic definition (★).

It is well known that a convex function on an open interval is continuous on I and differentiable at almost everywhere (see below).

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Here is a typical example and application of a convex function.

(Relation between the arithmetic means and the geometric means) *Let* $f(x) = -\log x$ *for* $x \in (0, +\infty)$. *Then* f *is strictly convex on* $(0, +\infty)$, *hence*

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

holds for any n *and* $x_1, x_2, \ldots, x_n > 0$. *There is equality if and only if* $x_1 = x_2 = \cdots = x_n$.

On the other hand a function f defined on I is called Q-*convex* function on I if there exists an *integer* $p \geq 2$ such that

$$(\star) \qquad f\Big(\frac{x_1 + x_2 + \cdots + x_p}{p}\Big) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_p)}{p} \quad \text{for any} \quad x_1, x_2, \ldots, x_p \in I.$$

Particularly f is said to be a *middle convex function* if it satisfies the above condition for $p = 2$.

The term Q means Quotient, that is if f is Q-convex on I then $f(kx + (1-k)y) \leq kf(x) + (1-k)f(y)$ holds for any *rational* number $0 < k < 1$ and any $x, y \in I$.

It is well known that the middle convex function becomes convex on I if and only if it is continuous on I. In [5], Kazamaki gave an example of a middle convex function on the real line which is not continuous almost everywhere, hence it is not convex.

A function f is said to be *absolutely continuous* on I if it satisfies the following property :

For any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever a finite family of pairwise disjoint sub-intervals $\{[a_k,\, b_k) \subseteq I\}$ satisfies $\sum_k |a_k - b_k| < \delta$ then

$$\sum_k |f(a_k) - f(b_k)| < \varepsilon.$$

The absolutely continuous function has important properties as follows.

Let f *be an absolutely continuous function on a finite interval* I. *Then there exists a Lebesgue integrable function* g *on* I *such that*

$$f(x) = \int_c^x g(s)\, ds + f(c)$$

for any $x, c \in I$, *that is* f *is differentiable at almost everywhere.*

Remark that this proposition does not hold if I is infinite (*e.g.* $f(x) = x$).

The author is interested in the problem of absolute continuity of a convex and Q-convex functions. For example the convex function $f(x) = -x^\alpha$ $(0 < \alpha < 1,\ x > 0)$ is absolutely continuous on $(0,\ 1)$. If $\alpha > 1$ then the convex function $f(x) = x^\alpha$ $(x > 0)$ is Lipschitz continuous on $(0,\ 1)$, consequently it is absolutely continuous on there. If $\alpha < 0$ then this function is not Lipschitz continuous, nor is absolutely continuous (*cf.* the examples of the next section).

The first aim of this paper is to prove the following :

Theorem 1.1 *Let* f *be a real-valued function defined on an open interval* I *in the real line.*

Assume f *is convex on* I. *If* f *is bounded on* I *then* f *is absolutely continuous on* I. *Moreover if* I *is finite then* f *is absolutely continuous on* I *if and only if* f *is bounded on there. Even if* f *is* Q-*convex, the comparable results are obtained on* I.

By the way, it is well known that a convex function has the following properties.

Theorem 1.2 (A. Zygmund) *Let f be a convex function on a finite interval $I = (a,\, b)$. Then there exists a right-continuous and non-decreasing function g such that*

$$f(x) = \int_c^x g(s)\, ds + f(c)$$

for any $x, c \in I$, that is f has a right-continuous and non-decreasing derivative.

This theorem claimed that the convex function on $I = (a,\, b)$ is Lipschitz continuous on any closed interval $[c,\, d] \subseteq I$. Unfortunately, in general, neither Lipschitz continuity nor absolute continuity does not hold on I, as seen in the above.

On the other hand, combining these results the following theorem can be obtained.

Theorem 1.3 *A convex function f on an interval I has a non-decreasing and right-continuous derivative. If I is closed then the derivative is integrable on I. If I is open and finite then the derivative is integrable on I if and only if f is bounded on I.*

The second aim of this paper is to prove the following.

Theorem 1.4 *Let f be a real-valued function defined on an infinite interval I.*

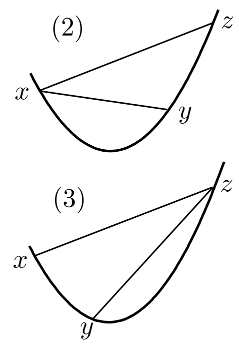
If f is bounded and Q-convex on I then the derivative of it is Lebesgue integrable on there. Assume f is convex on I. Then the derivative of f is Lebesgue integrable on I if and only if f is bounded on there.

