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The values of Hilbert-Eisenstein series at cusps

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ヒルベルト・アイゼンシュタイン級数の尖点での値

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Abstract. The Fourier coefficients, in particular the constant terms, of Hilbert-Eisenstein series have the number theoretic importance. The value at a cusp gives the constant terms of the Fourier expansion centered at the cusp. We give the values at all the cusps equivalent to $\sqrt{-1}\infty$, of some specific Hilbert-Eisenstein series whose Fourier coefficients of higher terms are in rather simple form. The result may be useful to obtain the special values of L-functions or to investigate the Shimura lifting for elliptic modular forms.

1. Introduction

Let K be a totally real algebraic number field of degree g over $\mathbf{Q}$, and let $\mathcal{O}_K$ be the ring of algebraic integers. Hilbert-Eisenstein series have essentially the special values of L-function of K as the values at cusps. Siegel [3] obtained the explicit arithmetic expressions of the special values of the zeta function of K at negative integers by exploiting the values of Hilbert-Eisenstein series for $\mathrm{SL}_2(\mathcal{O}_K)$ at the cusp $\sqrt{-1}\infty$. This method is generalized in [5], where for an integral ideal $\mathfrak{N}$, the values of Hilbert-Eisenstein series for $\Gamma_0(\mathfrak{N})_K := \{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) \mid \gamma \in \mathfrak{N}\}$ at the cusps equivalent to $\sqrt{-1}\infty$ are necessary

In [6],[7], the Shimura lifts of the product of the theta series and Eisenstein series are investigated. They are given as the restrictions to the diagonal, of Hilbert-Eisenstein series of real quadratic fields. To make a close investigation of the lifts, we also need to explicit formula for the values at cusps, of Hilbert-Eisenstein series. This is the purpose of the present paper.

We denote by $\mathfrak{d}_K$ and D_K, the different of K and the discriminant respectively. Let μ_K denote the Möbius function on K and let φ_K denote the Euler function on K. If $\mathfrak{P}$ is an prime ideal, then $v_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-adic valuation. If $\mathfrak{M}$ is an integral ideal, then $\{\mathfrak{M}\}_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-part of $\mathfrak{M}$, namely, $\{\mathfrak{M}\}_{\mathfrak{P}} = \mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{M})}$. Let $\mathfrak{N}$ be an integral ideal of K. Then $\mathcal{E}_{\mathfrak{N}}$ denotes the group of units $\varepsilon \succ 0$ congruent to 1 modulo $\mathfrak{N}$ where $\varepsilon \succ 0$ means that ε is totally positive. We denote by $C_{\mathfrak{N}}$, the narrow ray class group modulo $\mathfrak{N}$, and by $C_{\mathfrak{N}}^*$, the group of characters. A character $\psi \in C_{\mathfrak{N}}^*$ is called *even* (resp. *odd*) if it satisfies $\psi(\mu) = 1$ (resp. $\psi(\mu) = \mathrm{sgn}(\mathrm{N}(\mu))$ for $\mu \in \mathcal{O}_K, \neq 0$ congruent 1 modulo $\mathfrak{N}$. We exclusively work with even or odd characters in the present paper. The identity element of $C_{\mathfrak{N}}^*$ is denoted by $\mathbf{1}_N$, for which $\mathbf{1}_{\mathfrak{N}}(\mathfrak{A})$ is 1 or 0 according as an integral ideal $\mathfrak{A}$ is coprime to $\mathfrak{N}$ or not. For

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a character ψ, e_ψ is define to be 0 or 1 according as ψ is even or odd. We define the value of the characters at non-integral ideals to be 0. The conductor of a character ψ is denoted by $\mathfrak{f}_\psi$. For an integral ideal $\mathfrak{M}$, $\mathcal{R}(\mathfrak{M},\psi)$ denotes the set of all the products of primes divisors $\mathfrak{P}$ of $\mathfrak{M}$ coprime to $\mathfrak{f}_\psi$ with multiplicity one or zero. We note that $\mathcal{R}(\mathfrak{M},\psi)\neq\emptyset$ because $\mathcal{O}_K$ is always in it. If $\psi\in C^*_{\mathfrak{N}}$, then we denote by $\mathfrak{R}_{\mathfrak{N},\psi}$, the minimal divisor of $\mathfrak{N}$ satisfying $(\mathfrak{R}_{\mathfrak{N},\psi},\mathfrak{f}_\psi)=\mathcal{O}_K$. We denote by $\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}$, its radical, which is in $\mathcal{R}(\mathfrak{N},\psi)$. The primitive character associated with $\psi\in C_{\mathfrak{N}}$ is denoted by $\widetilde{\psi}$. For any integral ideal $\mathfrak{M}$ we define $\psi_{\mathfrak{M}}:=\widetilde{\psi}\mathbf{1}_{\mathfrak{M}}$. Then $\psi_{\mathfrak{M}}=\widetilde{\psi}$ for an integral ideal $\mathfrak{M}$ with $\mathfrak{M}|\mathfrak{f}_\psi$, and $\psi_{\mathfrak{N}}=\psi$.

For $\alpha\in K$, $\alpha^{(1)},\cdots,\alpha^{(g)}$ denotes the conjugates of α in a fixed order. We denote by N and tr, the norm map and the trace map of K over $\mathbf{Q}$ respectively, namely $\mathrm{N}(\alpha)=\prod_{i=1}^g\alpha^{(i)}$ and $\mathrm{tr}(\alpha)=\sum_{i=1}^g\alpha^{(i)}$. For $\psi\in C^*_{\mathfrak{N}}$, $L_K(s,\psi)$ denotes the Hecke L-function, that is,

$$L_K(s,\psi):=\sum_{\mathfrak{A}}\frac{\psi(\mathfrak{A})}{\mathrm{N}(\mathfrak{A})^s}$$

where $\mathfrak{A}$ runs over the set of all the integral ideal. Let $\mathfrak{H}^g$ denote the product of g copies of the upper half plane $\mathfrak{H}=\{z\in\mathbf{C}\mid\Im z>0\}$, $\Im z$ being the imaginary part of z. For $\gamma,\delta\in K$ and for $\mathfrak{z}=(z_1,\cdots,z_g)\in\mathfrak{H}$, $\mathrm{N}(\gamma\mathfrak{z}+\delta)$ stands for $\prod_{i=1}^g(\gamma^{(i)}z_i+\delta^{(i)})$, and for $\nu\in K$, $\mathrm{tr}(\nu\mathfrak{z})$ stands for $\sum_{i=1}^g\nu^{(i)}z_i$. Let

$$A=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\mathrm{SL}_2(\mathcal{O}_K).\tag{1}$$

Then we put $A\mathfrak{z}=\big(\frac{\alpha^{(1)}z_1+\beta^{(1)}}{\gamma^{(1)}z_1+\delta^{(1)}},\cdots,\frac{\alpha^{(g)}z_g+\beta^{(g)}}{\gamma^{(g)}z_g+\delta^{(g)}}\big)$.

Let $\mathfrak{N},\mathfrak{N}'$ be two fixed integral ideals of K. Let $\mathfrak{A}$ be an integral ideal of K and let k be a natural number. Let $\gamma_0\in\mathfrak{A}\mathfrak{d}_K^{-1},\delta_0\in\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}$. We define

$$E_{k,\mathfrak{A}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}'):=\mathrm{N}(\mathfrak{A})^k\Big(\sideset{}{'}\sum_{\substack{\gamma\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\\delta\equiv\delta_0(\mathfrak{A}\mathfrak{d}_K^{-1})\\(\gamma,\delta)/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}|\mathrm{N}(\gamma\mathfrak{z}+\delta)|^{-s}\Big)\Big|_{s=0}$$

where $\gamma\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})$ implies that $\gamma\equiv\gamma_0$ modulo $\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}$ and where $\sum'$ implies that the term corresponding to $(\gamma,\delta)=(0,0)$ is omitted in the summation. For a set S, $\Delta(x,S)$ is define to be 1 or 0 according as $x\in S$ or not. For $z\in\mathbf{C}$, we put $\mathbf{e}(z)=e^{2\pi\sqrt{-1}z}$. Then we have the Fourier expansion

$$\begin{aligned}&E_{k,\mathfrak{A}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')\\&=\Delta(\gamma_0,\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\mathrm{N}(\mathfrak{A})^k\Big(\sum_{\substack{\mu\equiv\delta_0(\mathfrak{A}\mathfrak{d}_K^{-1})\\\mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}\Big)\Big|_{s=0}\\&\quad+\Big(\frac{(-2\pi\sqrt{-1})^k}{(k-1)!}\Big)^gD_K^{1/2}\mathrm{N}(\mathfrak{A})^{k-1}\sum_{0\prec\nu\in\mathfrak{d}_K^{-1}}\sum_{\substack{\nu/\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\\mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathbf{e}(\mathrm{tr}(\delta_0\mu))\\&\qquad\times\mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))\end{aligned}$$

where there is the additional term

$$(-\pi\sqrt{-1})^g D_K^{1/2}\Big(\sum_{\substack{\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}\Big)|_{s=0}$$

when $k=1$, and where there is the additional term $-\frac{\pi\sqrt{-1}}{\Im z}$ when $g=1$ and $k=2$.

