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The values of Hilbert-Eisenstein series at cusps, II

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ヒルベルト・アイゼンシュタイン級数の尖点での値, II

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Abstract. In our previous paper [2], we obtain the values of some specific Hilbert-Eisenstein series at cusps equivalent to $\sqrt{-1}\infty$. In the present paper we obtain the values at all the cusps.

1. Introduction

In our previous paper [2], we obtain the values of some specific Hilbert-Eisenstein series for $\Gamma_0(\mathfrak{N}\mathfrak{N}')$ at cusps equivalent to $\sqrt{-1}\infty$. The result is useful in the study of the Shimura lifting maps [3], or of quadratic forms [4]. In the present paper, we consider the Hilbert-Eisenstein series for $\Gamma_0(\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{N}'\mathfrak{D})$ with any fractional ideal $\mathfrak{D}$, and obtain their values at all the cusps. The argument is parallel to [2]. We note that we make use of different notations from the previous paper [2].

Let K be a totally real algebraic number field of degree g over $\mathbf{Q}$, and let $\mathcal{O}_K$ be the ring of algebraic integers. We denote by $\mathfrak{d}_K$ and D_K, the different of K and the discriminant respectively. For $\alpha \in K$, $\alpha^{(1)}, \cdots, \alpha^{(g)}$ denotes the conjugates of α in a fixed order. If $\alpha^{(i)}$ is positive for every i, then we call α *totally positive*, and denote it by $\alpha \succ 0$. We denote by N and tr, the norm map and the trace map of K over $\mathbf{Q}$ respectively, namely $\mathrm{N}(\alpha) = \prod_{i=1}^{g} \alpha^{(i)}$ and $\mathrm{tr}(\alpha) = \sum_{i=1}^{g} \alpha^{(i)}$. Let μ_K denote the Möbius function on K and let φ_K denote the Euler function on K. If $\mathfrak{P}$ is a prime ideal, then $v_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-adic valuation. If $\mathfrak{M}$ is an integral ideal, then $\{\mathfrak{M}\}_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-part of $\mathfrak{M}$, namely, $\{\mathfrak{M}\}_{\mathfrak{P}} = \mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{M})}$. Let $\mathfrak{N}$ be an integral ideal of K. Then $\mathcal{E}_{\mathfrak{N}}$ denotes the group of totally positive units congruent to 1 modulo $\mathfrak{N}$. We denote by $C_{\mathfrak{N}}$, the class group modulo $\mathfrak{N}$ in the narrow sense, and denote by $C_{\mathfrak{N}}^*$, the group of characters of $C_{\mathfrak{N}}$. A character $\psi \in C_{\mathfrak{N}}^*$ is called *even* (resp. *odd*) if it satisfies $\psi(\mu) = 1$ (resp. $\psi(\mu) = \mathrm{sgn}(\mathrm{N}(\mu))$ for $\mu \in \mathcal{O}_K, \neq 0$ congruent 1 modulo $\mathfrak{N}$. The identity element of $C_{\mathfrak{N}}^*$ is denoted by $\mathbf{1}_{\mathfrak{N}}$, for which $\mathbf{1}_{\mathfrak{N}}(\mathfrak{A})$ is 1 or 0 according as an integral ideal $\mathfrak{A}$ is coprime to $\mathfrak{N}$ or not. In the present paper we consider even or odd characters exclusively. For a character ψ, e_ψ is define to be 0 or 1 according as ψ is even or odd. The value of the characters at a fractional ideal whose denominator is coprime to $\mathfrak{N}$, is naturally defined, namely $\psi(\mathfrak{A}\mathfrak{B}^{-1}) = \psi(\mathfrak{A})\overline{\psi}(\mathfrak{B})$ for integral ideals $\mathfrak{A}, \mathfrak{B}$ with $(\mathfrak{B}, \mathfrak{N}) = \mathcal{O}_K$, $\overline{\psi}$ being the complex conjugate of ψ. Let $\mathcal{I}_K$ be the function on the set of fractional ideals defined by $\mathcal{I}_K(\mathfrak{A})$ is 1 or 0 according as $\mathfrak{A}$ is integral or not.

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We denote by $\mathfrak{f}_\psi$, the conductor of a character ψ, and denote by $\mathfrak{e}_\psi$, the ideal given by

$$\mathfrak{e}_\psi := \mathfrak{f}_\psi \prod_{\psi(\mathfrak{P})=0, \mathfrak{P}\nmid \mathfrak{f}_\psi} \mathfrak{P}. \tag{1}$$

The primitive character associated with $\psi \in C_{\mathfrak{N}}$ is denoted by $\widetilde{\psi}$. For any integral ideal $\mathfrak{M}$, we define $\psi_{\mathfrak{M}} := \widetilde{\psi}\mathbf{1}_{\mathfrak{M}}$. Then $\psi_{\mathfrak{M}} = \widetilde{\psi}$ for an integral ideal $\mathfrak{M}$ with $\mathfrak{M}|\mathfrak{f}_\psi$, and $\psi_{\mathfrak{N}} = \psi$. Let $\mathcal{R}(\mathfrak{M}, \psi)$ denote the set of all the products of primes divisors $\mathfrak{P}$ of $\mathfrak{M}$ coprime to $\mathfrak{f}_\psi$ with multiplicity at most one.

For $\psi \in C^*_{\mathfrak{N}}$, $L_K(s, \psi)$ denotes the Hecke L-function, that is,

$$L_K(s, \psi) := \sum_{\mathfrak{A}} \frac{\psi(\mathfrak{A})}{\mathrm{N}(\mathfrak{A})^s}$$

where $\mathfrak{A}$ runs over the set of all the integral ideals. Let $\mathfrak{H}^g$ denote the product of g copies of the upper half plane $\mathfrak{H} = \{z \in \mathbf{C} \mid \Im z > 0\}$, $\Im z$ being the imaginary part of z. For $\gamma, \delta \in K$ and for $\mathfrak{z} = (z_1, \cdots, z_g) \in \mathfrak{H}$, $\mathrm{N}(\gamma\mathfrak{z}+\delta)$ stands for $\prod_{i=1}^g (\gamma^{(i)} z_i + \delta^{(i)})$, and $\mathrm{tr}(\gamma\mathfrak{z})$ stands for $\sum_{i=1}^g \gamma^{(i)} z_i$. For a matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(K), \tag{2}$$

we put

$$A\mathfrak{z} = \Big(\frac{\alpha^{(1)} z_1 + \beta^{(1)}}{\gamma^{(1)} z_1 + \delta^{(1)}}, \cdots, \frac{\alpha^{(g)} z_g + \beta^{(g)}}{\gamma^{(g)} z_g + \delta^{(g)}}\Big).$$

We define

$$\Gamma_0(\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{D}) := \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(K) \mid \alpha, \delta \in \mathcal{O}_K, \beta \in \mathfrak{D}^{-1}, \gamma \in \mathfrak{N}\mathfrak{D}\}$$

for a fractional ideal $\mathfrak{D}$ and for an integral ideal $\mathfrak{N}$.

2. Gauss sums

Let ψ be a primitive character of an ideal class group of K. The Gauss sum of ψ is defined by

$$\tau_K(\psi) := \psi(\rho \mathfrak{f}_\psi \mathfrak{d}_K) \sum_{\substack{\xi \succ 0 \\ \xi : \mathcal{O}_K / \mathfrak{f}_\psi}} \psi(\xi) \mathbf{e}(\mathrm{tr}(\rho\xi))$$

with $\rho \in K$, $\succ 0$, $(\rho\mathfrak{f}_\psi\mathfrak{d}_K, \mathfrak{f}_\psi) = \mathcal{O}_K$ where $\mathbf{e}(x)$ stands for $\exp(2\pi\sqrt{-1}x)$. The value $\tau_K(\psi)$ is determined up to the choices of ρ. We note that $\tau_K(\psi) = \psi(\mathfrak{d}_K)$ if $\mathfrak{f}_\psi = \mathcal{O}_K$.

In this section, we state some formulas related to the Gauss sums for the later use. Their proofs are found in [2].