The composition of this paper is as follows. In the next section we shall analyze the slope of the secant line of the curve $y = f(x)$, where f is a convex function, and characterize the convexity by the slope. In Section 3 we shall prove Theorem1.1 and Theorem1.4 concerning convex function, and give examples of convex functions which is not differentiable at a denumerable set of points. In Section 4 we shall study the properties of a Q-convex function and prove the remaining parts of Theorem1.1 and Theorem1.4.

2 Slope of the secant line

The following lemmas may provide some fundamental properties, for instance continuity, differentiability and monotonicity, of a convex function in the next section.

Lemma 2.1 *Let f be a real-valued function on an interval I. Then the Following three conditions are equivalent.*



(1) *f is convex on I.*

(2) *For $x < y < z,\ x, y, z \in I$ we have* $\dfrac{f(y) - f(x)}{y - x} \leq \dfrac{f(z) - f(x)}{z - x}$.

(3) *For $x < y < z,\ x, y, z \in I$ we have* $\dfrac{f(z) - f(x)}{z - x} \leq \dfrac{f(z) - f(y)}{z - y}$.

proof. First we assume that f is convex on I.

Let $x < y < z,\ x, y, z \in I$ and put $k = \dfrac{z - y}{z - x}$. Then $y = kx + (1 - k)z$, and the assumption implies

$$f(y) \leq kf(x) + (1 - k)f(z).$$

Hence we have

$$\frac{f(y)-f(x)}{y-x} \leq \frac{kf(x)+(1-k)f(z)-f(x)}{kx+(1-k)z-x} = \frac{f(z)-f(x)}{z-x}$$

and

$$\frac{f(z)-f(y)}{z-y} \geq \frac{f(z)-kf(x)-(1-k)f(z)}{z-kx-(1-k)z} = \frac{f(z)-f(x)}{z-x},$$

which prove (2) and (3).

Next we assume (2). For $x<y$, $x,y \in I$ and $0<k<1$ put $c = kx+(1-k)y$ $(0<k<1)$. Then $x<c<y$ and

$$\frac{f(c)-f(x)}{c-x} \leq \frac{f(y)-f(x)}{y-x}$$

by the assumption. Thus

$$f(c) \leq \frac{f(y)-f(x)}{y-x}(c-x)+f(x) = kf(x)+(1-k)f(z),$$

which mean that f is convex on I. Similarly we may prove that (1) is deduced by (3). □

Lemma 2.2 *Let f be a real-valued function on an interval I. Then the Following three conditions are equivalent.*

(1) *f is convex on I.*

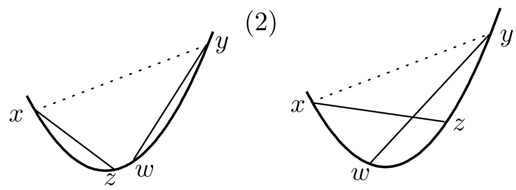
(2) *For $x<y$, $x,y \in I$ and $z,w \in (x,y)$ we have*

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(w)}{y-w}.$$

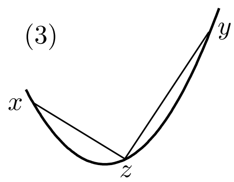
(3) *For $x<z<y$, $x,z,y \in I$ we have*

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}.$$

(2)



(3)



proof. (1)⟹(2) Let f be a convex function on I and $x<y$, $x,y \in I$, $z,w \in (x,\ y)$. Then Lemma2.1 implies

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(y)-f(w)}{y-w},$$

thus (2) holds

(2)⟹(3) is trivial.

(3)⟹(1) Let $x<y$, $x,y \in I$ and put $c = kx+(1-k)y$, $0<k<1$. Then by the assumption we have

$$\frac{f(c)-f(x)}{c-x} \leq \frac{f(y)-f(c)}{y-c},$$

thus $f(c) \leq kf(x)+(1-k)f(y)$ and (1) holds. □

Corollary 2.3 (Cheraru) *A function f is convex on an interval I if and only if for any $y \in I$*

$$F(x) = \frac{f(y)-f(x)}{y-x}, \quad x \in I \backslash \{y\}$$

is nondecreasing on I.