Let $\psi\in C^*_{\mathfrak{N}}, \psi'\in C^*_{\mathfrak{N}'}$ be even or odd characters so that $k\in\mathbf{N}$ and $\psi\psi'$ have the same parity. We assume that

$$(\mathfrak{N}, \mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\psi',\mathfrak{N}'}^{-1}) = \mathcal{O}_K. \tag{2}$$

We put

$$\begin{aligned}&\widetilde{\lambda}^{\psi'}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}(\mathfrak{z})\\ &:=\Big(\frac{(k-1)!}{(-2\sqrt{-1}\pi)^k}\Big)^g D_K^{-1/2}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\psi})^{-1}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sum_{\substack{\gamma_0:\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}}\\ &\qquad\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)E_{k,\mathfrak{A}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')\end{aligned}\tag{3}$$

where τ_K denotes the Gauss sum defined in (5), and where we assume that either $\psi\neq\mathbf{1}_{\mathfrak{N}}$ or $\psi'\neq\mathbf{1}_{\mathfrak{N}'}$ when $g=1$ and $k=2$. Further let

$$\begin{aligned}\widetilde{\Lambda}^{\psi'}_{k,\psi}(\mathfrak{z}) &= \mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_{\psi}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\\ &\quad\times\sum_{\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}}\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M})\widetilde{\lambda}^{\psi'}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}(\mathfrak{z}).\end{aligned}\tag{4}$$

We determine the values at cusps equivalent to $\sqrt{-1}\infty$, of this Hilbert-Eisenstein series (4) as well as the Fourier expansion at the cusp $\sqrt{-1}\infty$.

2. Gauss sums

Let ψ be a primitive character of an ideal class group of K. The Gauss sum of ψ is defined by

$$\tau_K(\psi) := \psi(\rho\mathfrak{f}_\psi\mathfrak{d}_K)\sum_{\substack{\xi\succ 0\\ \xi:\mathcal{O}_K/\mathfrak{f}_\psi}}\psi(\xi)\mathbf{e}(\mathrm{tr}(\rho\xi)) \tag{5}$$

with $\rho\in K$, $\succ 0$, $(\rho\mathfrak{f}_\psi\mathfrak{d}_K,\mathfrak{f}_\psi)=\mathcal{O}_K$. The value $\tau_K(\psi)$ is determined up to the choices of ρ.

Lemma 1. *Let $\mathfrak{A}$ be a non-zero integral ideal. Let $\psi\in C^*_{\mathfrak{N}}$, which is not necessarily primitive.*

(i) *Let $\mu\in\mathfrak{A}^{-1}$. Then*

$$\begin{aligned}&\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))\\ &=\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\tau_K(\overline{\widetilde{\psi}})\sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}}\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\widetilde{\psi}}(\mathfrak{R})\psi_{\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\mathfrak{R}).\end{aligned}\tag{6}$$

In the summation of the right hand side, at most one term survives. If there is $\mathfrak{R}\in\mathcal{R}(\mathfrak{N},\psi)$ satisfying $(\mu\mathfrak{A}\mathfrak{N}^{-1},\mathcal{O}_K)=\mathfrak{f}_\psi^{-1}\mathfrak{R}^{-1}$, then the term associated with $\mathfrak{R}$ survives.

(ii) *Let* $\mu \in \mathfrak{A}^{-1}\mathfrak{N}^{-1}$. *Then*

$$\sum_{\delta_0:\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \psi(\delta_0\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\delta_0\mu))$$

$$= \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\tau_K(\widetilde{\psi})\sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}} \mu_K(\mathfrak{R})\frac{\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})}{\varphi_K(\mathfrak{R})}\widetilde{\psi}(\mathfrak{R})\overline{\psi}_{\mathfrak{R}}(\mu\mathfrak{N}\mathfrak{A}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{R}). \quad (7)$$

In the summation of the right hand side, at most one term survives. If there is $\mathfrak{R} \in \mathcal{R}(\mathfrak{N},\psi)$ *satisfying* $(\mu\mathfrak{A}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}, \mathcal{O}_K) = \mathfrak{R}^{-1}$, *then the term associated with* $\mathfrak{R}$ *survives.*

Proof. (i) At first we prove the following; if ψ is primitive, then

$$\sum_{\substack{\xi\succ 0\\ \xi:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{f}_\psi\mathfrak{A}\mathfrak{d}_K^{-1}}} \overline{\psi}(\xi\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\xi\mu)) = \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\psi(\mu\mathfrak{A}\mathfrak{f}_\psi)\tau_K(\overline{\psi}) \quad (8)$$

for $\mu \in \mathfrak{A}^{-1}\mathfrak{f}_\psi^{-1}, \neq 0$. Let $\alpha \in \mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0$ with $(\alpha\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{A}\mathfrak{f}_\psi\mathfrak{d}_K) = \mathcal{O}_K$. Then $\sum_{\substack{\xi\succ 0\\ \xi:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{f}_\psi\mathfrak{A}\mathfrak{d}_K^{-1}}} \overline{\psi}(\xi\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\xi\mu)) = \sum_{\substack{\xi\succ 0\\ \xi:\mathcal{O}/\mathfrak{f}_\psi}} \overline{\psi}(\alpha\xi\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\alpha\xi\mu)) = \overline{\psi}(\alpha\mathfrak{A}^{-1}\mathfrak{d}_K)\sum_{\substack{\xi\succ 0\\ \xi:\mathcal{O}/\mathfrak{f}_\psi}} \overline{\psi}(\xi)\mathrm{e}(\mathrm{tr}(\xi(\alpha\mu)))$ where the summation of the extreme right hand side is equal to $\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\psi(\alpha\mu\mathfrak{f}_\psi\mathfrak{d}_K)\tau_K(\overline{\psi})$ by [4] Theorem 13. This shows the equality (8).

Let us take $\alpha \in \mathfrak{N}^{-1}, \succ 0$ such that $\alpha\mathfrak{N} \subset \mathcal{O}_K$ and $(\alpha\mathfrak{N},\mathfrak{N}) = \mathcal{O}_K$. Let $\alpha\mu = \mu_1 + \mu_2$ where all the prime factors of the denominator of μ_1 (resp. μ_2) are divisors of $\mathfrak{f}_\psi$ (resp. $\mathfrak{R}_{\mathfrak{N},\psi}$). Then

$$\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\delta_0\mu))$$

$$= \sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\alpha\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\delta_0\alpha\mu))$$

$$= \overline{\psi}(\alpha\mathfrak{N})\sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\delta_0\mu_1))\mathrm{e}(\mathrm{tr}(\delta_0\mu_2)),$$

which is 0 unless $\mu_1 \in \mathfrak{A}^{-1}\mathfrak{f}_\psi^{-1}$ and $\mu_2 \in \mathfrak{A}^{-1}\mathfrak{R}$ for some $\mathfrak{R} \in \mathfrak{R}_{\mathfrak{N},\psi}$. In such case the above is equal to

$$\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\psi}(\alpha\mathfrak{N})\sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{f}_\psi\mathfrak{R}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\delta_0\mu_1))\mathrm{e}(\mathrm{tr}(\delta_0\mu_2))$$

$$= \mu_K(\mathfrak{R})\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\psi}(\alpha\mathfrak{N})\sum_{\delta_0:\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{f}_\psi\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\widetilde{\psi}}(\delta_0\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\delta_0\mu_1))$$

$$= \mu_K(\mathfrak{R})\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\overline{\psi}(\alpha\mathfrak{N})\mathrm{sgn}(\mathrm{N}(\mu_1))^{e_\psi}\widetilde{\psi}(\mu_1\mathfrak{A}\mathfrak{f}_\psi)\tau_K(\overline{\widetilde{\psi}}) \qquad \text{(by (8))}$$

$$= \mu_K(\mathfrak{R})\tfrac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\overline{\widetilde{\psi}}(\mathfrak{R})\widetilde{\psi}(\mu\mathfrak{A}\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R})\tau_K(\overline{\widetilde{\psi}}).$$

The factor $\psi_{\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\mathfrak{R})$ appearing in the right hand side of (6) is equal to $\widetilde{\psi}(\mu\mathfrak{A}\mathfrak{N}^{-1}\mathfrak{f}_\psi\mathfrak{R})$ if $(\mu\mathfrak{A}\mathfrak{N}^{-1}, \mathcal{O}_K) = \mathfrak{f}_\psi^{-1}\mathfrak{R}^{-1}$ with $\mathfrak{R} \in \mathcal{R}(\mathfrak{N},\psi)$, and it is 0 if otherwise. This shows (6).

(ii) It is enough in the left hand side of (7), to take the summation over the representatives δ_0 so that $(\delta_0\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$. Such δ_0 are written as products $\delta_0 = \delta_1\delta_2$ where δ_1 are the representatives of $\mathfrak{f}_\psi^{-1}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}$ modulo $\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}$ with $(\delta_0\mathfrak{f}_\psi\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$, and δ_2 are the representatives of $\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}$ modulo $\mathcal{O}_K$ with $(\rho_2\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}, \mathcal{O}_K) = \mathcal{O}_K$. We can take δ_1 (resp. δ_2) so that they are totally positive and that the differences of δ_1's (resp. δ_2's) are in $\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}$ (resp. $\mathfrak{f}_\psi$). We write μ in the form $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \mathfrak{A}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}^{-1}$ and $\mu_2 \in \mathfrak{A}^{-1}\mathfrak{f}_\psi\mathfrak{N}^{-1}$. Then the left hand side of (7) is equal to

$$\sum_{\delta_1,\delta_2} \psi(\delta_1\delta_2\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_1))\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2)).$$

Since $\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2))$ is independent of δ_1, this is equal to

$$\begin{aligned}
&\sum_{\delta_2}\left\{\sum_{\delta_1} \psi(\delta_1\delta_2\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_1))\right\}\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2))\\
&= \sum_{\delta_2}\mathrm{sgn}(\mathrm{N}(\mu_1))^{e_\psi}\overline{\widetilde{\psi}}(\mu_1\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}\mathfrak{A})\tau_K(\widetilde{\psi})\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2)) \qquad \text{(by (8))}\\
&= \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\overline{\widetilde{\psi}}(\mu\mathfrak{N}\mathfrak{A})\tau_K(\widetilde{\psi})\sum_{\delta_2}\mathbf{e}(\mathrm{tr}(\delta_1\delta_2\mu_2)).
\end{aligned}$$