Lemma 1. *Let $\mathfrak{A}$ be a fractional ideal. Let $\psi \in C^*_{\mathfrak{N}}$, which is not necessarily primitive.*

(i) *Let $\mu \in \mathfrak{A}^{-1}$. Then*

$$\sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1} / \mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \overline{\psi}(\delta_0 \mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K) \mathbf{e}(\mathrm{tr}(\delta_0\mu))$$

$$= \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi} \tau_K(\overline{\widetilde{\psi}}) \sum_{\mathfrak{R} \in \mathcal{R}(\mathfrak{N}, \psi)} \mu_K(\mathfrak{R}) \frac{\varphi_K(\mathfrak{N})}{\varphi_K(\mathfrak{f}_\psi \mathfrak{R})} \overline{\widetilde{\psi}}(\mathfrak{R}) (\psi_{\mathfrak{R}} \mathcal{I}_K)(\mu \mathfrak{N}^{-1} \mathfrak{f}_\psi \mathfrak{R}\mathfrak{A}).$$

In the summation of the right hand side, at most one term survives. If there is $\mathfrak{R} \in \mathcal{R}(\mathfrak{N}, \psi)$ *satisfying* $(\mu\mathfrak{A}\mathfrak{N}^{-1}, \mathcal{O}_K) = \mathfrak{f}_\psi^{-1}\mathfrak{R}^{-1}$*, then the term associated with* $\mathfrak{R}$ *survives.*

(ii) *Let* $\mu \in \mathfrak{A}^{-1}\mathfrak{N}^{-1}$*. Then*

$$\sum_{\delta_0 : \mathfrak{e}_\psi^{-1}\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \psi(\delta_0\mathfrak{e}_\psi\mathfrak{N}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$$

$$= \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\tau_K(\widetilde{\psi}) \sum_{\mathfrak{R} \in \mathcal{R}(\mathfrak{N},\psi)} \mu_K(\mathfrak{R}) \frac{\varphi_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})}{\varphi_K(\mathfrak{R})} \widetilde{\psi}(\mathfrak{R})(\overline{\psi}_{\mathfrak{R}}\mathcal{I}_K)(\mu\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{R}\mathfrak{A}).$$

In the summation of the right hand side, at most one term survives. If there is $\mathfrak{R} \in \mathcal{R}(\mathfrak{N}, \psi)$ *satisfying* $(\mu\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{R}\mathfrak{A}, \mathcal{O}_K) = \mathfrak{R}^{-1}$*, then the term associated with* $\mathfrak{R}$ *survives.*

Let X be a function on the set $\{\mathfrak{N}\mathfrak{M}^{-1} \mid \mathfrak{M}|\mathfrak{N}, (\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K\}$ of ideals. Then we define $\Lambda_k(\mathfrak{N}, \psi)$ by

$$\Lambda_k(\mathfrak{N}, \psi)X := \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}$$
$$\times \sum_{\mathfrak{M}|\mathfrak{N}, (\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K} \Big(\prod_{\mathfrak{P}|\mathfrak{M}} (1 - \mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M})X(\mathfrak{N}\mathfrak{M}^{-1}). \qquad (3)$$

Proposition 1. *Let* $\mathfrak{N}$ *be an integral ideal and let* $\psi \in C_{\mathfrak{N}}^*$*. Let* $\mathfrak{A}$ *be a fractional ideal. Let* $X_\mu(\mathfrak{M}) = \sum_{\delta_0 : \mathfrak{M}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \overline{\psi}_{\mathfrak{M}}(\delta_0\mathfrak{M}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$ *for* $\mu \in \mathfrak{A}^{-1}$ *and for an integral ideal* $\mathfrak{M}$ *contained in* $\mathfrak{f}_\psi$*. Then*

$$\Lambda_k(\mathfrak{N}, \psi)X_\mu = \mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}\tau_K(\overline{\widetilde{\psi}})\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}(\psi\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A}).$$

3. Eisenstein series

Let $k \in \mathbf{N}$. Let $\mathfrak{N}, \mathfrak{N}'$ be fixed integral ideals of K and let $\mathfrak{D}$ be a fixed fractional ideal. Let $\mathfrak{A}, \mathfrak{B}$ be fractional ideals of K. Let $\gamma_0 \in \mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}, \delta_0 \in \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}$. We define

$$E_{k,\mathfrak{A},\mathfrak{D},\mathfrak{B}}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}') := \mathrm{N}(\mathfrak{A})^k \sum_{\substack{\gamma \equiv \gamma_0 (\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}) \\ \delta \equiv \delta_0 (\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}) \\ (\gamma,\delta)/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}}{}' \mathrm{N}(\gamma\mathfrak{z} + \delta)^{-k}|\mathrm{N}(\gamma\mathfrak{z} + \delta)|^{-s}|_{s=0}$$

where $\gamma \equiv \gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1})$ implies that $\gamma \equiv \gamma_0$ modulo $\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}$ and where $\sum'$ implies that the term corresponding to $(\gamma, \delta) = (0, 0)$ is omitted in the summation. For a set S, $\Delta(x, S)$ is define to be 1 or 0 according as $x \in S$ or not. Then we have the Fourier expansion

$$E_{k,\mathfrak{A},\mathfrak{D},\mathfrak{B}}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')$$
$$= \Delta(\gamma_0, \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1})\mathrm{N}(\mathfrak{A})^k \sum_{\substack{\mu \equiv \delta_0 (\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}) \\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$
$$+ \left(\frac{(-2\pi\sqrt{-1})^k}{(k-1)!}\right)^g D_K^{1/2}\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}(\mathfrak{B}) \sum_{0 \prec \nu \in \mathfrak{B}^2\mathfrak{D}\mathfrak{d}_K^{-1}} \sum_{\substack{\nu/\mu \equiv \gamma_0 (\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}) \\ \mu : \mathfrak{A}^{-1}\mathfrak{B}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathbf{e}(\mathrm{tr}(\delta_0\mu))$$
$$\times \mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

where there is the additional term

$$(-\pi\sqrt{-1})^g D_K^{1/2}\mathrm{N}(\mathfrak{B}) \sum_{\substack{\mu\equiv\gamma_0(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1})\\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

when $k=1$, and where there is the additional term $-\pi/(\mathrm{N}(\mathfrak{N}'\mathfrak{D})\Im z)$ when $g=1$ and $k=2$.

Let $\psi \in C_{\mathfrak{N}}^*, \psi' \in C_{\mathfrak{N}'}^*$ be even or odd characters so that $k \in \mathbf{N}$ and $\psi\psi'$ have the same parity, where we assume that either $\psi \neq \mathbf{1}_{\mathfrak{N}}$ or $\psi' \neq \mathbf{1}_{\mathfrak{N}'}$ when $g=1$ and $k=2$. We assume that

$$(\mathfrak{N}, \mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}) = \mathcal{O}_K. \tag{4}$$

Then we put

$$\begin{aligned} &\widetilde{\lambda}_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}) = \widetilde{\lambda}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}^{\psi'}(\mathfrak{z};\mathfrak{D}) \\ &:= \left(\tfrac{(k-1)!}{(-2\sqrt{-1}\pi)^k}\right)^g D_K^{-1/2}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1} \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \\ &\quad \times \sum_{\gamma_0,\delta_0} \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)E_{k,\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(\mathfrak{z},\gamma_0,\delta_0;\mathfrak{N},\mathfrak{N}') \end{aligned} \tag{5}$$

where in the second summation, γ_0 runs over the set of totally positive representatives of $\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}$ modulo $\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}$ with $(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K, \mathfrak{N}') = \mathcal{O}_K$, and δ_0 runs over the set of totally positive representatives of $\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}$ modulo $\mathfrak{A}\mathfrak{d}_K^{-1}$ with $(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$. Further let

$$\begin{aligned} \widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z}) &:= \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k} \\ &\times \sum_{\mathfrak{M}|\mathfrak{N},(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K} \Big(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\Big)\widetilde{\overline{\psi}}(\mathfrak{M})\widetilde{\lambda}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}^{\psi'}(\mathfrak{z};\mathfrak{D}). \end{aligned} \tag{6}$$

4. Constant terms of Hilbert Eisenstein series

In this section we use the following result due to Hecke [1] a number of times.

Lemma 2. *Let $\mathfrak{M}$ be a fractional ideal, and let $\mathcal{E}$ be a subgroup of finite index in the group of all units. Let $\mu_0 \in K$ and $k \in \mathbf{Z}$. Then there holds the functional equations*

$$\begin{aligned} &\sum_{\substack{\mu\equiv\mu_0(\mathfrak{M})\\ \mu/\mathcal{E}}}{}' \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0} \\ &= \left(\tfrac{(-2\sqrt{-1}\pi)^k}{2\cdot(k-1)!}\right)^g D_K^{-1/2}\mathrm{N}(\mathfrak{M})^{-1} \sum_{\mu:\mathfrak{M}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}}{}' \mathbf{e}(\mathrm{tr}(\mu_0\mu)) \\ &\qquad \times \mathrm{sgn}(\mathrm{Nm}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}, \\ &\sum_{\substack{\mu\equiv\mu_0(\mathfrak{M})\\ \mu/\mathcal{E}}}{}' \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0} \\ &= (-\sqrt{-1}\pi^{-1})^g D_K^{-1/2}\mathrm{N}(\mathfrak{M})^{-1} \sum_{\mu:\mathfrak{M}^{-1}\mathfrak{d}_K^{-1}/\mathcal{E}}{}' \mathbf{e}(\mathrm{tr}(\mu_0\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0} \end{aligned}$$

where $\sum'$ implies that the term corresponding to $\mu=0$ is omitted in the summation.