Corollary 2.4[4] *Let g be a non-decreasing function on an interval I in the real line. Suppose that the function f satisfies*

$$f(x) = \int_a^x g(t)\,dt + f(a)$$

for any $a, x \in I$. Then f is convex on I.

proof. Let $x < z < y,\ x, y, z \in I$. Since g is non-decreasing we have

$$\frac{f(z) - f(x)}{z - x} = \frac{\int_x^z g(s)\,ds}{z - x} \leq g(z)$$

and

$$\frac{f(y) - f(z)}{y - z} = \frac{\int_z^y g(s)\,ds}{y - z} \geq g(z),$$

thus the statement (3) in Lemma2.2 holds. Hence f is a convex function on I. □

Examples. Here we give some examples of the convex functions on $I = (0,\ 1)$.

(1) Let $0 < \alpha < 1$ and put $f(x) = -x^\alpha$, $g(x) = -\alpha x^{\alpha-1}$. Since g is increasing on I and since $f(x) = \int_a^x g(s)\,ds + f(a),\ x, a \in I$, f is a convex function on I. For $x < y,\ x, y \in I$

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{\int_x^y |g(s)|\,ds}{|x - y|} \geq g(y) \ \rightarrow\ +\infty \quad (y \rightarrow 0),$$

thus f is *not* Lipschitz continuous on I. On the other hand, f is absolutely continuous on I because g is integrable on I.

(2) Let $\alpha < 0$ or $\alpha > 1$ and put $f(x) = x^\alpha$, $g(x) = \alpha x^{\alpha-1}$, then f is convex on I. For $\alpha > 1$ f is Lipschitz continuous on I, hence it is also absolutely continuous on there. On the other hand, for $\alpha < 0$ f is *not* absolutely continuous on I because g is not integrable on I.

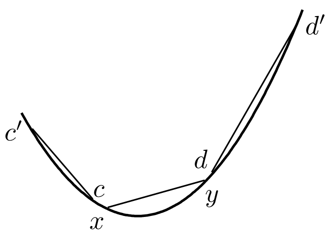
3 Absolute continuity of the convex function

In this section we always consider a convex function f on an open interval $I = (a,\ b)$.

Fix any $c, d \in I,\ c < d$ and choose $c', d' \in I$ such that $c' < c < d < d'$.

For $x, y \in [c,\ d],\ x < y$, we have by Lemma2.2

$$\frac{f(c) - f(c')}{c - c'} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(d') - f(d)}{d' - d}.$$



By Lemma2.1, since $\dfrac{f(y) - f(x)}{y - x}$ is non-decreasing as $x \nearrow y$, and since $\dfrac{f(y) - f(x)}{y - x}$ is non-increasing as $y \searrow x$, the one-sided limits

$$D^- f(y) = \lim_{x \nearrow y} \frac{f(y) - f(x)}{y - x} \qquad \textit{and} \qquad D^+ f(x) = \lim_{y \searrow x} \frac{f(y) - f(x)}{y - x}$$

exist and bounded on $[c,\ d]$. On the other hand Lemma2.2 implies that $D^- f(x)$ and $D^+ f(x)$ are non-decreasing on $[c,\ d]$. Moreover for any $x \in [c,\ d]$ we have $D^- f(x) \leq D^+ f(x)$ by Lemma2.2.

Hence for $c < x < y < d$ we have the following well known results.

$$(\bigstar\bigstar) \quad \frac{f(c) - f(c')}{c - c'} \leq D^- f(x) \leq D^+ f(x) \leq \frac{f(y) - f(x)}{y - x} \leq D^- f(y) \leq D^+ f(y) \leq \frac{f(d') - f(d)}{d' - d}.$$

Then we may prove the following result immediately.

Proposition 3.1 *Let f be a convex function on an open interval $(a,\ b)$. Then f is Lipschitz continuous on $(a,\ b)$ if and only if*

$$\limsup_{y\searrow a}\left(\limsup_{x\searrow a}\left|\frac{f(y)-f(x)}{y-x}\right|\right)<+\infty \quad \textit{and} \quad \limsup_{x\nearrow b}\left(\limsup_{y\nearrow b}\left|\frac{f(y)-f(x)}{y-x}\right|\right)<+\infty.$$

Now we shall prove the following theorem which provides Theorem1.1 concerning convex function.