The last summation is equal to $\mu_K(\mathfrak{R})\frac{\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})}{\varphi_K(\mathfrak{R})}$ if $(\mu_2\mathfrak{A}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}, \mathcal{O}_K) = \mathfrak{R}^{-1}, \mathfrak{R} \in \mathcal{R}(\mathfrak{N},\psi)$ or equivalently if $(\mu\mathfrak{A}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{N}, \mathcal{O}_K) = \mathfrak{R}^{-1}, \mathfrak{R} \in \mathcal{R}(\mathfrak{N},\psi)$. Thus for this $\mathfrak{R}$, the left hand side of (7) is equal to

$$\mu_K(\mathfrak{R})\frac{\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})}{\varphi_K(\mathfrak{R})}\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\tau_K(\widetilde{\psi})\widetilde{\psi}(\mathfrak{R})\overline{\widetilde{\psi}}(\mu\mathfrak{N}\mathfrak{A}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{R}).$$

The similar argument of the last part of the proof of (i) shows our assertion. □

Let X be some function on the set of ideals. We define $\Lambda_k(\mathfrak{N},\psi)$ for $\psi \in C_{\mathfrak{N}}^*$ and for $k \in \mathbf{N}$ by

$$\begin{aligned}
\Lambda_k(\mathfrak{N},\psi)X := \mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\\
\times\sum_{\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}}\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M})X(\mathfrak{N}\mathfrak{M}\mathfrak{M}^{-1}). \qquad (9)
\end{aligned}$$

Proposition 1. *Let $\mathfrak{N}$ be an integral ideal and let $\psi \in C_{\mathfrak{N}}^*$. Let $\mathfrak{A}$ be an integral ideal so that $(\mathfrak{A},\mathfrak{N}) = \mathcal{O}_K$. Let $X_\mu(\mathfrak{M}) = \sum_{\delta_0:\mathfrak{M}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\overline{\psi}_{\mathfrak{M}}(\delta_0\mathfrak{M}\mathfrak{A}^{-1}\mathfrak{d}_K)\times$ $\mathbf{e}(\mathrm{tr}(\delta_0\mu))$ for $\mu \in \mathfrak{A}^{-1}$ and for an integral ideal $\mathfrak{M}$ containing $\mathfrak{f}_\psi$. Then*

$$\Lambda(\mathfrak{N},\psi)X_\mu = \mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k+1}\tau_K(\overline{\widetilde{\psi}})\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\psi(\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}), \qquad (10)$$

where we note that an ideal class character is define to be 0 at non-integral ideals.

Proof. Unless $(\mu\mathfrak{f}_\psi\mathfrak{N}^{-1}, \mathfrak{f}_\psi) = \mathcal{O}_K$, then the both sides of (10) are 0, and the equality holds. We assume that $(\mu\mathfrak{f}_\psi\mathfrak{N}^{-1}, \mathfrak{f}_\psi) = \mathcal{O}_K$. Let $\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}$ and put $\mathfrak{R} = \mathfrak{f}_\psi^{-1}(\mu\mathfrak{N}^{-1}\mathfrak{M}, \mathcal{O}_K)^{-1}$. Then by (6), $X(\mathfrak{N}\mathfrak{M}\mathfrak{M}^{-1}) = \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N}\mathfrak{M}\mathfrak{M}^{-1})}{\varphi_K(\mathfrak{f}_\psi\mathfrak{R})}\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}$ $\times\overline{\widetilde{\psi}}(\mathfrak{R})\widetilde{\psi}(\mu\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{f}_\psi\mathfrak{R})\tau_K(\overline{\widetilde{\psi}})$. Let $Y(\mathfrak{N}\mathfrak{M}\mathfrak{M}^{-1}) = \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})}{\varphi_K(\mathfrak{R})}\overline{\widetilde{\psi}}(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})$.

Then $X(\mathfrak{N}\mathfrak{M}^{-1})$ is equal to the product of $\frac{\varphi_K(\mathfrak{N}\mathfrak{R}_{\mathfrak{N},\psi}^{-1})}{\varphi_K(\mathfrak{f}_\psi)}\operatorname{sgn}(\mathrm{N}(\mu))^{e_\psi}\widetilde{\psi}(\mu\mathfrak{N}^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{A}\mathfrak{f}_\psi)$ $\times\tau_K(\overline{\widetilde{\psi}})$ and $Y(\mathfrak{N}\mathfrak{M}^{-1})$, where the former is the constant on $\mathfrak{M}$. We must compute $\Lambda_k(\mathfrak{N},\psi)Y$. We note that $\mathfrak{R}=\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K$. Then

$$\begin{aligned}
&\Lambda_k(\mathfrak{N},\psi)Y\\
&=\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\sum_{\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}}\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\\
&\qquad\times\overline{\widetilde{\psi}}(\mathfrak{M})\mu_K(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K)\frac{\varphi_K(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})}{\varphi_K(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K)}\overline{\widetilde{\psi}}(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})\\
&=\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\sum_{\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\cap\mathcal{O}_K|\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}}\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\\
&\qquad\times\mu_K(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K)\frac{\varphi_K(\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1})}{\varphi_K(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{M}^{-1}\cap\mathcal{O}_K)}\\
&=\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\prod_{\mathfrak{P}|\mathfrak{R}_{\mathfrak{N},\psi}}Z(\mathfrak{P}),
\end{aligned}$$

where

$$Z(\mathfrak{P})=\sum_{i=\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-1,0\}}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\prod_{\mathfrak{P}|\mathfrak{P}^i}(1-\mathrm{N}(\mathfrak{P}))\cdot\mu_K(\mathfrak{P}^{\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-i,0\}})\times\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})}{\varphi_K(\mathfrak{P}^{\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-i,0\}})}.$$

If $v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})\le 0$, then

$$\begin{aligned}
Z(\mathfrak{P})&=\sum_{i=0}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\prod_{\mathfrak{P}|\mathfrak{P}^i}(1-\mathrm{N}(\mathfrak{P}))\cdot\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})\\
&=\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})})+(1-\mathrm{N}(\mathfrak{P}))\sum_{i=1}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})=0.
\end{aligned}$$

If $v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})=1$, then

$$\begin{aligned}
Z(\mathfrak{P})&=\sum_{i=0}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\prod_{\mathfrak{P}|\mathfrak{P}^i}(1-\mathrm{N}(\mathfrak{P}))\cdot\mu_K(\mathfrak{P}^{\max\{1-i,0\}})\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})}{\varphi_K(\mathfrak{P}^{\max\{1-i,0\}})}\\
&=-\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})})}{\varphi_K(\mathfrak{P})}+(1-\mathrm{N}(\mathfrak{P}))\sum_{i=1}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})\\
&=-\mathrm{N}(\mathfrak{P})^{v_P(\mathfrak{R}_{\mathfrak{N},\psi})}.
\end{aligned}$$

If $v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})>1$, then

$$\begin{aligned}
&Z(\mathfrak{P})\\
&=(1-\mathrm{N}(\mathfrak{P}))\sum_{i=v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-1}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})}\mu_K(\mathfrak{P}^{\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-i,0\}})\frac{\varphi_K(\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})-i})}{\varphi_K(\mathfrak{P}^{\max\{v_\mathfrak{P}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})-i,0\}})}
\end{aligned}$$

$$= (1-\mathrm{N}(\mathfrak{P}))\{-\frac{\varphi_K(\mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N},\psi})+1-v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})})}{\varphi_K(\mathfrak{P})} + \sum_{i=v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N},\psi})}^{v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N},\psi})} \varphi_K(\mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N},\psi})-i})\} = 0.$$

Thus

$$\Lambda_k(\mathfrak{N},\psi)Y = \begin{cases} \mathrm{N}(\mathfrak{R}_{\mathfrak{N},\psi}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k} & (v_{\mathfrak{P}}(\mu) = v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N},\psi}) - 1 \text{ for } \mathfrak{P}|\mathfrak{R}_{\mathfrak{N},\psi}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Then

$$\begin{aligned} &\Lambda_k(\mathfrak{N},\psi)X_\mu \\ &= \tfrac{\varphi_K(\mathfrak{N}\mathfrak{R}_{\mathfrak{N},\psi}^{-1})}{\varphi_K(\mathfrak{f}_\psi)} \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\widetilde{\psi}(\mu\mathfrak{N}^{-1}\mathfrak{R}_{\mathfrak{N},\psi}\mathfrak{A}\mathfrak{f}_\psi)\tau_K(\overline{\widetilde{\psi}}) \\ &\quad\times \begin{cases} \mathrm{N}(\mathfrak{R}_{\mathfrak{N},\psi}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k} & (v_{\mathfrak{P}}(\mu) = v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N},\psi}) - 1 \text{ for } \mathfrak{P}|\mathfrak{R}_{\mathfrak{N},\psi}) \\ 0 & (\text{otherwise}) \end{cases} \\ &= \mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k+1}\tau_K(\overline{\widetilde{\psi}})\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\psi(\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}). \end{aligned}$$