Let A be as in (2) with $\alpha \in \mathcal{O}_K, \gamma \in \mathfrak{D}$. If $f(\mathfrak{z})$ is a Hilbert modular form of weight k for $\Gamma_0(\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{N}'\mathfrak{D})$, then the value $\kappa(\alpha/\gamma, f)$ of $f(\mathfrak{z})$ at the cusp α/γ is defined by

$$\kappa(\alpha/\gamma, f) := \lim_{\mathfrak{z}\to\sqrt{-1}\infty} \mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k} f(A\mathfrak{z}) \times \begin{cases} \mathrm{sgn}(\mathrm{N}(\delta))^k & (\delta \neq 0), \\ 1 & (\delta = 0). \end{cases} \tag{7}$$

We determine the value at each cusp, of the Hilbert-Eisenstein series (6) as well as the Fourier expansion at the cusp $\sqrt{-1}\infty$.

For a cusp $\alpha/\gamma \in K \cup \{\infty\}$, we can take $\alpha \in \mathcal{O}_K, \gamma \in \mathfrak{D}$ so that

$$\mathfrak{B} := (\alpha, \gamma\mathfrak{D}^{-1})$$

satisfies

$$(\mathfrak{B}, \mathfrak{N}\mathfrak{N}') = \mathcal{O}_K. \tag{8}$$

Since $A \in \mathrm{SL}_2(K)$, there holds $\mathfrak{B}^{-1} = (\beta\mathfrak{D}, \delta)$. Further we can take β, δ so that $(\beta, \mathfrak{N}\mathfrak{N}') = (\delta, \mathfrak{N}\mathfrak{N}') = \mathcal{O}_K$. Then the equality $\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k} E_{k,\mathfrak{N}'\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(A\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}') = E_{k,\mathfrak{N}'\mathfrak{A},\mathfrak{D},\mathfrak{B}}(\mathfrak{z}, \alpha\gamma_0+\gamma\delta_0, \beta\gamma_0+\delta\delta_0; \mathfrak{N}, \mathfrak{N}')$ holds, and the constant term of the Fourier expansion of $E_{k,\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')$ at α/γ, is equal to

$$\mathrm{N}(\mathfrak{A})^k \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'), \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1})$$
$$\times \sideset{}{'}\sum_{\substack{\mu\equiv\beta\gamma_0+\delta(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}) \\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

where there is the additional term

$$(-\pi\sqrt{-1})^g D_K^{1/2}\mathrm{N}(\mathfrak{N}'^{-1}\mathfrak{B}) \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \sideset{}{'}\sum_{\substack{\mu\equiv\alpha\gamma_0+\gamma(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}) \\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

when $k = 1$. The modular form $\widetilde{\lambda}^{\psi'}_{k,\psi_\mathfrak{N},\mathfrak{N}}(\mathfrak{z};\mathfrak{D})$ is a linear combination of $E_{k,\mathfrak{A},\mathfrak{D},\mathcal{O}_K}(\mathfrak{z}, \gamma_0, \delta_0; \mathfrak{N}, \mathfrak{N}')$'s by (5), and we obtain the following;

Lemma 3. *Let $A, \alpha, \beta, \gamma, \delta$ be as above. Assume the condition (8). Let $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$ denote the constant term of $\mathrm{N}(\gamma\mathfrak{z}+\delta)^{-k}\widetilde{\lambda}^{\psi'}_{k,\psi_\mathfrak{N},\mathfrak{N}}(A\mathfrak{z};\mathfrak{D})$. Then it is given by*

$$C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_\mathfrak{N}, \psi')$$
$$= \left(\frac{(k-1)!}{(-2\sqrt{-1}\pi)^k}\right)^g D_K^{-1/2}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1} \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1},\succ 0 \\ \delta_0:\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}}$$
$$\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^k \sum_{\delta':\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}}$$
$$\Delta(\alpha\gamma_0+\gamma(\delta_0+\delta'), \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}) \sideset{}{'}\sum_{\substack{\mu\equiv\beta\gamma_0+\delta(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}) \\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{N}(\mu)^{-k}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$

where in the second summation, γ_0 and δ_0 satisfy $(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K, \mathfrak{N}') = \mathcal{O}_K$, $(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$ respectively. When $k = 1$, there is the additional term

$C^1_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ *with*

$$
\begin{aligned}
&C^1_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi') \\
:=&\, 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1} \tau_K(\widetilde{\overline{\psi}})^{-1} \mathrm{N}(\mathfrak{N}'^{-1}\mathfrak{B}) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}, \succ 0 \\ \delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0}} \\
&\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K) \sum_{\delta' : \mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{d}_K^{-1}} \sideset{}{'}\sum_{\substack{\mu\equiv\alpha\gamma_0+\gamma(\delta_0+\delta')(\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}) \\ \mu/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \\
&\quad \mathrm{sgn}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s}|_{s=0}.
\end{aligned}
$$

For $\gamma \in \mathcal{O}_K$, we put

$$\mathfrak{M}'_\gamma := \mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}(\gamma\mathfrak{D}^{-1}, \mathfrak{N}')^{-1}.$$

By the assumption (4), $\mathfrak{M}'_\gamma$ is coprime to $\mathfrak{N}$ if it is integral. The purpose of this section is to prove the following;

Theorem 1. *Let* $\alpha \in \mathcal{O}_K, \gamma \in \mathfrak{D}, \mathfrak{B} = (\alpha, \gamma\mathfrak{D}^{-1})$ *with* $(\mathfrak{B}, \mathfrak{N}\mathfrak{N}') = \mathcal{O}_K$. *Put* $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}) := C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1}, k, \psi_{\mathfrak{N}\mathfrak{M}^{-1}}, \psi')$ *for a divisor* $\mathfrak{M}$ *of* $\mathfrak{N}$ *with* $(\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K$. *Let* $\Lambda_k(\mathfrak{N}, \psi)$ *be as in (3). If there is no divisor* $\mathfrak{M}_\gamma$ *of* $\mathfrak{N}$ *with* $(\mathfrak{M}_\gamma, \mathfrak{f}_\psi) = \mathcal{O}_K$ *and* $(\gamma\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$ *for* $\mathfrak{M}'_\gamma$ *integral, then* $\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma} = 0$. *Suppose otherwise. Let* $\mathfrak{M}_\gamma$ *be the largest ideal satisfying* $(\gamma\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$. *Then*

$$
\begin{aligned}
&\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma} \\
=&\, \mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}'))\widetilde{\overline{\psi}}(\alpha\mathfrak{B}^{-1}\mathfrak{M}_\gamma\mathfrak{M}'_\gamma(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')^{-1}) \\
&\times \psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k \\
&\times \mathrm{N}(\mathfrak{M}_\gamma^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{M}_\gamma\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1}\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau_K(\widetilde{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M}_\gamma^{-1}) \\
&\times \prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\ L_K(1-k, \widetilde{\psi\overline{\psi}'}) \prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}, \mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}} (1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-k}) \\
&\times \prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}, \mathfrak{P}\nmid\mathfrak{M}_\gamma\mathfrak{N}'} (1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{k-1}).
\end{aligned}
\tag{9}
$$

If $\gamma = 0$, *then* $\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma}$ *is non-zero only when* $\mathfrak{N}' = \mathcal{O}_K$, *and the value is obtained by replacing* γ *in (9) by* $\mathrm{N}(\mathfrak{N})$. *If* $\alpha = 0$, *then then* $\Lambda_k(\mathfrak{N}, \psi)C_{\alpha/\gamma}$ *is non-zero only when* $\mathfrak{f}_\psi = \mathcal{O}_K$, *and the value is obtained by replacing* α *in (9) by* 1.

Several preparations are necessary to give the proof.

Lemma 4. *Unless* $(\gamma\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{N}') = \mathfrak{N}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$ *for* $\mathfrak{M}'_\gamma$ *integral, then* $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ *vanishes. Suppose the equality. Then* $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ *equals*

$$
\begin{aligned}
&2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)) \\
&\times \mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \\
&\times \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} (\overline{\psi}\psi')(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))
\end{aligned}
$$

$$\times\, \mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Proof. Since $(\alpha\mathfrak{B}^{-1}, \gamma\mathfrak{B}^{-1}\mathfrak{D}^{-1}) = \mathcal{O}_K$ and since $(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K, \mathfrak{N}') = (\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K, \mathfrak{N}) = \mathcal{O}_K$ in the equation for $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ in Lemma 3, it is possible that $\Delta(\alpha\gamma_0 + \gamma(\delta_0 + \delta'), \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}) \neq 0$ only when $(\gamma\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{N}') = \mathfrak{N}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$ for $\mathfrak{M}'_\gamma$ integral. This shows the first assertion of Lemma 4. In particular if $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi') \neq 0$, then $\gamma\mathfrak{D}^{-1} \subset \mathfrak{N}$ and $(\alpha, \mathfrak{N}) = \mathcal{O}_K$. When $(\gamma\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{N}') = \mathfrak{N}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$ for $\mathfrak{M}'_\gamma$ integral, $C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi')$ is equal to

$$2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \sum_{\substack{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}'^{-1}_\gamma\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}, \succ 0 \\ \delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0}}$$