Theorem 3.2 *Let f be a convex function on an open interval $(a,\ b)$. If*

$$\limsup_{x\searrow a}|f(x)|<+\infty \quad \textit{and} \quad \limsup_{x\nearrow b}|f(x)|<+\infty$$

then f is absolutely continuous on $(a,\ b)$. If the interval is finite then f is absolutely continuous on $(a,\ b)$ if and only if f is bounded on there.

proof. Assume $\limsup_{x\searrow a}|f(x)|<+\infty$ and $\limsup_{x\nearrow b}|f(x)|<+\infty$. There are four cases :

(1) $Df_+(x)$ is bounded on $(a,\ b)$, (2) $\lim_{x\searrow a} D^+f(x)=-\infty$ and $\lim_{x\nearrow b} D^+f(x)=+\infty$,

(3) $\lim_{x\searrow a} D^+f(x)>-\infty$ and $\lim_{x\nearrow b} D^+f(x)=+\infty$, (4) $\lim_{x\searrow a} D^+f(x)=-\infty$ and $\lim_{x\nearrow b} D^+f(x)<+\infty$.

If (1) is true then f is Lipschitz continuous on $(a,\ b)$ by (★★), so f is absolutely continuous on $(a,\ b)$. Next we shall consider the case (2). Then there exist $c,d\in(a,\ b),\ c<d$ such that

$$D^+f(x)<0 \quad \textit{for} \quad x\in(a,\ c)$$

and

$$D^+f(x)>0 \quad \textit{for} \quad x\in(d,\ b).$$

Since

$$\frac{f(y)-f(x)}{y-x}\le D^+f(y)<0 \quad \textit{for} \quad x,y\in(a,\ c),\ y>x$$

and since

$$0<D^+f(x)\le\frac{f(y)-f(x)}{y-x} \quad \textit{for} \quad x,y\in(d,\ b),\ y>x,$$

it is obtained that f is non-increasing on $(a,\ c)$ and is non-decreasing on $(d,\ b)$. Hence $\lim_{x\searrow a} f(x)>-\infty$ and $\lim_{x\nearrow b} f(x)<+\infty$ by the assumption (2).

Fix any $\varepsilon>0$. Choose $a_\varepsilon\in(a,\ c)$ and $b_\varepsilon\in(d,\ b)$ such that

$$0<\lim_{x\searrow a} f(x)-f(a_\varepsilon)<\frac{\varepsilon}{3} \quad \textit{and} \quad 0<\lim_{x\nearrow b} f(x)-f(b_\varepsilon)<\frac{\varepsilon}{3}.$$

Put $K=\max\{\,|D^+f(a_\varepsilon)|,\ |D^+f(b_\varepsilon)|\,\}$ then by (★★) we have

$$|f(x)-f(y)|\le K|x-y| \quad \textit{for} \quad x,y\in[a_\varepsilon,\ b_\varepsilon].$$

Denote $\delta=\dfrac{\varepsilon}{3K}$ and consider a finite family of pairwise disjoint sub-intervals $\{[a_k,\ b_k)\subseteq I\}$ such that $\sum_k|b_k-a_k|<\delta$. Then

$$\cup_k[a_k,\ b_k)\subseteq(a,\ a_\varepsilon) \quad\Longrightarrow\quad \sum_k|f(b_k)-f(a_k)|\le\lim_{x\searrow a} f(x)-f(a_\varepsilon)<\frac{\varepsilon}{3},$$

$$\cup_k[a_k,\ b_k)\subseteq[a_\varepsilon,\ b_\varepsilon] \quad\Longrightarrow\quad \sum_k|f(b_k)-f(a_k)|<K\delta=\frac{\varepsilon}{3},$$

$$\cup_k[a_k,\ b_k)\subseteq(b_\varepsilon,\ b) \quad\Longrightarrow\quad \sum_k|f(b_k)-f(a_k)|\le\lim_{x\nearrow b} f(x)-f(b_\varepsilon)<\frac{\varepsilon}{3}.$$

Therefore f is absolutely continuous on $(a,\ b)$. By the similar arguments, it is clear that f is absolutely continuous on $(a,\ b)$ in other cases.