□

3. Constant terms of Hilbert Eisenstein series

Let A be as in (1). If $f(\mathfrak{z})$ is a Hilbert modular form of weight k for some congruence subgroup, then the value of $f(\mathfrak{z})$ at a cusp α/γ is given by $\lim_{\mathfrak{z}\to\sqrt{-1}\infty}\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}f(A\mathfrak{z})$. For fixed α,γ, we can take β,δ so that $(\beta,\mathfrak{N}\mathfrak{N}') = (\delta,\mathfrak{N}\mathfrak{N}') = \mathcal{O}_K$. Then the transformation formula of $\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}E_{k,\mathfrak{A}}(A\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')$ is readily obtained and it shows that the constant term of the Fourier expiation of $E_{k,\mathfrak{A}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')$ at α/γ is

$$\mathrm{N}(\mathfrak{A})^k \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'),\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}) \sideset{}{'}\sum_{\substack{\mu\equiv\beta\gamma_0+\delta(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

where there is the additional term

$$(-\pi\sqrt{-1})^g D_K^{1/2}\mathrm{N}(\mathfrak{A})\mathrm{N}(\mathfrak{N}'\mathfrak{A})^{-1} \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \sideset{}{'}\sum_{\substack{\mu\equiv\alpha\gamma_0+\gamma(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

when $k=1$. Since $\widetilde{\lambda}^{\psi'}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}(\mathfrak{z})$ is a linear combination of $E_{k,\mathfrak{A}}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}')$'s by (3), we obtain the following;

Lemma 2. *Let A be as in* (1). *The constant term of* $\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}\widetilde{\lambda}^{\psi'}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}(A\mathfrak{z})$ *is equal to $C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ with*

$$\begin{aligned} &C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi') \\ &:= \Big(\tfrac{(k-1)!}{(-2\sqrt{-1}\pi)^k}\Big)^g D_K^{-1/2}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1} \sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}} \\ &\quad\times\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^k \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \end{aligned}$$

$$\Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'),\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\sum_{\substack{\mu\equiv\beta\gamma_0+\delta(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}{}' \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

where when $k=1$, there is the additional term $C^1_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ with

$$\begin{aligned}&C^1_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')\\ &:=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{N}(\mathfrak{N}')^{-1}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sum_{\substack{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}}\\ &\quad\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}\\ &\quad\times\sum_{\substack{\mu\equiv\alpha\gamma_0+\gamma(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}{}'\mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}.\end{aligned}$$

For $\gamma\in\mathcal{O}_K$, we put

$$\mathfrak{M}'_\gamma:=\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}(\gamma,\mathfrak{N}')^{-1}.$$

By the assumption (2), $\mathfrak{M}'_\gamma$ is coprime to $\mathfrak{N}$ if it is integral. The purpose of this section is to prove the following;

Theorem 1. *Put $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}):=C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$ for a divisor $\mathfrak{M}$ of $\mathfrak{R}_{\mathfrak{N},\psi}$. Let $\Lambda_k(\mathfrak{N},\psi)$ be as in (9). If there is no divisor $\mathfrak{M}$ of $\mathfrak{R}_{\mathfrak{N},\psi}$ with $(\gamma,\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$, then $\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}=0$. Suppose otherwise. Then*

$$\begin{aligned}&\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}\\ &=\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\overline{\widetilde{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\widetilde{\psi}((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\\ &\quad\times\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{M}^{-1}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1}\\ &\quad\times\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{M})^{-1}\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau_K(\overline{\widetilde{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\\ &\quad\times L_K(1-k,(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}})\prod_{\mathfrak{P}|\mathfrak{N}',\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\frac{\widetilde{\overline{\psi}\psi'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})^k})\end{aligned}\tag{11}$$

where $\mathfrak{M}$ denotes the largest ideal satisfying $(\gamma,\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma\mathfrak{N}'$.

Several preparations are necessary to give the proof.

Step 1. *Unless $(\gamma,\mathfrak{N}\mathfrak{N}')=\mathfrak{N}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$ for an integral $\mathfrak{M}'_\gamma$, then $C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ vanishes. Suppose the equality. Then $C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ equals*

$$\begin{aligned}&2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\\ &\times\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}((\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma,\mathcal{O}_K))\sum_{\mu'':\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma,\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\\ &\overline{\psi}(\mu''\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(-\gamma\mu''\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\end{aligned}$$

$$\times \sum_{\mu:(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'_\gamma,\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}' \mathbf{e}(\mathrm{tr}(\mu''\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Proof. Since $(\alpha,\gamma)=(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\mathfrak{N}')=(\delta_0\mathfrak{N},\mathfrak{N})=\mathcal{O}_K$ in the equation of $C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ in Lemma 2, it is possible that $\Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'),\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\neq 0$ only when $(\gamma,\mathfrak{N}\mathfrak{N}')=\mathfrak{N}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ for $\mathfrak{M}'_\gamma$ integral. This shows the first assertion of the Step 1. In particular if $C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')\neq 0$, then $\gamma\in\mathfrak{N}$ and $(\alpha,\mathfrak{N})=\mathcal{O}_K$. When $(\gamma,\mathfrak{N}\mathfrak{N}')=\mathfrak{N}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$, $C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ is equal to

$$2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\psi})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K))\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}$$

$$\sum_{\substack{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}} \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\times\sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}} \Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'),\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})$$

$$\times\sum_{\mu:(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}' \mathbf{e}(\mathrm{tr}((\beta\gamma_0+\delta(\delta_0+\delta'))\mu))$$

$$\times\,\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Since the map of $\begin{pmatrix}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}\\ \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}\end{pmatrix}$ to itself given by multiplication by the matrix $\begin{pmatrix}\alpha&\gamma\\ \beta&\delta\end{pmatrix}$ is bijective since $\alpha\gamma_0\equiv-\gamma(\delta_0+\delta') \pmod{\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}$ if and only if $\gamma_0\equiv-\gamma(\beta\gamma_0+\delta(\delta_0+\delta')) \pmod{\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}$, where β,δ are the algebraic integers given at the beginning of this section. We have

$$\begin{aligned}&C_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')\\ &=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\psi})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K))\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\end{aligned}$$

$$\sum_{\substack{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}} \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^{k-1}\sum_{\substack{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}\\ \gamma_0\equiv 0 \pmod{\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}}}$$

$$\mathrm{sgn}(\mathrm{N}(-\gamma(\delta_0+\delta')))^{e_{\psi'}}\psi'(-\gamma(\delta_0+\delta')\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)$$

$$\times\sum_{\mu:(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}' \mathbf{e}(\mathrm{tr}((\delta_0+\delta')\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0},$$

here replacing $\delta_0+\delta'$ by totally positive μ'' which is congruent to $\delta_0+\delta'$ modulo $(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}$,

$$=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\psi})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\times \mathrm{N}((\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)) \sum_{\mu'':\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$\overline{\psi}(\mu''\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(-\gamma\mu''\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K) \sum'_{\mu:(\mathfrak{N}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}$$

$$\times \mathbf{e}(\mathrm{tr}(\mu''\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

□

Step 2. *Unless* $(\gamma,\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$ *for* $\mathfrak{M}_\gamma'$ *integral, then* $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$ *vanishes. Suppose the equality. Then it equals*

$$2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\overline{\psi}(\mathfrak{M}_\gamma')$$

$$\times\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}_\gamma')\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K))$$

$$\times\sum'_{\mu:(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\ \sum_{\mu'':\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$(\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}\psi')(\mu''\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}_\gamma'^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\mu''\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Proof. Substituting $\mathfrak{N}\mathfrak{M}^{-1}$ for $\mathfrak{N}$ in the equation in Step 1, we have

$$\begin{aligned}&C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')\\
&=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\\
&\quad\times\mathrm{N}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K))\sum_{\mu'':\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\\
&\quad\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}(\mu''\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(-\gamma\mu''\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\\
&\quad\times\sum'_{\mu:(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\mathbf{e}(\mathrm{tr}(\mu''\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.\end{aligned}$$

It is readily checked that this is equal to the one given in Step 2. □

Let $\mathfrak{M}$ be the largest ideal dividing $\mathfrak{R}_{\mathfrak{N},\psi}$ with $(\gamma,\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\times\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$. If $\mathfrak{M}'$ is an integral ideal dividing $\mathfrak{R}_{\mathfrak{N},\psi}$ with $\mathfrak{M}\nmid\mathfrak{M}'$, then $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}'^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}'^{-1}},\psi')=0$ by Step 2. Then

$$\begin{aligned}&\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}\\
&=\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\overline{\widetilde{\psi}}(\mathfrak{M})\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\\
&\quad\times\sum_{\mathfrak{M}'|\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi},(\mathfrak{M}',\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}|\mathfrak{M}'\\ \mathfrak{P}\nmid\mathfrak{M}}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M}')\\
&\quad\times C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}},\psi').\end{aligned}$$

Noticing that $(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)=(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)$ for $\mathfrak{M}'$ with $(\mathfrak{M}',\mathfrak{N}')=\mathcal{O}_K$ and that $\mathrm{N}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1},\mathcal{O}_K)=\mathrm{N}(\mathfrak{M}_\gamma')^{-1}$

$$\times \mathrm{N}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}, \mathcal{O}_K),$$

$$\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}$$

$$= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})$$

$$\times \mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_{\overline{\psi}}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})^{-k}\overline{\widetilde{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\Big(\prod_{\mathfrak{P}\mid\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)$$

$$\times \psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}, \mathcal{O}_K))$$

$$\times \sideset{}{'}\sum_{\mu:(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma,\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}\times D(\mu)$$

with

$$D(\mu) = \mathrm{N}(\mathfrak{M}'_\gamma)^{-1}\sum_{\mathfrak{M}'\mid\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi},(\mathfrak{M}',\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}\mid\mathfrak{M}'\\ \mathfrak{P}\nmid\mathfrak{M}}}(1-\mathrm{N}(\mathfrak{P}))\Big)(\overline{\widetilde{\psi}}\psi')(\mathfrak{M}')$$

$$\times \sum_{\mu'':\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{M}'\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma,\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$(\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}}\psi')(\mu''\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\mathfrak{M}'^{-1}_\gamma\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\mu''\mu)).$$