$$\Delta(\alpha\gamma_0 + \gamma\delta_0, \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1})\overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)(\psi'\mathcal{I}_K)(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}(\mathfrak{B})$$
$$\times \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)\mathfrak{A}^{-1}\mathfrak{B}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \mathbf{e}(\mathrm{tr}((\beta\gamma_0 + \delta\delta_0)\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0},$$

which is obtained by Lemma 2 and by Lemma 1 (ii). The map

$$\begin{pmatrix} \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}'^{-1}_\gamma\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1} \\ \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}'^{-1}_\gamma\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1} \\ \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1} \end{pmatrix}$$

obtained by multiplying by $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, is bijective. Using this bijection, we have

$$\begin{aligned}
&C_{\alpha/\gamma}(\mathfrak{N}, k, \psi_{\mathfrak{N}}, \psi') \\
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \\
&\qquad \sum_{\substack{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{M}'^{-1}_\gamma\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1}, \succ 0 \\ \delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{B}^{-1}\mathfrak{d}_K^{-1}, \succ 0}} \Delta(\gamma_0, \mathfrak{N}'\mathfrak{A}\mathfrak{B}\mathfrak{D}\mathfrak{d}_K^{-1})\mathrm{sgn}(\mathrm{N}(-\beta\gamma_0 + \alpha\delta_0))^{e_\psi} \\
&\quad \times \overline{\psi}((-\beta\gamma_0 + \alpha\delta_0)\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathrm{sgn}(\mathrm{N}(\delta\gamma_0 - \gamma\delta_0))^{e_{\psi'}}(\psi'\mathcal{I}_K)((\delta\gamma_0 - \gamma\delta_0)\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K) \\
&\quad \times \mathrm{N}(\mathfrak{A})^{k-1}\mathrm{N}(\mathfrak{B}) \sideset{}{'}\sum_{\mu : (\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma, \mathcal{O}_K)\mathfrak{A}^{-1}\mathfrak{B}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}
\end{aligned}$$

replacing $\mathfrak{A}$ by $\mathfrak{A}\mathfrak{B}$,

$$\begin{aligned}
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}'_\gamma)^{-1} \\
&\quad \times \overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma) \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \\
&\quad \times \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{M}'_\gamma(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} (\overline{\psi}\psi'\mathcal{I}_K)(\delta_0\mathfrak{N}\mathfrak{M}'^{-1}_\gamma\mathfrak{A}^{-1}\mathfrak{d}_K) \\
&\quad \times \sideset{}{'}\sum_{\mu : \mathfrak{M}'^{-1}_\gamma(\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}} \mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0} \\
&= 2^{-g}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{M}'_\gamma)^{-1}
\end{aligned}$$

$$\times\,\overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\times\sum'_{\mu:\mathfrak{M}'^{-1}_\gamma(\mathfrak{N}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\ \sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}'_\gamma\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{M}'_\gamma(\mathfrak{N}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$(\overline{\psi}\psi')(\delta_0\mathfrak{N}\mathfrak{M}'^{-1}_\gamma\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Replacing $\mathfrak{A}$ by $\mathfrak{M}'^{-1}_\gamma\mathfrak{A}$, we obtain the result of the lemma. □

Just replacing $\mathfrak{N}$ by $\mathfrak{N}\mathfrak{M}^{-1}$ in the lemma, we obtain the following;

Corollary. *Let $\mathfrak{M}$ be a divisor of $\mathfrak{N}$ with $(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K$. Unless $(\gamma\mathfrak{D}^{-1},\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{N}'\mathfrak{e}^{-1}_{\psi'}\mathfrak{M}'^{-1}_\gamma$ for $\mathfrak{M}'_\gamma$ integral, then $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$ vanishes. Suppose the equality. Then it equals*

$$2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K))$$

$$\times\,\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\sum'_{\mu:(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\ \sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$(\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}\psi')(\delta_0\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Let $\mathfrak{M}_\gamma$ be the largest ideal with $\mathfrak{M}_\gamma|\mathfrak{N},(\mathfrak{M}_\gamma,\mathfrak{f}_\psi)=\mathcal{O}_K$ satisfying $(\gamma\mathfrak{D}^{-1},\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}^{-1}_{\psi'}\mathfrak{N}'\mathfrak{M}'^{-1}_\gamma$. Then $C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')=0$ for $\mathfrak{M}$ with $\mathfrak{M}_\gamma\nmid\mathfrak{M}$. Suppose that $\mathfrak{M}$ is a divisor of $\mathfrak{N}\mathfrak{M}_\gamma^{-1}$ coprime to $\mathfrak{f}_\psi$ satisfying $(\gamma\mathfrak{D}^{-1},\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{N}')=\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{e}^{-1}_{\psi'}\mathfrak{N}'\mathfrak{M}'^{-1}_\gamma$. Then $(\mathfrak{M},\mathfrak{N}')=\mathcal{O}_K$, from which there holds $(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)=(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)$. For such $\mathfrak{M}$, we have

$$C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}},\psi')$$

$$=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})^k\mathrm{N}((\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K))$$

$$\times\,\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\times\sum'_{\mu:(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\ \sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{e}^{-1}_{\psi'},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0}$$

$$\overline{\psi}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\delta_0\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$$

$$\times\,\mathrm{sgn}(\mathrm{N}(\mu))^k|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}.$$

Then

$$\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma}$$

$$=\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k}\overline{\widetilde{\psi}}(\mathfrak{M}_\gamma)\Big(\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)$$

$$\sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}|\mathfrak{M}\\ \mathfrak{P}\nmid\mathfrak{M}_\gamma}}(1-\mathrm{N}(\mathfrak{P}))\Big)\overline{\widetilde{\psi}}(\mathfrak{M})C_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}},\psi')$$

$$=2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\widetilde{\psi}})^{-1}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})$$

$$\times \mathrm{N}(\mathfrak{e}_\psi \mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}((\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1}, \mathcal{O}_K))\mathrm{N}(\mathfrak{B})^k \mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{M}'_\gamma)^{-k}\Big(\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)$$

$$\times \widetilde{\overline{\psi}}(\mathfrak{M}_\gamma)\overline{\psi}(\alpha\mathfrak{B}^{-1}\mathfrak{M}'_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{k-1}$$

$$\times \sum_{\mu:(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}' \mathrm{sgn}(\mathrm{N}(\mu))^k D(\mu)|\mathrm{N}(\mu)|^{k-1}|\mathrm{N}(\mu)|^{-s}|_{s=0} \tag{10}$$

with

$$\begin{aligned} &D(\mu) \\ :=& \sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\substack{\mathfrak{P}|\mathfrak{M}\\ \mathfrak{P}\nmid\mathfrak{M}_\gamma}}(1-\mathrm{N}(\mathfrak{P}))\Big)\widetilde{\overline{\psi}}(\mathfrak{M})\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \\ &\overline{\psi}_{\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}(\delta_0\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\psi'(\delta_0\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu)). \end{aligned}$$

Lemma 5. *Let* $\mu\in(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)\mathfrak{A}^{-1}$. *Then* $D(\mu)$ *is equal to*

$$\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\cap\mathfrak{e}_{\psi'})\mathrm{N}(\mathfrak{f}_{\overline{\psi}\psi'})^{-1}\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\mathrm{N}(\mathfrak{P})^{-1})$$

$$\times\mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R})$$

$$\times((\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'}). \tag{11}$$

Proof. There holds

$$\begin{aligned} &D(\mu) \\ =& \sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\mathfrak{P}|\mathfrak{M},\mathfrak{P}\nmid\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)(\widetilde{\overline{\psi}}\psi')(\mathfrak{M})\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{e}_{\psi'}^{-1},\mathcal{O}_K)^{-1}\mathfrak{A}\mathfrak{d}_K^{-1},\succ 0} \\ &\quad(\overline{\psi}_{\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}}\psi')(\delta_0\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu)) \\ =&\,\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})(\widetilde{\psi\overline{\psi}'}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\sum_{\mathfrak{M}|\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1},(\mathfrak{M},\mathfrak{f}_\psi\mathfrak{N}')=\mathcal{O}_K}\Big(\prod_{\mathfrak{P}|\mathfrak{M},\mathfrak{P}\nmid\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\Big)\times \\ &\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\cap\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1}\mathfrak{M}^{-1}\cap\mathfrak{e}_{\psi'})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}\times\begin{cases}1 & (\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{M}\mathfrak{A},\mathcal{O}_K)=\mathfrak{f}_{\overline{\psi}\psi'}^{-1}\mathfrak{R}^{-1}) \\ 0 & (\text{otherwise})\end{cases} \end{aligned}$$

by Lemma 1. Then

$$D(\mu)=\mathrm{sgn}(\mathrm{N}(\mu))^k\tau_K(\widetilde{\overline{\psi}\psi'})(\widetilde{\psi\overline{\psi}'}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1},\mathfrak{P}\nmid\mathfrak{f}_\psi\mathfrak{N}'}Z(\mathfrak{P})$$

$$\times\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{e}_{\psi'}\cap\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'}}\mathfrak{P}^{v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1})})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})}$$