Assume that f is absolutely continuous on the finite interval $I=(a,\ b)$. Then there exists $\delta>0$ such that if the finite family of pairwise disjoint sub-intervals $\{[a_k,\ b_k)\subseteq I\}$ satisfies $\sum_k |b_k-a_k|<\delta$ then

$$\sum_k |f(a_k)-f(b_k)|<1.$$

Choose an integer n such that $\dfrac{b-a}{n}<\delta$ and put $x_k=a+\dfrac{b-a}{n}k$ $(k=0,1,2,\ldots,n)$. Then

$$|\,f(x)-f(x_k)|<1 \quad for \quad x\in(x_{k-1},\ x_k),\ \ k=1,2,\ldots,n$$

Denote

$$M=\max\left\{\,|f(x_1)|\,,\,|f(x_2)|\,,\,\cdots\,,\,|f(x_{n-1})|\,\right\}+1$$

Then we have $|f(x)|<M$ for any $x\in(a,\ b)$, hence f is bounded on I. □

Examples. Here we construct examples of the convex functions which are not differentiable at a denumerable set of points.

(1) Let $a_n=1-\dfrac{1}{n}=\dfrac{n-1}{n}$, $n=1,2,3,\ldots$ and $\{b_n\}$ a *convergence* sequence of positive numbers. Denote for $s\in I=(0,\ 1)$

$$g(s)=\sum_{n=1}^{\infty}b_n 1_{[a_n,\,1)}(s)=\begin{cases} b_1, & 0<s<a_2\\ b_1+b_2, & a_2\le s<a_3\\ \vdots & \vdots\\ b_1+b_2+\cdots+b_n, & a_n\le s<a_{n+1}\\ \vdots & \vdots \end{cases}$$

Then g is non-decreasing on I. Hence the following function f defined in I is convex on I by Corollary2.4.

$$f(x)=\int_0^x g(s)\,ds=\begin{cases} b_1x, & 0<x<a_2\\ \dfrac{b_1}{2}+(b_1+b_2)(x-\dfrac{1}{2}), & a_2\le x<a_3\\ \vdots & \vdots\\ \displaystyle\sum_{k=1}^{n-1}\sum_{l=1}^{k}\frac{b_l}{k(k+1)}+\sum_{k=1}^{n}b_k(x-\frac{n-1}{n}), & a_n\le x<a_{n+1}\\ \vdots & \vdots \end{cases}$$

By Proposition3.1 and Theorem3.2, we have

① f is Lipschitz continuous on I if and only if $\sum_n b_n<+\infty$,

② f is absolutely continuous on I if and only if $\sum_n \dfrac{b_n}{n}<+\infty$.

On the other hand we have

③ $f'(x)=b_1+b_2+\cdots+b_n \quad for \quad a_n<x<a_{n+1}$,

④ f is not differentiable at $x=a_2,\ a_3,\ a_4,\ \ldots$. □

(2) Let $\{a_n\}$ be a non-decreasing sequence of positive numbers. Denote $b_1=a_1,\ b_n=a_n-a_{n-1}\ (n\ge 2)$.

Also denote for $x \in I = (0, +\infty)$

$$g(x) = \sum_{n=1}^{\infty} b_n 1_{[n-1,\, n)}(x) \qquad \textit{and} \qquad f(x) = \int_0^x g(s)\, ds.$$

Then f is convex on I by Corollary2.4. By Proposition3.1 and Theorem3.2, we have

① f is Lipschitz continuous on I if and only if $\sup_n a_n < +\infty$,

② g is integrable on I if and only if $\sum_n a_n < +\infty$.

On the other hand f is not differentiable at the set of natural numbers. □

Remark. By (★★) we get the following results which was proved by Zygmund[4].

Let f be a convex function on a finite open interval $(a,\, b)$.

Then f is Lipschitz continuous on $[c,\, d]$ for any $c, d \in I$, $c < d$, thus it is absolutely continuous on there.

f has the unilateral derivatives $D^+f(x)$ and $D^-f(x)$ for any $x \in (a,\, b)$ and they are non-decreasing on $(a,\, b)$, hence they are continuous almost everywhere. Moreover they satisfies for any $x, y \in (a,\, b)$, $x < y$

$$D^-f(x) \le D^+f(x) \le D^-f(y).$$

Thus $D^-f(x) = D^+f(x)$ almost everywhere, that is f is differentiable almost everywhere.