Step 3. *Let* $\mu \in (\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)\mathfrak{A}^{-1}$. *Then* $D(\mu)$ *is equal to*

$$\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M}'_\gamma)^{-1}\sum_{\mathfrak{R}\in\mathcal{R}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}^{-1}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}),\overline{\psi}\psi')}\mu_K(\mathfrak{R})$$

$$\times \frac{\varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}^{-1}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1})$$

$$\times \mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'})(\widetilde{\overline{\psi}\psi'})(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1}\mathfrak{R})$$

$$\times (\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1}\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1}\mathfrak{R}\mathfrak{M}'_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})$$

where

$$\mathfrak{R}_{\mathfrak{N},\psi,\mathfrak{N}'} := \prod_{\mathfrak{P}\mid\mathfrak{N},\mathfrak{P}\nmid\mathfrak{f}_\psi\mathfrak{N}'}\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{R}_{\mathfrak{N},\psi})},\ \widetilde{\mathfrak{R}}_{\mathfrak{N},\psi,\mathfrak{N}'} := \prod_{\mathfrak{P}\mid\mathfrak{N},\mathfrak{P}\nmid\mathfrak{f}_\psi\mathfrak{N}'}\mathfrak{P}.$$

Proof. We have

$$D(\mu)$$

$$= \mathrm{N}(\mathfrak{M}'_\gamma)^{-1}\sum_{\mathfrak{M}'\mid\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi},(\mathfrak{M}',\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}\mid\mathfrak{M}'\\ \mathfrak{P}\nmid\mathfrak{M}}}(1-\mathrm{N}(\mathfrak{P}))\Big)(\overline{\widetilde{\psi}}\psi')(\mathfrak{M}')$$

$$\times \sum_{\mu'':\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{M}'\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma,\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$(\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}}\psi')(\mu''\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\mathfrak{M}'^{-1}_\gamma\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{e}(\mathrm{tr}(\mu''\mu))$$

$$= \mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})(\widetilde{\psi\overline{\psi}'})(\mu\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{M}'_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\sum_{\mathfrak{M}'\mid\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi},(\mathfrak{M}',\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}\mid\mathfrak{M}'\\ \mathfrak{P}\nmid\mathfrak{M}}}(1-\mathrm{N}(\mathfrak{P}))\Big)$$

$$\times \mathrm{N}(\mathfrak{M}'_\gamma)^{-1} \sum_{\mathfrak{R}\in\mathcal{R}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}),\overline{\psi}\psi')} \mu_K(\mathfrak{R})\frac{\varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}$$
$$\times\begin{cases}1 & (\mu\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}\mathfrak{M}'\mathfrak{M}'_\gamma\mathfrak{A},\mathcal{O}_K)=\mathfrak{f}^{-1}_{\overline{\psi}\psi'}\mathfrak{R}^{-1})\\ 0 & (\text{otherwise})\end{cases}$$

by using Lemma 1 (i). Then

$$\begin{aligned}&D(\mu)\\ =&\operatorname{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})(\psi\overline{\psi}')(\mu\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}\mathfrak{M}'_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M}'_\gamma)^{-1}\times Z\\ &\times\sum_{\mathfrak{R}\in\mathcal{R}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}),\overline{\psi}\psi')}\mu_K(\mathfrak{R})\frac{\varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}\\ &\times\begin{cases}1 & (\mu\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\mathfrak{M}'_\gamma\mathfrak{A},\mathcal{O}_K)=\mathfrak{f}^{-1}_{\overline{\psi}\psi'}\mathfrak{R}^{-1})\\ 0 & (\text{otherwise})\end{cases}\end{aligned}$$

where Z is written as the product $Z=\prod_{\mathfrak{P}|\mathfrak{N}\mathfrak{M}^{-1},\mathfrak{P}\nmid\mathfrak{f}_\psi\mathfrak{N}'}Z(\mathfrak{P})$ with

$$Z(\mathfrak{P})=\begin{cases}\displaystyle\sum_{i=v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})-1}^{v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})}(1-\mathrm{N}(\mathfrak{P}))^{\min\{1,i\}}\\ \qquad\qquad\times\mu_K(\{\mathfrak{R}\}_{\mathfrak{P}})\frac{\varphi_K(\{\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{P}^{-i}\}_{\mathfrak{P}})}{\varphi_K(\{\mathfrak{R}\}_{\mathfrak{P}})} & (\mathfrak{P}\nmid\mathfrak{M}),\\ \displaystyle\sum_{i=v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})-1}^{v_{\mathfrak{P}}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})}\mu_K(\{\mathfrak{R}\}_{\mathfrak{P}})\frac{\varphi_K(\{\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{P}^{-i}\}_{\mathfrak{P}})}{\varphi_K(\{\mathfrak{R}\}_{\mathfrak{P}})} & (\mathfrak{P}|\mathfrak{M})\end{cases}$$

$\mathfrak{P}^i$ being associated with $\{\mathfrak{M}'\}_{\mathfrak{P}}$. A simple calculation leads to the following;

(i) The case that $\mathfrak{P}\nmid\mathfrak{M}$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})>1$: $Z(\mathfrak{P})=0$.
(ii) The case that $\mathfrak{P}\nmid\mathfrak{M}$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})=1$: $Z(\mathfrak{P})=-\mathrm{N}(\mathfrak{P})^{v_{\mathfrak{P}}(\mathfrak{N}\mathfrak{M}^{-1})}$.
(iii) The case that $\mathfrak{P}\nmid\mathfrak{M}$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})\le 0$: $Z(\mathfrak{P})=0$.
(iv) The case that $\mathfrak{P}|\mathfrak{M}$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})>1$: $Z(\mathfrak{P})=0$.
(v) The case that $\mathfrak{P}|\mathfrak{M}$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})=1$: $Z(\mathfrak{P})=0$.
(vi) The case that $\mathfrak{P}|\mathfrak{M}$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi})\le 0$: $Z(\mathfrak{P})=\mathrm{N}(\mathfrak{P})^{v_{\mathfrak{P}}(\mathfrak{N}\mathfrak{M}^{-1})}$.

From this our assertion follows. □

Proof of Theorem 1. By Step 2 and Step 3, $\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}$ is equal to

$$\begin{aligned}&2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\operatorname{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\operatorname{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\tau_K(\widetilde{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\widetilde{\overline{\psi}}(\mathfrak{M}\mathfrak{N}\mathfrak{M}'_\gamma)\\ &\times\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{M}'_\gamma)^{-1}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{N}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})^{-1}\\ &\times\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k}\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\mathfrak{f}^{-1}_{\psi'}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K))\\ &\times\sideset{}{'}\sum_{\mu:\mathfrak{M}'^{-1}_\gamma(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{M}'^{-1}\mathfrak{f}^{-1}_{\psi'}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}\end{aligned}$$

$$\times \sum_{\mathfrak{R}\in\mathcal{R}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}),\overline{\psi}\psi')} \mu_K(\mathfrak{R})\frac{\varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}$$

$$\times \mu_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1})\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'})$$

$$\times (\widetilde{\overline{\psi}\psi'})(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1}\mathfrak{R})$$

$$\times (\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1}\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'},\mathfrak{M})^{-1}\mathfrak{M}'_\gamma\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})$$

$$=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\tau_K(\overline{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})$$

$$\times \mu_K((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\overline{\overline{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{M}'_\gamma)^{-1}$$

$$\times \mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k}\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'})\mathrm{N}(\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K))(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P})))$$

$$\times \varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})$$

$$\times \frac{(\widetilde{\overline{\psi}\psi'})(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sum'_{\mu:\mathfrak{M}'^{-1}_\gamma(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}$$

$$(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{M}'_\gamma\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\mathrm{N}(\mu\mathfrak{A})^{k-1-s}|_{s=0}.$$

Here we note that there holds $\mathfrak{N}\mathfrak{M}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')\mathfrak{M}'^{-1}_\gamma\mathfrak{R}^{-1}\mathfrak{f}^{-1}_{\overline{\psi}\psi'} = \mathfrak{M}'^{-1}_\gamma\times(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}\mathfrak{f}^{-1}_{\overline{\psi}\psi'},\mathfrak{N}\mathfrak{N}'\mathfrak{R}^{-1}\mathfrak{f}^{-1}_{\overline{\psi}\psi'})\subset\mathfrak{M}'^{-1}_\gamma(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}^{-1}_{\psi'}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K)$ for $\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\overline{\psi}\psi')$ and that $(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{M}'^{-1}_\gamma\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})$ is 0 if $\mu\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{M}'_\gamma\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'}$ is not integral. Then $\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}$ is equal to

$$2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\tau_K(\overline{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mu_K((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))$$

$$\times \widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\overline{\overline{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k}\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'})$$

$$\times \mathrm{N}(\mathfrak{M}'_\gamma)^{-1}\mathrm{N}(\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K))(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))))$$

$$\times \varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})$$

$$\times \frac{(\widetilde{\overline{\psi}\psi'})(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sum'_{\mu:\mathfrak{N}\mathfrak{M}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}\mathfrak{M}'^{-1}_\gamma(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')\mathfrak{R}^{-1}\mathfrak{f}^{-1}_{\overline{\psi}\psi'}\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}$$

$$(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R}}(\mu\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{M}'_\gamma\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\mathrm{N}(\mu\mathfrak{A})^{k-1-s}|_{s=0}$$

$$=\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\tau_K(\overline{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mu_K((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\widetilde{\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})$$

$$\times \overline{\overline{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k}\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'})$$

$$\times \mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K))(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P})))$$