$$\times\begin{cases}1 & (\mu\mathfrak{A}\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'}}\mathfrak{P}^{-v_\mathfrak{P}(\mathfrak{N}\mathfrak{M}\mathfrak{M}_\gamma^{-1})},\mathcal{O}_K)=\mathfrak{f}_{\overline{\psi}\psi'}^{-1}\mathfrak{R}^{-1}) \\ 0 & (\text{otherwise})\end{cases}$$

where

$$Z(\mathfrak{P}) = \begin{cases} \displaystyle\sum_{i=v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})-1}^{v_{\mathfrak{P}}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})} (1-\mathrm{N}(\mathfrak{P}))^{\min\{1,i\}}\mu_K(\{\mathfrak{R}\}_{\mathfrak{P}})\frac{\varphi_K(\{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{P}^{-i}\}_{\mathfrak{P}})}{\varphi_K(\{\mathfrak{R}\}_{\mathfrak{P}})} & (\mathfrak{P}\nmid\mathfrak{M}_\gamma), \\ \displaystyle\sum_{i=v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1})-1}^{v_{\mathfrak{P}}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})} \mu_K(\{\mathfrak{R}\}_{\mathfrak{P}})\frac{\varphi_K(\{\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{P}^{-i}\}_{\mathfrak{P}})}{\varphi_K(\{\mathfrak{R}\}_{\mathfrak{P}})} & (\mathfrak{P}|\mathfrak{M}_\gamma). \end{cases}$$

A simple calculation leads to the following;

(i) The case that $\mathfrak{P}\nmid\mathfrak{M}_\gamma$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1}) > 1$: $Z(\mathfrak{P}) = 0$.

(ii) The case that $\mathfrak{P}\nmid\mathfrak{M}_\gamma$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1}) = 1$: $Z(\mathfrak{P}) = -\mathrm{N}(\mathfrak{P})^{v_{\mathfrak{P}}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})}$.

(iii) The case that $\mathfrak{P}\nmid\mathfrak{M}_\gamma$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1}) \le 0$: $Z(\mathfrak{P}) = 0$.

(iv) The case that $\mathfrak{P}|\mathfrak{M}_\gamma$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1}) > 1$: $Z(\mathfrak{P}) = 0$.

(v) The case that $\mathfrak{P}|\mathfrak{M}_\gamma$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1}) = 1$: $Z(\mathfrak{P}) = 0$.

(vi) The case that $\mathfrak{P}|\mathfrak{M}_\gamma$ and $v_{\mathfrak{P}}(\mu^{-1}\mathfrak{N}\mathfrak{M}_\gamma^{-1}) \le 0$: $Z(\mathfrak{P}) = \mathrm{N}(\mathfrak{P})^{v_{\mathfrak{P}}(\mathfrak{N}\mathfrak{M}_\gamma^{-1})}$.

Then

$$\begin{aligned} &D(\mu) \\ &= \mathrm{sgn}(\mathrm{N}(\mu))^k \tau_K(\widetilde{\overline{\psi}\psi'}) \sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')} \mu_K(\mathfrak{R})\frac{\varphi_K(\mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{R}^{-1}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi,\mathfrak{N}'}\cap\mathfrak{e}_{\psi'})}{\varphi_K(\mathfrak{f}_{\overline{\psi}\psi'}\mathfrak{R})} \\ &\quad\times \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\mathrm{N}(\mathfrak{R}_{\mathfrak{N}\mathfrak{M}_\gamma^{-1},\psi,\mathfrak{N}'})(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}) \\ &\quad\times ((\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}}\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}\mathfrak{A}\mathfrak{f}_{\overline{\psi}\psi'}), \end{aligned}$$

which is equal to (11). □

Proof of Theorem 1. By Lemma 5 and by (10), we have

$$\begin{aligned} &\Lambda_k(\mathfrak{N},\psi)C_{\alpha/\gamma} \\ &= 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\widetilde{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')) \\ &\quad\times\widetilde{\overline{\psi}}(\alpha\mathfrak{B}^{-1}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{M}'_\gamma\mathfrak{M}_\gamma)\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_{\overline{\psi}\psi'})^{-k} \\ &\quad\times\mathrm{N}(\mathfrak{f}_\psi\mathfrak{M}_\gamma^{-1})\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\mathrm{N}(\mathfrak{P})^{-1})\mathrm{N}(\mathfrak{M}_\gamma^{-1}\mathfrak{f}_\psi(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))^{k-1} \\ &\quad\times(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\mathrm{N}(\mathfrak{R})^{-k+1}\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{R}) \\ &\quad\times\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\sum_{\mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}{}' ((\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}})(\mu\mathfrak{A})\mathrm{N}(\mu\mathfrak{A})|^{k-1-s}|_{s=0} \\ &= \tau_K(\widetilde{\overline{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{sgn}(\mathrm{N}(\alpha))^{e_\psi}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_{\psi'}}\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))\widetilde{\overline{\psi}}(\alpha\mathfrak{B}^{-1}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{M}'_\gamma\mathfrak{M}_\gamma) \\ &\quad\times\psi'(-\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma)\mathrm{N}(\mathfrak{B})^k\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{M}_\gamma^{-1})^k\mathrm{N}(\mathfrak{f}_{\overline{\psi}\psi'})^{-k} \\ &\quad\times\prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P}))\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\mathrm{N}(\mathfrak{P})^{-1})\mathrm{N}((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))^{k-1} \\ &\quad\times(\widetilde{\overline{\psi}\psi'})(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1})\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\mathrm{N}(\mathfrak{R})^{-k+1}\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{R}) \end{aligned}$$

$$\times L_K(1-k,(\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}}).$$

Here

$$\sum_{\mathfrak{R}\in\mathcal{R}(\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\overline{\psi}\psi')}\mu_K(\mathfrak{R})\mathrm{N}(\mathfrak{R})^{-k+1}\varphi_K(\mathfrak{R})^{-1}(\widetilde{\overline{\psi}\psi'})(\mathfrak{R})L_K(1-k,(\psi\overline{\psi}')_{\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{R}})$$

$$=L_K(1-k,\widetilde{\psi\overline{\psi}'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{P}\nmid\mathfrak{M}_\gamma\mathfrak{N}'}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{k-1})\prod_{\mathfrak{P}|\mathfrak{f}_\psi\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}\mathrm{N}(\mathfrak{P})(\mathrm{N}(\mathfrak{P})-1)^{-1}$$

$$\times\prod_{\mathfrak{P}|\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-k}),$$

from which, the theorem follows. □

5. The case of weight 1

We compute the additional term which appears when $k=1$. As in the preceding section , we put $\mathfrak{B}:=(\alpha,\gamma\mathfrak{D}^{-1})$ for $\alpha\in\mathcal{O}_K$, $\gamma\in\mathfrak{D}$, and assume the condition (8). From Lemma 3 and Lemma 2, we have for $\mathfrak{M}|\mathfrak{N}$ with $(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K$,

$$C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$$
$$=(-\sqrt{-1}\pi^{-1})^g2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau_K(\overline{\psi})^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{1/2}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{-1}$$
$$\times\sum'_{\mu:(\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1},\mathfrak{N}')^{-1}\mathfrak{A}^{-1}\mathfrak{B}^{-1}\mathfrak{D}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\ \sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1},\succ0}$$
$$\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}$$
$$\times\sum_{\delta_0:\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1},\succ0}\overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}(\delta_0\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\gamma\delta_0\mu)).\qquad(12)$$

The purpose of this section is to prove the following;

Theorem 2. *Let $\alpha\in\mathcal{O}_K,\gamma\in\mathfrak{D},\mathfrak{B}=(\alpha,\gamma\mathfrak{D}^{-1})$ with $(\mathfrak{B},\mathfrak{N}\mathfrak{N}')=\mathcal{O}_K$. Let $C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1})$ denote $C^1_{\alpha/\gamma}(\mathfrak{N}\mathfrak{M}^{-1},k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\psi')$. Put*

$$\mathfrak{L}_\gamma:=\gamma\mathfrak{D}^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}.$$

If there is no divisor $\mathfrak{R}$ of $\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ so that the numerator of $\mathfrak{L}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$, then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ vanishes. Suppose that such $\mathfrak{R}$ exists. Let $\widetilde{\mathfrak{R}}_\gamma$ be the divisor of $(\mathfrak{N},\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$ satisfying $v_{\mathfrak{P}}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})=0$ for any prime divisor $\mathfrak{P}$ of $(\mathfrak{N},\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$. Put $\mathfrak{L}'_\gamma:=(\gamma\mathfrak{D}^{-1}\mathfrak{N}'^{-1},\mathfrak{N}^{-1})\cap\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$. Then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is equal to

$$\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mu_K(\widetilde{\mathfrak{R}}_\gamma)\mathrm{N}(\mathfrak{B})\overline{\psi}(\mathfrak{B})\psi((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1})\cap\mathcal{O}_K)\overline{\widetilde{\psi}'}(\alpha\mathfrak{B}^{-1})\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)$$
$$\times\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1},\mathcal{O}_K)^{-1})\varphi_K(\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}'^{-1}_\gamma)\mathrm{N}((\mathfrak{L}_\gamma,\widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}'_\gamma)\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\psi\overline{\psi}'})\tau_K(\widetilde{\overline{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})$$
$$\times L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi,\mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})).\qquad(13)$$

If $\gamma = 0$, then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is non-zero only when $\mathfrak{N} = \mathcal{O}_K$, and the value is obtained by replacing γ in (13) by $\mathrm{N}(\mathfrak{N}')$. If $\alpha = 0$, then then $\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}$ is non-zero only when $\mathfrak{f}_{\psi'} = \mathcal{O}_K$, and the value is obtained by replacing α in (13) by 1.