Denote $g(x) = \lim_{y \searrow x} D^-f(x)$, $x \in (a,\, b)$, then g is non-decreasing right-continuous on $(a,\, b)$ and we have for any $x, c \in (a,\, b)$

$$f(x) = \int_c^x g(s)\, ds + f(c).$$

Combining this results and Theorem3.2, we understand that a convex function on a closed interval $[a,\, b]$ is absolutely continuous on there.

Proof of Theorem1.4(*concerning convex function*) Let f be a convex function on an infinite interval $I = (a,\, b)$ and f' be the derivative of f. Since f' is non-decreasing there exist $a < a' < b' < b$ such that $f'(x)$ has same sign on $(a,\, a')$ and $(b',\, b)$. Then we have

$$\left| f(a') - f(x) \right| = \left| \int_x^{a'} f'(s)\, ds \right| = \int_x^{a'} |f'(s)|\, ds \quad \textit{for any } x \in (a,\, a')$$

and

$$\left| f(x) - f(b') \right| = \left| \int_{b'}^{x} f'(s)\, ds \right| = \int_{b'}^{x} |f'(s)|\, ds \quad \textit{for any } x \in (b',\, b).$$

Furthermore since f is bounded on $[a',\, b']$ the proposition may be proved. □

4 Q-convex function

Recall that a Q-convex function is a function defined on I which satisfies

$$(\text{☆}) \qquad f\Big(\frac{x_1 + x_2 + \cdots + x_p}{p}\Big) \le \frac{f(x_1) + f(x_2) + \cdots + f(x_p)}{p} \quad \textit{for any} \quad x_1, x_2, \ldots, x_p \in I.$$

for some integer $p \ge 2$

Lemma 4.1 *Let f be a Q-convex function on an interval I which satisfies (☆). Then we have for any positive integer n*

$$(\text{☆☆}) \qquad f\Big(\frac{x_1 + x_2 + \cdots + x_{p^n}}{p^n}\Big) \le \frac{f(x_1) + f(x_2) + \cdots + f(x_{p^n})}{p^n} \quad \textit{for any } x_1, x_2, \ldots, x_{p^n} \in I.$$

proof. The condition (☆☆) is trivial for $n = 1$. Assume (☆☆) hods for n. Then for $x_1, x_2, \dots, x_{p^{n+1}} \in I$

$$
\begin{aligned}
& f\Big(\frac{x_1 + x_2 + \cdots + x_{p^{n+1}}}{p^{n+1}}\Big) \\
&= f\left(\frac{\frac{x_1+\cdots+x_{p^n}}{p^n} + \frac{x_{p^n+1}+\cdots+x_{2p^n}}{p^n} + \cdots + \frac{x_{(p-1)p^n+1}+\cdots+x_{p^{n+1}}}{p^n}}{p}\right) \\
&\leq \frac{f\big(\frac{x_1 + \cdots + x_{p^n}}{p^n}\big) + f\big(\frac{x_{p^n+1} + \cdots + x_{2p^n}}{p^n}\big) + \cdots + f\big(\frac{x_{(p-1)p^n+1} + \cdots + x_{p^{n+1}}}{p^n}\big)}{p} \\
&\leq \frac{\frac{f(x_1) + \cdots + f(x_{p^n})}{p^n} + \frac{f(x_{p^n+1}) + \cdots + f(x_{2p^n})}{p^n} + \cdots + \frac{f(x_{(p-1)p^n+1}) + \cdots + f(x_{p^{n+1}})}{p^n}}{p} \\
&= \frac{f(x_1) + f(x_2) + \cdots + f(x_{p^{n+1}})}{p^{n+1}}.
\end{aligned}
$$

Thus we complete the proof by induction. □

The following result is well known.

Lemma 4.2 *Let f be a Q-convex function on an interval I. Then f is convex on I if and only if f is continuous on there.*

proof. If f is convex on I then f is continuous on I by Zygmund result, and Q-convex trivially.