$$\times\,\varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})$$
$$\times\,\frac{\widetilde{(\overline{\psi}\psi')}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'},\mathfrak{R})}\mathrm{N}(\mathfrak{N}\mathfrak{M}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')\mathfrak{R}^{-1}\mathfrak{f}^{-1}_{\overline{\psi}\psi'})^{k-1}$$
$$\times\,L_K(1-k,(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{R}})$$
$$=\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\tau_K(\widetilde{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mu_K((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\widetilde{\overline{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)$$
$$\times\,\widetilde{\psi}((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)$$
$$\times\,\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{M}^{-1}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}^{-1}_{\overline{\psi}\psi'})^{k-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-1}\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'})$$
$$\times\,\mathrm{N}(\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{f}^{-1}_{\psi'}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'},\mathcal{O}_K))\Big(\prod_{\mathfrak{P}|\mathfrak{N}',\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}\mathrm{N}(\mathfrak{P})(\mathrm{N}(\mathfrak{P})-1)^{-1}\Big)$$
$$\times\,\frac{\varphi_K((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}))}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'})}\Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)$$
$$\times\,L_K(1-k,(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}})\prod_{\mathfrak{P}|\mathfrak{N}',\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}\Big(1-\frac{\widetilde{\overline{\psi}\psi'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})^k}\Big),$$

which is equal to the left hand side of (11). □

4. The case of weight 1

We compute the additional term which appears when $k=1$. If $\mathfrak{B}$ is an ideal and if $\mathcal{E}$ is a some group in $\mathcal{E}_{\mathcal{O}_K}$ of finite index, then there holds the functional equation

$$\sum_{\substack{\mu\equiv\mu_0(\mathfrak{B}),\mu\neq0\\ \mu/\mathcal{E}}}\mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}$$
$$=(-\sqrt{-1}\pi^{-1})^gD_K^{-1/2}\mathrm{N}(\mathfrak{B})^{-1}\sum_{\mu:\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E},\mu\neq0}\mathbf{e}(\mathrm{tr}(\mu_0\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}$$

by Hecke [2]. By applying this to the equation in Lemma 2 (ii), $C^1_{\alpha/\gamma}(\mathfrak{N},k,\psi_{\mathfrak{N}},\psi')$ is shown to to be equal to

$$2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{N}(\mathfrak{N}')^{-1}(-\sqrt{-1}\pi^{-1})^gD_K^{1/2}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{N}'\mathfrak{A})^{-1}$$
$$\times\sum_{\gamma_0:\mathfrak{N}'\mathfrak{f}^{-1}_{\psi'}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ0}\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}\ \sideset{}{'}\sum_{\mu:(\mathfrak{N}'\mathfrak{A})^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}$$
$$\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ0}\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\gamma\delta_0\mu))\mathbf{e}(\mathrm{tr}((\alpha\gamma_0+\gamma\delta')\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

From this we obtain, by substituting $\mathfrak{N}\mathfrak{M}^{-1}$ for $\mathfrak{N}$,

$$C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$$
$$=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{N}(\mathfrak{N}')^{-1}(-\sqrt{-1}\pi^{-1})^gD_K^{1/2}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{N}'\mathfrak{A})^{-1}$$

$$\times \sum_{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K) \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}$$
$$\sum_{\mu:(\mathfrak{N}'\mathfrak{A})^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}' \sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\gamma\delta_0\mu))$$
$$\times\, \mathbf{e}(\mathrm{tr}((\alpha\gamma_0+\gamma\delta')\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

We put

$$\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi',\mathfrak{N}} := \prod_{\mathfrak{P}|\mathfrak{R}_{\mathfrak{N}',\psi'},\mathfrak{P}\nmid\mathfrak{N}} \mathfrak{P}.$$

The purpose of this section is to prove the following;

Theorem 2. *Let $C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1})$ denote $C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$. Put $\mathfrak{M}_\gamma := \gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}'^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}$, $\mathfrak{L}_\gamma := (\gamma\mathfrak{N}'^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\cap\mathcal{O}_K, \widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\mathfrak{N})^{-1})$. Then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is possibly not zero only when there is the divisor $\mathfrak{R}$ of $\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}$ such that the numerator of $\mathfrak{M}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$. Let $\widetilde{\mathfrak{R}}_\gamma$ be the divisor of $(\mathfrak{N},\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$ satisfying $v_{\mathfrak{P}}(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})=0$ for any prime divisor $\mathfrak{P}$ of $(\mathfrak{N},\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$. Then*

$$\begin{aligned}
&\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}\\
&=\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{N}(\mathfrak{f}_{\psi'})\mathrm{N}(\mathfrak{f}_{\psi\overline{\psi}'})^{-1}\tau_K(\widetilde{\overline{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})\mu_K(\widetilde{\mathfrak{R}}_\gamma)\\
&\quad\times\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}_\gamma^{-1})\mathrm{N}(\mathfrak{N}'^{-1}(\gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}_\gamma)\\
&\quad\times\psi(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi'}_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},\mathcal{O}_K)^{-1})L_K(0,(\overline{\psi}\psi')_{\mathfrak{L}_\gamma})\\
&\quad\times\prod_{\mathfrak{P}|\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{P}\nmid\mathfrak{f}_{\psi'}}\Big(1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})}\Big)\prod_{\mathfrak{P}|\mathfrak{f}_\psi,\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}\Big(1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})}\Big). \qquad (12)
\end{aligned}$$

Proof. By (10), we have

$$\begin{aligned}
&\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}\\
&=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}')^{-1}(-\sqrt{-1}\pi^{-1})^gD_K^{1/2}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{N}'\mathfrak{A})^{-1}\\
&\quad\times\sum_{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}\\
&\quad\times\sum_{\mu:(\mathfrak{N}'\mathfrak{A})^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}'\mathrm{sgn}(\mathrm{N}(\gamma\mu))^{e_\psi}\psi(\gamma\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathbf{e}(\mathrm{tr}((\alpha\gamma_0+\gamma\delta')\mu))\\
&\qquad\qquad\times\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}\\
&=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}(-\sqrt{-1}\pi^{-1})^gD_K^{1/2}\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{N}'\mathfrak{A})^{-1}\sum_{\mu:(\mathfrak{N}'\mathfrak{A})^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}'\\
&\quad\psi(\gamma\mu\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{A})\sum_{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)
\end{aligned}$$

$$\times\, \mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$
$$= 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}(-\sqrt{-1}\pi^{-1})^g D_K^{1/2}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\tau_K(\widetilde{\psi}')$$
$$\times\, \varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})\mathrm{N}(\mathfrak{N}')^{-1}\sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}}\frac{\mu_K(\mathfrak{R})\widetilde{\psi}'(\mathfrak{R})}{\varphi_K(\mathfrak{R})}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sideset{}{'}\sum_{\mu:(\mathfrak{N}'\mathfrak{A})^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}$$
$$\psi(\gamma\mu\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{A})\overline{\psi'}_{\mathfrak{R}}(\mu\mathfrak{N}'\mathfrak{A}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{R})\mathrm{N}(\mu\mathfrak{A})^{-1}|\mathrm{N}(\mu\mathfrak{A})|^{-s}|_{s=0},$$

where the last equality follows from Lemma 1 (ii). The element $\mu\in(\mathfrak{N}'\mathfrak{A})^{-1}$ which contributes the summation, is in $\gamma^{-1}\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1}\mathfrak{A}^{-1}\cap(\mathfrak{N}'\mathfrak{A})^{-1}$. Then

$$\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$$
$$= 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}(-\sqrt{-1}\pi^{-1})^g D_K^{1/2}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\tau_K(\widetilde{\psi}')\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$$
$$\times\, \mathrm{N}(\mathfrak{N}')^{-1}\sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}}\frac{\mu_K(\mathfrak{R})\widetilde{\psi}'(\mathfrak{R})}{\varphi_K(\mathfrak{R})}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sideset{}{'}\sum_{\mu:(\gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{R})^{-1}\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}$$
$$\psi(\gamma\mu\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{A})\overline{\psi'}_{\mathfrak{R}}(\mu\mathfrak{N}'\mathfrak{A}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{R})\mathrm{N}(\mu\mathfrak{A})^{-1}|\mathrm{N}(\mu\mathfrak{A})|^{-s}|_{s=0}.$$

For a fixed $\mathfrak{R}$, the corresponding term vanishes unless the numerator of $\mathfrak{M}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$, namely unless

$$(\mathfrak{M}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K,\mathfrak{N})=((\mathfrak{M}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1},\mathfrak{f}_{\psi'}\mathfrak{R})=\mathcal{O}_K. \qquad (13)$$

Hence the value of $\Lambda(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ at α/γ is possibly not zero only when there are $\mathfrak{R}$ with $\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}$ satisfying (13). For such $\mathfrak{R}$, $(\mathfrak{R},\mathfrak{N})$ is uniquely determined. Then $\widetilde{\mathfrak{R}}_\gamma=(\mathfrak{R},\mathfrak{N})$, and $\mathfrak{R}$ satisfying (13) are written as products of $\widetilde{\mathfrak{R}}_\gamma$ and divisors of $\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi',\mathfrak{N}}$. Then