Proof. By (12) and by Proposition 1, we have

$$\begin{aligned}
&\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}\\
&=(-\sqrt{-1}\pi^{-1})^g 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{1/2}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{-1}\\
&\qquad\sideset{}{'}\sum_{\mu:(\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1},\mathfrak{N}')^{-1}\mathfrak{A}^{-1}\mathfrak{B}^{-1}\mathfrak{D}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\ \sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1},\succ 0}\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\\
&\qquad\times\mathrm{sgn}(\mathrm{N}(\gamma\mu))^{e_\psi}(\psi\mathcal{I}_K)(\gamma\mu\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu))\mathrm{N}(\mu)^{-1}|\mathrm{N}(\mu)|^{-s}|_{s=0}\\
&=(-\sqrt{-1}\pi^{-1})^g 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{1/2}\mathrm{sgn}(\mathrm{N}(\gamma))^{e_\psi}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\mathrm{N}(\mathfrak{A})^{-1}\\
&\qquad\sideset{}{'}\sum_{\mu:(\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1},\mathcal{O}_K)^{-1}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{B}^{-1}\mathfrak{D}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\psi(\gamma\mu\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi}\mathrm{N}(\mu)^{-1}\\
&\qquad\times|\mathrm{N}(\mu)|^{-s}|_{s=0}\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1},\succ 0}\psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\alpha\gamma_0\mu)).
\end{aligned}$$

Since $\tau_K(\widetilde{\psi}') = (-1)^{e_{\psi'}g}\mathrm{N}(\mathfrak{f}_{\psi'})\tau_K(\overline{\widetilde{\psi}'})^{-1}$, Lemma 1 leads to

$$\begin{aligned}
&\Lambda_1(\mathfrak{N},\psi)C^1_{\alpha/\gamma}\\
&=(\sqrt{-1}\pi^{-1})^g 2^{-g}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{1/2}\mathrm{N}(\mathfrak{f}_{\psi'})\tau_K(\overline{\widetilde{\psi}'})^{-1}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\\
&\qquad\times\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\sum_{\mathfrak{R}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}\\
&\qquad\sideset{}{'}\sum_{\mu:(\alpha\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{R},\gamma\mathfrak{D}^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi)^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}\psi(\gamma\mu\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\\
&\qquad\times\widetilde{\psi}'(\mathfrak{R})\overline{\psi}'_{\mathfrak{R}}(\alpha\mu\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{A}\mathfrak{D}\mathfrak{R})\mathrm{N}(\mu\mathfrak{A})^{-1-s}|_{s=0}\\
&=(\sqrt{-1}\pi^{-1})^g\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}D_K^{1/2}\mathrm{N}(\mathfrak{f}_{\psi'})\tau_K(\overline{\widetilde{\psi}'})^{-1}\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\\
&\qquad\times\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\sum_{\mathfrak{R}|(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1},\gamma\mathfrak{D}^{-1})}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\psi(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\mathfrak{R})\\
&\qquad\times\overline{\psi}'_{\mathfrak{R}}((\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})\mathrm{N}((\alpha\mathfrak{R},\mathfrak{L}_\gamma)\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{D})L_K(1,\widetilde{\psi\overline{\psi}'})\\
&\qquad\times\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\\
&=\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\tau_K(\overline{\widetilde{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}(\mathfrak{N}'\mathfrak{D})^{-1}\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\psi\overline{\psi}'})\\
&\qquad\sum_{\mathfrak{R}|(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1},\gamma\mathfrak{D}^{-1})}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\psi(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\mathfrak{R})\overline{\psi}'_{\mathfrak{R}}((\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1},\mathcal{O}_K)^{-1})\\
&\qquad\times\mathrm{N}((\alpha\mathfrak{R},\mathfrak{L}_\gamma)\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{D})L_K(0,\widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}),
\end{aligned}$$

where we use the functional equation of the L-function at the last equality.

Since $\mathfrak{B} = (\alpha, \gamma\mathfrak{D}^{-1})$ is coprime to $\mathfrak{N}\mathfrak{N}'$, we have $(\alpha\mathfrak{N}\mathfrak{e}_{\psi}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{R}, \gamma\mathfrak{D}^{-1}) = (\alpha, \gamma\mathfrak{D}^{-1})(\mathfrak{N}\mathfrak{e}_{\psi}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{R}, \gamma\mathfrak{D}^{-1}) = \mathfrak{B}(\mathfrak{R}, \mathfrak{L}_\gamma)\mathfrak{N}\mathfrak{e}_{\psi}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$ for $\mathfrak{R}$ dividing $(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}, \gamma\mathfrak{D}^{-1})$. Then $(\alpha^{-1}\mathfrak{L}_\gamma\mathfrak{R}^{-1}, \mathcal{O}_K) = \alpha^{-1}\mathfrak{B}(\mathfrak{L}_\gamma\mathfrak{R}^{-1}, \mathcal{O}_K)$ follows. Then

$$\begin{aligned}
&\Lambda_1(\mathfrak{N}, \psi)C^1_{\alpha/\gamma}\\
&= \mathrm{N}(\mathfrak{B})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\overline{\psi\psi'}})\tau_K(\overline{\widetilde{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\overline{\psi}(\mathfrak{B})\\
&\quad\times\overline{\widetilde{\psi'}}(\alpha\mathfrak{B}^{-1})\sum_{\mathfrak{R}|(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}, \gamma\mathfrak{D}^{-1})}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\mathrm{N}((\mathfrak{L}_\gamma, \mathfrak{R})\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(\mathfrak{L}_\gamma\mathfrak{R}^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\mathfrak{R})\\
&\quad\times\overline{\psi}'_{\mathfrak{R}}((\mathfrak{L}_\gamma\mathfrak{R}^{-1}, \mathcal{O}_K)^{-1})L_K(0, \widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}).
\end{aligned}$$

In the summation, the term corresponding to $\mathfrak{R}$ survives if the numerator of $\mathfrak{L}_\gamma\mathfrak{R}^{-1}$ is coprime to $\mathfrak{N}$ and the denominator is coprime to $\mathfrak{f}_{\psi'}\mathfrak{R}$. Suppose that such $\mathfrak{R}$ exists. Then

$$\widetilde{\mathfrak{R}}_\gamma := \prod_{\mathfrak{P}|(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}),\ v_{\mathfrak{P}}(\mathfrak{L}_\gamma)=1}\mathfrak{P}$$

is the largest such ideal, and $\mathfrak{R}$ is written as the product of $\widetilde{\mathfrak{R}}_\gamma$ and a divisor of $\mathfrak{L}'_\gamma\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ where $\mathfrak{L}'_\gamma := (\gamma\mathfrak{D}^{-1}\mathfrak{N}'^{-1}, \mathfrak{N}^{-1})\cap\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$. Then