Let f be a continuous function on I and satisfies (☆☆). Let $x, y \in I$, $x < y$ and $0 < k < 1$. Choose a sequence of integers $\{k_n\}$, $0 < k_n \leq p^n$ such that

$$\Big|k - \frac{k_n}{p^n}\Big| \leq \frac{1}{p^n}.$$

Then by Lemma4.1 we have

$$f(kx + (1-k)y) = \lim_{n\to\infty} f\big(\frac{k_n}{p^n}x + (1 - \frac{k_n}{p^n})y\big) \leq \lim_{n\to\infty} \Big\{\frac{k_n}{p^n} f(x) + (1 - \frac{k_n}{p^n}) f(y)\Big\} = kf(x) + (1-k)f(y),$$

which complete the proof. □

Proof of Theorem1.1 and Theorem1.4 (*concerning Q-convex function*) If f is absolutely continuous on a finite interval I then it is bounded on there by the proof of Theorem3.2.

Let f be a Q-convex function on a finite or infinite interval $I = (a,\, b)$. Assume $\sup_{x\in I} |f(x)| = K < +\infty$. We shall prove that f is continuous on I.

Fix any $x \in I$ and any $\varepsilon > 0$ and choose $h > 0$ and integer n such that

$$(x - h,\ x + h) \subseteq I \quad \textit{and} \quad \frac{2K}{p^n} < \varepsilon.$$

Put $\delta = \dfrac{h}{p^n}$ then if $|x - y| < \delta$ we have

$$
\begin{aligned}
f(y) - f(x) &= f\Big(\frac{(p^n - 1)x + \{\, x + p^n(y - x)\,\}}{p^n}\Big) - f(x) \\
&\leq \frac{(p^n - 1)}{p^n} f(x) + \frac{1}{p^n} f\big(x + p^n(y - x)\big) - f(x) \\
&= \frac{1}{p^n}\big\{f\big(x + p^n(y - x)\big) - f(x)\big\} \leq \frac{2K}{p^n} < \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
f(y)-f(x) &= f(y)-f\Big(\frac{(p^n-1)y+\{\,x-(p^n-1)(y-x)\,\}}{p^n}\Big)\\
&\geq f(y)-\frac{(p^n-1)}{p^n}f(y)-\frac{1}{p^n}f\big(x-(p^n-1)(y-x)\big)\\
&= \frac{1}{p^n}\{f(y)-f\big(x-(p^n-1)(y-x)\big)\}\geq\frac{-2K}{p^n}>-\varepsilon,
\end{aligned}
$$

hence $|f(y)-f(x)|<\varepsilon$, so that f is continuous on I.

Then Lemma4.2 implies that f is convex on I, hence f is absolutely continuous on I by Theorem3.2. Moreover Theorem1.1 implies that the derivative of f is Lebesgue integrable on I. □

The following Proposition explain why we use the term *Q*uotient.

Proposition 4.3[5] *Let f be a Q-convex function on an interval I. Then for any integer $n\geq 2$ we have*

$$
f(\frac{x_1+x_2+\cdots+x_n}{n})\leq\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\quad \textit{for any } x_1,x_2,\ldots,x_n\in I,
$$

hence $f(kx+(1-k)y)\leq kf(x)+(1-k)y)$ for any rational number $0<k<1$ and any $x,y\in I$.

proof. Assume

$$
f(\frac{x_1+x_2+\cdots+x_{p^n}}{p^n})\leq\frac{f(x_1)+f(x_2)+\cdots+f(x_{p^n})}{p^n}\quad \textit{for any } x_1,x_2,\ldots,x_{p^n}\in I,
$$

where $p\geq 2$ and $n\geq 1$. Fix any integer $n\geq 2$ and $x_1,x_2,\ldots,x_n\in I$. Denote

$$
y_k=\begin{cases} x_k & k=1,2,\ldots,n\\ \dfrac{x_1+x_2+\cdots+x_n}{n} & k=n+1,n+2,\ldots,p^n\end{cases}
$$

then by the assumption

$$
\begin{aligned}
f\Big(\frac{y_1+y_2+\cdots+y_{p^n}}{p^n}\Big) &\leq \frac{f(y_1)+f(y_2)+\cdots+f(y_{p^n})}{p^n}\\
&= \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{p^n}+\frac{p^n-n}{p^n}f\Big(\frac{x_1+x_2+\cdots+x_n}{n}\Big).
\end{aligned}
$$

On the other hand

$$
\frac{y_1+y_2+\cdots+y_{p^n}}{p^n}=\frac{x_1+x_2+\cdots+x_n}{n},
$$

thus

$$
f(\frac{x_1+x_2+\cdots+x_n}{n})\leq\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n},
$$

which complete the proof. □

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