$$\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$$
$$= 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}(-\sqrt{-1}\pi^{-1})^g D_K^{1/2}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\tau_K(\widetilde{\psi}')$$
$$\times\, \varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})\mathrm{N}(\mathfrak{N}')^{-1}\sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}}\tfrac{\mu_K(\mathfrak{R})\widetilde{\psi}'(\mathfrak{R})}{\varphi_K(\mathfrak{R})}\mathrm{N}((\gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{R}))$$
$$\times\, \psi(\mathfrak{M}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K)\overline{\psi'}_{\mathfrak{R}}((\mathfrak{M}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})$$
$$\times\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\sideset{}{'}\sum_{\mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\psi(\mu\mathfrak{A})\overline{\psi'}_{\mathfrak{R}}(\mu\mathfrak{A})\mathrm{N}(\mu\mathfrak{A})^{-1}|\mathrm{N}(\mu\mathfrak{A})|^{-s}|_{s=0}$$
$$=(-\sqrt{-1}\pi^{-1})^g D_K^{1/2}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\tau_K(\widetilde{\psi}')\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})\mathrm{N}(\mathfrak{N}')^{-1}$$
$$\times\sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}}\tfrac{\mu_K(\mathfrak{R})\widetilde{\psi}'(\mathfrak{R})}{\varphi_K(\mathfrak{R})}\mathrm{N}((\gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{R}))\psi(\mathfrak{M}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K)$$
$$\times\,\overline{\psi'}_{\mathfrak{R}}((\mathfrak{M}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})L_K(1,\psi\overline{\psi'}_{\mathfrak{R}})$$
$$=(-1)^g\mathrm{N}(\mathfrak{f}_{\psi\overline{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\tau_K(\widetilde{\psi}')\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$$
$$\times\,\mathrm{N}(\mathfrak{N}')^{-1}\tfrac{\mu_K(\widetilde{\mathfrak{R}}_\gamma)\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)}{\varphi_K(\widetilde{\mathfrak{R}}_\gamma)}L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{P}\nmid\mathfrak{f}_{\psi'}}(1-\tfrac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})})\prod_{\mathfrak{P}|\mathfrak{f}_\psi,\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}(1-\tfrac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})})$$

$$\times \sum_{\mathfrak{R}|\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi',\mathfrak{N}}} \frac{\mu_K(\mathfrak{R})\widetilde{\psi'}(\mathfrak{R})}{\varphi_K(\mathfrak{R})} \mathrm{N}((\gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}, \mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}, \widetilde{\mathfrak{R}}_\gamma\mathfrak{R}))\psi(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{R}^{-1}\cap\mathcal{O}_K)$$

$$\times \overline{\psi'}_{\widetilde{\mathfrak{R}}_\gamma\mathfrak{R}}((\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})\prod_{\mathfrak{P}|\mathfrak{R}}(1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})})$$

$$=(-1)^g\mathrm{N}(\mathfrak{f}_{\psi\overline{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\tau_K(\widetilde{\psi}')\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$$

$$\times \mathrm{N}(\mathfrak{N}')^{-1}\frac{\mu_K(\widetilde{\mathfrak{R}}_\gamma)\widetilde{\psi'}(\widetilde{\mathfrak{R}}_\gamma)}{\varphi_K(\widetilde{\mathfrak{R}}_\gamma)}L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{P}\nmid\mathfrak{f}_{\psi'}}(1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})})$$

$$\times \prod_{\mathfrak{P}|\mathfrak{f}_\psi,\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}(1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})})\mathrm{N}((\gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1},\widetilde{\mathfrak{R}}_\gamma))\psi(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K)$$

$$\times \overline{\psi'}_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},\mathcal{O}_K)^{-1})\frac{\mathrm{N}(\mathfrak{L}_\gamma)}{\varphi_K(\mathfrak{L}_\gamma)}\prod_{\mathfrak{P}|\mathfrak{L}_\gamma}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})),$$

which is equal to the right hand side of (12). □

5. Main Theorem and its Application

Main Theorem. *Let $\mathfrak{N},\mathfrak{N}'$ be integral ideals of K. Let $\psi\in C_{\mathfrak{N}}^*,\psi'\in C_{\mathfrak{N}'}^*$ be even or odd characters, which are not necessarily primitive. Let $\mathfrak{f}_\psi,\mathfrak{f}_{\psi'}$ be the conductors of ψ,ψ' respectively. Let $\mathfrak{R}_{\mathfrak{N},\psi}$ (resp. $\mathfrak{R}_{\mathfrak{N}',\psi'}$) be the smallest ideal dividing $\mathfrak{N}$ (resp. $\mathfrak{N}'$) so that $(\mathfrak{R}_{\mathfrak{N},\psi},\mathfrak{f}_\psi)=\mathcal{O}_K$ (resp. $(\mathfrak{R}_{\mathfrak{N}',\psi'},\mathfrak{f}_{\psi'})=\mathcal{O}_K$) and let $\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}$ (resp. $\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}$) be its radical. We assume that $(\mathfrak{N},\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\psi',\mathfrak{N}'}^{-1})=\mathcal{O}_K$. Let $\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z})$ be as in (4) where k is the natural number with the same parity as $\psi\psi'$. Then $\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z})$ is a Hilbert modular form for $\Gamma_0(\mathfrak{N}\mathfrak{N}')_K$ of weight k with character $\psi\psi'$, which has the Fourier expansion*

$$\begin{cases}\overline{\psi}'(\mathfrak{N}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}^{-1})\psi'(\mathfrak{d}_K)L_K(1-k,\psi\overline{\psi}') & (k>1 \text{ or } \mathfrak{N}\subsetneq\mathcal{O}_K,\ \text{and } \mathfrak{N}'=\mathcal{O}_K)\\ \overline{\psi}(\mathfrak{N}'\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1})\psi(\mathfrak{d}_K)L_K(0,\overline{\psi}\psi') & (k=1,\mathfrak{N}=\mathcal{O}_K,\mathfrak{N}'\subsetneq\mathcal{O}_K)\\ \psi'(\mathfrak{d}_K)L_K(0,\psi\overline{\psi}')+\psi(\mathfrak{d}_K)L_K(0,\overline{\psi}\psi') & (k=1,\mathfrak{N}=\mathfrak{N}'=\mathcal{O}_K)\\ 0 & (\text{otherwise})\end{cases}$$

$$+2^g\sum_{\nu\in\mathfrak{d}_K^{-1},\succ 0}\ \sum_{\mathcal{O}_K\supset\mathfrak{A}\supset\nu\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{d}_K}\psi(\mathfrak{A})$$

$$\times\psi'(\nu\mathfrak{A}^{-1}\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})). \tag{14}$$

Let $\alpha,\gamma\in\mathcal{O}_K$ with $(\alpha,\gamma)=\mathcal{O}_K$. The value of $\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z})$ at the cusp α/γ is 0 if there are no integral ideals $\mathfrak{M},\mathfrak{M}'_\gamma$ with $\mathfrak{M}|\mathfrak{R}_{\mathfrak{N},\psi}$, $\mathfrak{M}'_\gamma|\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}$ satisfying $(\gamma,\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}^{-1}\mathfrak{M}_\gamma'^{-1}$. Suppose otherwise and let $\mathfrak{M}$ be the largest such ideal. Then the value of $\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z})$ at the cusp α/γ is given by

$$\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\overline{\psi}(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))\overline{\widetilde{\psi}}(\mathfrak{M}\mathfrak{M}'_\gamma)\widetilde{\psi}((\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}'))$$

$$\times\psi'(-\gamma\mathfrak{N}^{-1}\mathfrak{M}\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{M}^{-1}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1}$$

$$\times\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau(\overline{\widetilde{\psi}})^{-1}\tau(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M})^{-1}\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))$$

$$\times L_K(1-k,(\psi\overline{\psi}')_{\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}(\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi},\mathfrak{M}\mathfrak{N}')^{-1}}) \prod_{\mathfrak{P}|\mathfrak{N}',\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}} (1-\frac{\widetilde{\overline{\psi}\psi'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})^k}).$$

Put $\mathfrak{M}_\gamma := \gamma\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}'^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}$, $\mathfrak{L}_\gamma := (\gamma\mathfrak{N}'^{-1}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\cap O_K, \widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'},\mathfrak{N})^{-1})$. *If* $k=1$ *and if there is the divisor* $\mathfrak{R}$ *of* $\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}$ *such that the numerator of* $\mathfrak{M}_\gamma\mathfrak{R}^{-1}$ *is coprime to* $\mathfrak{N}$ *and the denominator is coprime to* $\mathfrak{f}_{\psi'}\mathfrak{R}$, *then there is the additional term. Let* $\widetilde{\mathfrak{R}}_\gamma$ *be the divisor of* $(\mathfrak{N},\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$ *satisfying* $v_{\mathfrak{P}}(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})=0$ *for any prime divisor* $\mathfrak{P}$ *of* $(\mathfrak{N},\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'})$. *Then it is*

$$\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}'(\alpha)\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mu_K(\widetilde{\mathfrak{R}}_\gamma)\psi(\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap O_K)\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{M}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},O_K)^{-1})$$
$$\times\varphi_K(\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}_\gamma^{-1})\mathrm{N}(\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'}(\mathfrak{M}_\gamma,\widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}_\gamma)\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\psi\overline{\psi}'})\tau(\widetilde{\overline{\psi}'})^{-1}\tau(\widetilde{\psi\overline{\psi}'})$$
$$\times L_K(0,(\overline{\psi}\psi')_{\mathfrak{L}_\gamma})\prod_{\mathfrak{P}|\mathfrak{N},\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}(1-\frac{\widetilde{\psi\overline{\psi}'}(\mathfrak{P})}{\mathrm{N}(\mathfrak{P})}).$$