$$\begin{aligned}
&\Lambda_1(\mathfrak{N}, \psi)C^1_{\alpha/\gamma}\\
&= \mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\overline{\psi\psi'}})\tau_K(\overline{\widetilde{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\overline{\psi}(\mathfrak{B})\\
&\quad\overline{\widetilde{\psi}}'(\alpha\mathfrak{B}^{-1})\sum_{\mathfrak{R}|\mathfrak{L}'_\gamma\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}}\mu_K(\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})\varphi_K(\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})^{-1}\mathrm{N}((\mathfrak{L}_\gamma, \widetilde{\mathfrak{R}}_\gamma\mathfrak{R})\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{R}^{-1}\cap\mathcal{O}_K)\\
&\quad\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma\mathfrak{R})\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma\mathfrak{R}}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{R}^{-1}, \mathcal{O}_K)^{-1})L_K(0, \widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\\
&= \mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\overline{\psi\psi'}})\tau_K(\overline{\widetilde{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\overline{\psi}(\mathfrak{B})\overline{\widetilde{\psi}}'(\alpha\mathfrak{B}^{-1})\mu_K(\widetilde{\mathfrak{R}}_\gamma)\\
&\quad\varphi_K(\widetilde{\mathfrak{R}}_\gamma)^{-1}\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}((\mathfrak{L}_\gamma, \widetilde{\mathfrak{R}}_\gamma)\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}, \mathcal{O}_K)^{-1})\\
&\quad L_K(0, \widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\sum_{\mathfrak{R}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}\mu_K(\mathfrak{R})\varphi_K(\mathfrak{R})^{-1}\mathrm{N}(\mathfrak{R})\overline{\psi}(\mathfrak{R})\widetilde{\psi}'(\mathfrak{R})\\
&\quad\prod_{\mathfrak{P}|\mathfrak{R}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\\
&= \mathrm{sgn}(\mathrm{N}(-\gamma))^{e_\psi}\mathrm{sgn}(\mathrm{N}(\alpha))^{e_{\psi'}}\mathrm{N}(\mathfrak{B})\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}^{-1}_{\overline{\psi\psi'}})\tau_K(\overline{\widetilde{\psi'}})^{-1}\tau_K(\widetilde{\psi\overline{\psi'}})\overline{\psi}(\mathfrak{B})\overline{\widetilde{\psi}}'(\alpha\mathfrak{B}^{-1})\mu_K(\widetilde{\mathfrak{R}}_\gamma)\\
&\quad\varphi_K(\widetilde{\mathfrak{R}}_\gamma)^{-1}\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})\mathrm{N}((\mathfrak{L}_\gamma, \widetilde{\mathfrak{R}}_\gamma)\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'})\psi(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}\cap\mathcal{O}_K)\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}, \mathcal{O}_K)^{-1})\\
&\quad L_K(0, \widetilde{\overline{\psi}\psi'})\prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_{\psi'}}(1-\widetilde{\psi\overline{\psi'}}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1})\mathrm{N}(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma)\varphi_K(\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma)^{-1}\prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma}(1-\overline{\psi}\widetilde{\psi}'(\mathfrak{P})),
\end{aligned}$$

which is equal to (13). □

6. Main Theorem

We define

$$\sigma_{k-1,\psi}^{\psi'}(\nu;\mathfrak{M}) := \sum_{\nu\mathfrak{M}\subset\mathfrak{A}\subset\mathcal{O}_K} \psi(\mathfrak{A})\psi'(\nu\mathfrak{M}\mathfrak{A}^{-1})\mathrm{N}(\mathfrak{A})^{k-1}$$

for a totally positive $\nu \in K$ and for a fractional ideal $\mathfrak{M}$. We note that it is 0 if $\nu\mathfrak{M}$ is not integral.

Main Theorem. *Let $\mathfrak{N}, \mathfrak{N}'$ be integral ideals of K. Let $\psi \in C_{\mathfrak{N}}^*, \psi' \in C_{\mathfrak{N}'}^*$ be even or odd characters with the conductors $\mathfrak{f}_\psi, \mathfrak{f}_{\psi'}$ respectively. Let $\widetilde{\psi}$ denote the primitive character associated with ψ. Let e_ψ be 0 or 1 according as ψ is even or odd. Let $\mathfrak{e}_\psi, \mathfrak{e}_{\psi'}$ be as in (1). We assume $(\mathfrak{N}, \mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}) = \mathcal{O}_K$. For $k \in \mathbf{N}$ with the same parity as $\psi\psi'$ and for a fixed fractional ideal $\mathfrak{D}$, let*

$$\begin{aligned}\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z}) :=& \mu_K(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\widetilde{\psi}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})\mathrm{N}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1})^{-1}\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k} \\ &\times \sum_{\mathfrak{M}|\mathfrak{N},(\mathfrak{M},\mathfrak{f}_\psi)=\mathcal{O}_K} \left(\prod_{\mathfrak{P}|\mathfrak{M}}(1-\mathrm{N}(\mathfrak{P}))\right) \overline{\widetilde{\psi}}(\mathfrak{M})\widetilde{\lambda}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}^{\psi'}(\mathfrak{z};\mathfrak{D}),\end{aligned}$$

where we assume that $\psi \neq \mathbf{1}_{\mathfrak{N}}$ or $\psi' \neq \mathbf{1}_{\mathfrak{N}'}$ when $g = 1$ and $k = 2$. Then $\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z})$ is a Hilbert modular form for $\Gamma_0(\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{N}'\mathfrak{D})$ of weight k with character $\psi\psi'$, whose Fourier expansion is given by

$$\begin{cases} \overline{\psi}'(\mathfrak{N}\mathfrak{e}_\psi^{-1}\mathfrak{f}_\psi\mathfrak{D}\mathfrak{d}_K^{-1})L_K(1-k,\psi\overline{\psi}') & (k>1 \textit{ or } \mathfrak{N}\subsetneq\mathcal{O}_K, \textit{ and } \mathfrak{N}'=\mathcal{O}_K) \\ \overline{\psi}(\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}\mathfrak{D}\mathfrak{d}_K^{-1})L_K(0,\overline{\psi}\psi') & (k=1,\mathfrak{N}=\mathcal{O}_K,\mathfrak{N}'\subsetneq\mathcal{O}_K) \\ \overline{\psi}'(\mathfrak{D}\mathfrak{d}_K^{-1})L_K(0,\psi\overline{\psi}')+\overline{\psi}(\mathfrak{D}\mathfrak{d}_K^{-1})L_K(0,\overline{\psi}\psi') & (k=1,\mathfrak{N}=\mathfrak{N}'=\mathcal{O}_K) \\ 0 & (\textit{otherwise}) \end{cases}$$

$$+2^g \sum_{0\prec\nu\in\mathfrak{D}\mathfrak{d}_K^{-1}} \sigma_{k-1,\psi}^{\psi'}(\nu;\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{D}\mathfrak{d}_K^{-1})\ \mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})).$$

Let α/γ be a cusp with $\alpha \in \mathcal{O}_K$, $\gamma \in \mathfrak{D}$. We can take α, γ so that $\mathfrak{B} := (\alpha, \gamma\mathfrak{D}^{-1})$ is coprime to $\mathfrak{N}\mathfrak{N}'$. The value $\kappa(\alpha/\gamma, \widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z}))$ of $\widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z})$ at the cusp α/γ defined in (7) is 0 if there are not integral ideals $\mathfrak{M}_\gamma, \mathfrak{M}'_\gamma$ with $\mathfrak{M}_\gamma|\mathfrak{N}$, $(\mathfrak{M}_\gamma, \mathfrak{f}_\psi) = \mathcal{O}_K$, $\mathfrak{M}'_\gamma|\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}$ and with $(\gamma\mathfrak{D}^{-1}, \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}') = \mathfrak{N}\mathfrak{M}_\gamma^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{M}'^{-1}_\gamma$. Suppose otherwise, and let $\mathfrak{M}_\gamma$ be the largest such ideal. Then the value $\kappa(\alpha/\gamma, \widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z}))$ is given by

$$\begin{aligned}&(-1)^{ge_{\psi'}}\mathrm{sgn}(\mathrm{N}(\alpha\gamma))^{e_\psi}\mu_K((\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}'))\overline{\widetilde{\psi}}(\alpha\mathfrak{B}^{-1}\mathfrak{M}_\gamma\mathfrak{M}'_\gamma(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}) \\ &\times \psi'(\gamma\mathfrak{D}^{-1}\mathfrak{B}^{-1}\mathfrak{N}^{-1}\mathfrak{M}_\gamma\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')^{-1}\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{M}'_\gamma) \\ &\times \mathrm{N}(\mathfrak{M}_\gamma^{-1}(\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{M}_\gamma\mathfrak{N}')\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})^{k-1}\mathrm{N}(\mathfrak{M}'_\gamma)^{-k}\mathrm{N}(\mathfrak{f}_\psi\mathfrak{f}_{\overline{\psi}\psi'}^{-1})\tau_K(\overline{\widetilde{\psi}})^{-1}\tau_K(\widetilde{\overline{\psi}\psi'})\mathrm{N}(\mathfrak{M}_\gamma^{-1}) \\ &\times L_K(1-k,\widetilde{\psi\overline{\psi}'}) \prod_{\mathfrak{P}|\mathfrak{M}_\gamma}(1-\mathrm{N}(\mathfrak{P})) \prod_{\mathfrak{P}|\mathfrak{e}_{\psi'},\mathfrak{P}\nmid\mathfrak{f}_{\overline{\psi}\psi'}}(1-\widetilde{\overline{\psi}\psi'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-k}) \\ &\times \prod_{\mathfrak{P}|\mathfrak{e}_\psi\mathfrak{f}_\psi^{-1},\mathfrak{P}\nmid\mathfrak{M}_\gamma\mathfrak{N}'}(1-\widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{k-1}) \qquad (14)\end{aligned}$$

where if $\gamma = 0$*, then the value is non-zero only when* $\mathfrak{N}' = \mathcal{O}_K$ *and it is given by replacing* γ *in (14) by* $\mathrm{N}(\mathfrak{N})$*, and where if* $\alpha = 0$*, the value is non-zero only when* $\mathfrak{f}_\psi = \mathcal{O}_K$ *and it is given by replacing* α *in (14) by* 1*.*

Let $\mathfrak{L}_\gamma := \gamma\mathfrak{D}^{-1}\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ *and* $\mathfrak{L}'_\gamma := (\gamma\mathfrak{D}^{-1}\mathfrak{N}'^{-1}, \mathfrak{N}^{-1}) \cap \mathfrak{e}_{\psi'}^{-1}\mathfrak{f}_{\psi'}$*. If* $k = 1$ *and if there is an integral divisor of* $\mathfrak{R}$ *of* $\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}$ *so that the numerator of* $\mathfrak{L}_\gamma\mathfrak{R}^{-1}$ *is coprime to* $\mathfrak{N}$ *and the denominator is coprime to* $\mathfrak{f}_{\psi'}\mathfrak{R}$*, then there is the additional term. Let* $\widetilde{\mathfrak{R}}_\gamma$ *be the divisor of* $(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$ *satisfying* $v_{\mathfrak{P}}(\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}) = 0$ *for any prime divisor* $\mathfrak{P}$ *of* $(\mathfrak{N}, \mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1})$*. Then* $\kappa(\alpha/\gamma, \widetilde{\Lambda}_{k,\psi}^{\psi'}(\mathfrak{z}))$ *has the additional term*

$$\begin{aligned}&(-1)^{ge_\psi}\mathrm{sgn}(\mathrm{N}(\alpha\gamma))^{e_{\psi'}}\mu_K(\widetilde{\mathfrak{R}}_\gamma)\overline{\psi}(\mathfrak{B})\psi((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}) \cap \mathcal{O}_K)\overline{\widetilde{\psi}'}(\alpha\mathfrak{B}^{-1})\widetilde{\psi}'(\widetilde{\mathfrak{R}}_\gamma)\\ &\times \overline{\psi}'_{\widetilde{\mathfrak{R}}_\gamma}((\mathfrak{L}_\gamma\widetilde{\mathfrak{R}}_\gamma^{-1}, \mathcal{O}_K)^{-1})\varphi_K(\widetilde{\mathfrak{R}}_\gamma^{-1}\mathfrak{L}'^{-1}_\gamma)\mathrm{N}((\mathfrak{L}_\gamma, \widetilde{\mathfrak{R}}_\gamma)\mathfrak{L}'_\gamma)\mathrm{N}(\mathfrak{f}_{\psi'}\mathfrak{f}_{\psi\overline{\psi}'}^{-1})\tau_K(\overline{\widetilde{\psi}'})^{-1}\tau_K(\widetilde{\psi\overline{\psi}'})\\ &\times L_K(0, \widetilde{\overline{\psi}\psi'}) \prod_{\mathfrak{P}|\mathfrak{e}_\psi, \mathfrak{P}\nmid\mathfrak{f}_{\psi\overline{\psi}'}} (1 - \widetilde{\psi\overline{\psi}'}(\mathfrak{P})\mathrm{N}(\mathfrak{P})^{-1}) \prod_{\mathfrak{P}|\mathfrak{e}_{\psi'}\mathfrak{f}_{\psi'}^{-1}\mathfrak{L}'_\gamma} (1 - \widetilde{\overline{\psi}\psi'}(\mathfrak{P}))\end{aligned} \tag{15}$$

where if $\gamma = 0$*, then the value is non-zero only when* $\mathfrak{N} = \mathcal{O}_K$ *and it is given by replacing* γ *in (15) by* $\mathrm{N}(\mathfrak{N}')$*, and where if* $\alpha = 0$*, the value is non-zero only when* $\mathfrak{f}_{\psi'} = \mathcal{O}_K$ *and it is given by replacing* α *in (15) by* 1*.*

Proof. The values at cusps are investigated in the section 4 and the section 5. We compute the higher terms. Then

$$\begin{aligned}&\widetilde{\lambda}_{k,\psi}^{\psi'}(\mathfrak{z};\mathfrak{D}) = \widetilde{\lambda}_{k,\psi_{\mathfrak{N}},\mathfrak{N}}^{\psi'}(\mathfrak{z};\mathfrak{D})\\ &= C + [\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau(\overline{\widetilde{\psi}})^{-1} \sum_{0 \prec \nu \in \mathfrak{D}\mathfrak{d}_K^{-1}} \sum_{\mathfrak{A} \in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sum_{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}, \succ 0}\\ &\quad \psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K) \sum_{\substack{\nu/\mu \equiv \gamma_0 (\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1})\\ \mu : \mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))\\ &\quad \times \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \overline{\psi}(\delta_0\mathfrak{N}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu)),\end{aligned}$$

where C is the constant term. Let $X(\mathfrak{N}\mathfrak{M}^{-1}) := \widetilde{\lambda}_{k,\psi_{\mathfrak{N}\mathfrak{M}^{-1}},\mathfrak{N}\mathfrak{M}^{-1}}^{\psi'}(\mathfrak{z};\mathfrak{D})$ for $\mathfrak{M}|\mathfrak{N}$ with $(\mathfrak{M}, \mathfrak{f}_\psi) = \mathcal{O}_K$, and let $X_\mu(\mathfrak{N}\mathfrak{M}^{-1}) := \sum_{\delta_0 : \mathfrak{N}^{-1}\mathfrak{M}\mathfrak{A}\mathfrak{d}_K^{-1}/\mathfrak{A}\mathfrak{d}_K^{-1}, \succ 0} \overline{\psi}_{\mathfrak{N}\mathfrak{M}^{-1}}(\delta_0\mathfrak{N}\mathfrak{M}^{-1}\mathfrak{A}^{-1}\mathfrak{d}_K)\mathbf{e}(\mathrm{tr}(\delta_0\mu))$. Then

$$\begin{aligned}&\Lambda_k(\mathfrak{N}, \psi)X\\ &= C' + [\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1}\tau(\overline{\widetilde{\psi}})^{-1} \sum_{0 \prec \nu \in \mathfrak{D}\mathfrak{d}_K^{-1}} \sum_{\mathfrak{A} \in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1} \sum_{\gamma_0 : \mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}, \succ 0}\\ &\quad \sum_{\substack{\nu/\mu \equiv \gamma_0 (\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1})\\ \mu : \mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} \psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathrm{sgn}(\mathrm{N}(\mu))\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))\\ &\quad \times \Lambda_k(\mathfrak{N}, \psi)X_\mu(\mathfrak{N}\mathfrak{M}^{-1})\\ &= C' + \mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}[\mathcal{E}_{\mathcal{O}_K} : \mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1} \sum_{0 \prec \nu \in \mathfrak{D}\mathfrak{d}_K^{-1}} \sum_{\mathfrak{A} \in C_{\mathfrak{N}\mathfrak{N}'}} \mathrm{N}(\mathfrak{A})^{k-1}\end{aligned}$$

$$\sum_{\gamma_0:\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}/\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1},\succ 0} \psi'(\gamma_0\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K) \sum_{\substack{\nu/\mu\equiv\gamma_0 (\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1})\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} (\psi\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})$$

$$\times\, \mathrm{sgn}(\mathrm{N}(\mu))^{e_\psi-1}\mathrm{N}(\mu)^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

$$= C' + \mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1}[\mathcal{E}_{\mathcal{O}_K}:\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}]^{-1} \sum_{0\prec\nu\in\mathfrak{D}\mathfrak{d}_K^{-1}} \sum_{\mathfrak{A}\in C_{\mathfrak{N}\mathfrak{N}'}}$$

$$\sum_{\substack{\nu/\mu\in\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{A}\mathfrak{D}\mathfrak{d}_K^{-1}\\ \mu:\mathfrak{A}^{-1}/\mathcal{E}_{\mathfrak{N}\mathfrak{N}'}}} (\psi\mathcal{I}_K)(\mu\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})\psi'(\nu/\mu\cdot\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathrm{N}(\mu\mathfrak{A})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

$$= C' + 2^g\mathrm{N}(\mathfrak{N}\mathfrak{e}_\psi^{-1})^{-k+1} \sum_{0\prec\nu\in\mathfrak{D}\mathfrak{d}_K^{-1}} \sum_{\nu\mathfrak{N}'^{-1}\mathfrak{e}_{\psi'}\mathfrak{D}^{-1}\mathfrak{d}_K\subset\mathfrak{A}\subset\mathfrak{N}\mathfrak{e}_\psi} \psi(\mathfrak{N}^{-1}\mathfrak{e}_\psi\mathfrak{A})$$

$$\times\, \psi'(\nu\mathfrak{e}_{\psi'}\mathfrak{N}'^{-1}\mathfrak{A}^{-1}\mathfrak{D}^{-1}\mathfrak{d}_K)\mathrm{N}(\mathfrak{A})^{k-1}\mathbf{e}(\mathrm{tr}(\nu\mathfrak{z}))$$

$$= C' + 2^g \sum_{0\prec\nu\in\mathfrak{D}\mathfrak{d}_K^{-1}} \sigma_{k-1,\psi}^{\psi'}(\nu;\mathfrak{N}\mathfrak{e}_{\psi'}^{-1}\mathfrak{N}'\mathfrak{e}_{\psi'}^{-1}\mathfrak{D}\mathfrak{d}_K^{-1})\ \mathbf{e}(\mathrm{tr}(\nu\mathfrak{z})).$$

□

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