Proof. The values at each cusps are investigated in the section 3 and the section 4. We compute the higher terms. Then

$$\widetilde{\lambda}^{\psi'}_{k,\psi_\mathfrak{N},\mathfrak{N}}(\mathfrak{z})$$
$$=C+[\mathcal{E}_{O_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau(\widetilde{\overline{\psi}})^{-1}\sum_{\nu\in\mathfrak{d}_K^{-1},\succ 0}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\sum_{\substack{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0\\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}}$$
$$\times\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)$$
$$\times\sum_{\substack{\nu/\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})),$$

where C is the constant term. Put $X(\mathfrak{M})=\widetilde{\lambda}^{\psi'}_{k,\psi_\mathfrak{M},\mathfrak{M}}(\mathfrak{z})$ for $\mathfrak{M}$ with $\mathfrak{M}\supset\mathfrak{f}_\psi$. Then

$$\Lambda_k(\mathfrak{N},\psi)X$$
$$=C'+[\mathcal{E}_{O_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k+1}\sum_{\nu\in\mathfrak{d}_K^{-1},\succ 0}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$
$$\times\sum_{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}\psi'(\gamma_0\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)$$
$$\times\sum_{\substack{\nu/\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1})\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}\psi(\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi-1}\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$
$$=C'+2^g\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k+1}\sum_{\nu\in\mathfrak{d}_K^{-1},\succ 0}\sum_{\mathfrak{A}:C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}\sum_{\gamma_0:\mathfrak{N}'\mathfrak{f}_{\psi'}^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N}',\psi'}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$
$$\psi(\mu\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})\psi'((\nu/\mu)\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi+e_{\psi'}-1}$$
$$\times\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$
$$=C'+2^g\mathrm{N}(\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi})^{-k+1}\sum_{\nu\in\mathfrak{d}_K^{-1},\succ 0}\sum_{\mathfrak{N}\mathfrak{f}_\psi^{-1}\widetilde{\mathfrak{R}}^{-1}_{\mathfrak{N},\psi}\supset\mathfrak{B}\supset\nu\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{d}_K}\psi(\mathfrak{B}\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi})$$

$$\times\, \psi'(\nu\mathfrak{B}^{-1}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{d}_K)\mathrm{N}(\mathfrak{B})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

$$= C' + 2^g \sum_{\nu\in\mathfrak{d}_K^{-1},\succ 0} \; \sum_{\mathcal{O}_K\supset\mathfrak{B}\supset\nu\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{d}_K} \psi(\mathfrak{B})$$

$$\times\, \psi'(\nu\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{f}_\psi\widetilde{\mathfrak{R}}_{\mathfrak{N},\psi}\mathfrak{N}'^{-1}\mathfrak{f}_{\psi'}\widetilde{\mathfrak{R}}_{\mathfrak{N}',\psi'}\mathfrak{d}_K)\mathrm{N}(\mathfrak{B})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})).$$

□

Let the notation be as in the theorem. Let N be the minimal natural number contained in $\mathfrak{N}\mathfrak{N}'$. Let χ_0 be the Dirichlet character modulo N obtained by restricting $\psi\psi'$ to $\mathbf{Z}$. Then $\widetilde{\Lambda}^{\psi'}_{k,\psi}((z,\cdots,z))$ $(z\in\mathfrak{H})$ is an elliptic modular form for $\Gamma_0(N)$ of weight gk with character χ_0. The theorem particularly gives the values of $\widetilde{\Lambda}^{\psi'}_{k,\psi}((z,\cdots,z))$ at all the cusps of $\Gamma_0(N)$. Suppose that there are Dirichlet characters χ,χ' so that

$$\psi = \chi\circ\mathrm{N},\ \ \psi' = \chi'\circ\mathrm{N}.$$

Put

$$\Lambda^{\chi'}_{gk,\chi}(z) := \widetilde{\Lambda}^{\psi'}_{k,\psi}((z,\cdots,z)) \qquad (z\in\mathfrak{H}).$$

Then $\Lambda^{\chi'}_{gk,\chi}$ is an elliptic modular form for $\Gamma_0(N)$ of weight gk with character $(\chi\chi')^g$. Let a be a square free integer. Let a^* denote a or $4a$ according as $a\equiv 1 \pmod 4$ or not. We denote by χ_{a^*}, the Legendre-Jacobi-Kronecker symbol. We define $\sigma^{\chi'}_{k-1,\chi}$ by $\sigma^{\chi'}_{k-1,\chi}(n) = \sum_{d|n}\chi'(n/d)\chi(d)d^{k-1}$ for $n\in\mathbf{N}$ and $\sigma^{\chi'}_{k-1,\chi}(n)=0$ for $n\notin\mathbf{N}\cup\{0\}$. If K is real quadratic and if ψ,ψ' are primitive, then the higher term of $\Lambda^{\chi'}_{2k,\chi}(z)$ is computed from (14) by the method in [6] as

$$4\sum_{n=1}^{\infty}\sum_{d|n}(\chi_K\chi\chi')(d)\sum_{m\in\mathbf{Z}}\sigma^{\chi'}_{k-1,\chi}\left(\frac{(n/d)^2D_K-m^2}{4}\right)\mathbf{e}(nz)$$

where χ_K denotes the Kronecker-Jacobi-Legendre symbol for D_K.

As an application of the main theorem, we give some formulas for special values of the Dirichlet L-function, which is similar to Example 1 in the section 6 [6]. Let a be a square-free natural number with $(a,6)=1$. Then

$$L(0,\chi_{-24a}) = -2\sum_{m\in\mathbf{Z}}\sigma^{\chi_{-3}}_{0,\chi_8}(4a-m^2), \tag{15}$$

$$L(-1,\chi_{24a}) = 2^2 3^{-2}\sum_{m\in\mathbf{Z}}\sigma^{\chi_{-3}}_{1,\chi_{-8}}(4a-m^2) + 2\cdot 3^{-2}\sum_{m\in\mathbf{Z}}\sigma^{\chi_{-3}}_{1,\chi_{-8}}(16a-m^2). \tag{16}$$

Let us put $K=\mathbf{Q}(\sqrt{a})$. Put $\psi_{-3}:=\chi_{-3}\circ\mathrm{N}$, $\psi_{\pm 8}:=\chi_{\pm 8}\circ\mathrm{N}$. Then the conductor $\mathfrak{f}_{\psi_{-3}}$ is equal to 3, and the conductor $\mathfrak{f}_{\psi_{\pm 8}}$ is equal to 8 or 4 according as $a\equiv 1$ (mod 4) or not. A simple calculation leads to $\tau_K(\psi_{-3}) = -3$ and $\tau_K(\psi_{\pm 8}) = \pm 8$ $(a\equiv 1 \pmod 4)$, $\tau_K(\psi_{\pm 8}) = \pm 4$ $(a\equiv 3 \pmod 4)$. Then $\Lambda^{\chi_{-3}}_{2,\chi_8}(z)$ is an elliptic modular form for $\Gamma_0(3\mathfrak{f}_{\psi_8})$ of weight 2. The main theorem shows that the value at the cusp $1/\mathfrak{f}_{\psi_8}$ is $-3^{-1}L_K(0,\psi_{-3}\psi_8)$ and the value at cusp $1/3$ is $\mathfrak{f}_{\psi_8}^{-1}L_K(0,\psi_{-3}\psi_8)$, and the values at cusps equivalent neither to $1/\mathfrak{f}_{\psi_8}$ nor $1/3$ under $\Gamma_0(3\mathfrak{f}_{\psi_8})$, are 0.

Let U_m $(m\in\mathbf{N})$ be the operator on the elliptic modular forms defined by $U_m(\sum_{n=0}^{\infty}c_n\mathbf{e}(nz)) = \sum_{n=0}^{\infty}c_{mn}\mathbf{e}(nz)$. Then $U_2(\Lambda^{\chi_{-3}}_{2,\chi_8})$ is a modular form for $\Gamma_0(6)$ if $a\not\equiv 1 \pmod 4$, and $U_4(\Lambda^{\chi_{-3}}_{2,\chi_8})$ is a modular form for $\Gamma_0(6)$ if $a\equiv 1 \pmod 4$ ([1]). In either case, the modular form takes the value $2^{-1}L_K(0,\psi_{-3}\psi_8)$ at the cups $1/3$, the value $-3^{-1}L_K(0,\psi_{-3}\psi_8)$ at the cusps $1/2$ and the value 0 at any cusp equivalent to neither $1/3$ nor $1/2$. The space of modular forms for $\Gamma_0(6)$ of weight 2 is spanned by elliptic Eisenstein series whose values at cusps or Fourier expansions are easily obtained. Comparing with them, we obtain $L_K(0,\psi_{-3}\psi_8) = -4\sum_{m\in\mathbf{Z}}\sigma_{0,\chi_8}^{\chi_{-3}}(4a - m^2)$. Since $L_K(0,\psi_{-3}\psi_8) = L(0,\chi_{-24})L(0,\chi_{-24a}) = 2L(0,\chi_{-24a})$, we obtain the identity (15). We note that $L(0,\chi_{-24})$ gives the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{-24a})$.

The modular form $\Lambda_{4,\chi_{-8}}^{\chi_{-3}}(z)$ for $\Gamma_0(3\mathfrak{f}_{\psi_{-8}})$ of weight 4 takes the value $-3^{-3} \times L_K(-1,\psi_{-3}\psi_{-8})$ at the cusp $1/\mathfrak{f}_{\psi_{-8}}$, and 0 at any cusp not equivalent to $1/\mathfrak{f}_{\psi_{-8}}$. Then $U_2(\Lambda_{4,\chi_{-8}}^{\chi_{-3}}(z))$ $(a \not\equiv 1 \pmod 4)$ or $U_4(\Lambda_{4,\chi_{-8}}^{\chi_{-3}}(z))$ $(a \equiv 1 \pmod 4)$ is a modular form for $\Gamma_0(6)$ which takes the value $-3^{-3}L_K(-1,\psi_{-3}\psi_{-8})$ at the cusp $1/2$, and 0 at any cusp not equivalent to $1/2$. By the similar method as above shows that $3L_K(-1,\psi_{-3}\psi_{-8}) = -2^3\sum_{m\in\mathbf{Z}}\sigma_{1,\chi_{-8}}^{\chi_{-3}}(4a-m^2) - 2^2\sum_{m\in\mathbf{Z}}\sigma_{1,\chi_{-8}}^{\chi_{-3}}(16a-m^2)$. Since $L_K(-1,\psi_{-3}\psi_{-8}) = -6L(-1,\chi_{-24a})$, we obtain the identity (16).

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