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§0. Introduction

The purpose of the present paper is to give a central limit theorem for subadditive process in the sense of J. F. C. Kingman (cf. [3], [4]).

Throughout this article $(\Omega, \mathscr{B}, P)$ denotes a probability space on which all random variables are defined. Let T be a measure preserving transformation in what follows. According to Kingman, a family $(x_{s,t};\ s<t,\ s=0, 1,\dots, t=1, 2,\dots)$ of random variables is called a subadditive process, if the following three conditions $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ are satisfied.

S$_1$. $x_{s,u} \leqq x_{s,t} + x_{t,u}$ for all $s<t<u$.

S$_2$. $x_{s+1,t+1}(\omega) = x_{s,t}(T\omega)$ for all $s<t$.

S$_3$. The expectation $E(x_{0,t})$ exists and satisfies

$$E(x_{0,t}) \geqq -At,$$

for all $t \geqq 1$ with some constant A.

In §1 and §2 we show that the random variable

$$t^{-\frac{1}{2}}(x_{0,t} - E(x_{0,t}))$$

has an asymptotically normal distribution under suitable conditions. J. F. C. Kingman and D. L. Burkholder proved the decomposition of

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subadditive processes into additive ones (cf. [3], [4]). Hence we can reduce our problem to the central limit theorem for stationary sequences given in [2].

In §3 we treat an application to products of positive random matrices. H. Furstenberg and H. Kesten have already obtained a central limit theorem for those (cf. [1]). By using the result in §1, we can weaken their conditions on moments and "weak dependence".

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§1. Notations and Results

In order to state our results, we shall introduce some notations. Set $g_t=E(x_{0,t})$ and $\gamma=\lim_{t\to\infty} g_t/t$ (cf. [4], p. 883). Let $\{\mathscr{M}_a^b;\ a\leqq b,\ a=0, 1,\dots, \infty,\ b=0, 1,\dots, \infty\}$ be a family of sub-σ-fields of $\mathscr{B}$ satisfying the following two conditions $\mathbf{P}_1$ and $\mathbf{P}_2$:

$\mathbf{P}_1$. If $a\leqq c\leqq d\leqq b$, then $\mathscr{M}_c^d\subset\mathscr{M}_a^b$.

$\mathbf{P}_2$. For all $a\leqq b$, $T^{-1}\mathscr{M}_a^b=\mathscr{M}_{a+1}^{b+1}$.

We define $\phi(n)$ and $\alpha(n)$ by

$$(1\cdot1)\quad \phi(n)=\sup_{k\geqq 0}\sup\left\{\frac{|P(A\cap B)-P(A)P(B)|}{P(A)};\ A\in\mathscr{M}_0^k,\ B\in\mathscr{M}_{k+n}^\infty,\ P(A)\neq 0\right\},$$

and

$$(1\cdot2)\quad \alpha(n)=\sup_{k\geqq 0}\sup\{|P(A\cap B)-P(A)P(B)|;\ A\in\mathscr{M}_0^k,\ B\in\mathscr{M}_{k+n}^\infty\}.$$

Let us define $\Phi_\sigma(z)$ by

$$\Phi_\sigma(z)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^z \exp\left[-t^2/2\sigma^2\right]dt$$

for $\sigma>0$ and

$$\Phi_0(z)=\begin{cases}1, & (z>0),\\ 0, & (z\leqq 0).\end{cases}$$

The notation $\|*\|_\theta$ $(\theta \geqq 1)$ stands for $[E|*|^\theta]^{1/\theta}$.

We are now in a position to state our results.

Theorem 1. *Let $(x_{s,t})$ be a subadditive process and $\{\mathscr{M}_a^b\}$ be a family of sub-σ-fields of $\mathscr{B}$ satisfying* $\mathbf{P}_1$ *and* $\mathbf{P}_2$. *Suppose that the following four conditions are satisfied:*

(1) $(g_t - t\gamma)/\sqrt{t} \to 0$, *as* $t \to \infty$,

(2) $\sum_{n=1}^{\infty} [\phi(n)]^{\frac{1}{2}} < +\infty$,

(3) *there exists a random variable* Ψ *such that* $E|\Psi|^2 < +\infty$ *and*

$$|x_{0,t} - x_{1,t}| \leqq \Psi \qquad \text{(a.e.)}$$

for all $t \geqq 1$,

(4) $\sum_{n=1}^{\infty} \sup_{t \geqq 1} \|(x_{0,t} - x_{1,t}) - E(x_{0,t} - x_{1,t} | \mathscr{M}_0^n)\|_2 < +\infty$.

Then we have, for some $\sigma \geqq 0$,

$$\lim_{t \to \infty} P\left\{\frac{1}{\sqrt{t}}(x_{0,t} - g_t) < z\right\} = \Phi_\sigma(z)$$

at every continuity point of $\Phi_\sigma(z)$.

If we assume a stronger condition than (3) of Theorem 1, we can weaken the conditions (2) and (4); namely we have the following remarks.

Remark 1. The conclusion of Theorem 1 remains valid if the conditions (2)–(4) of Theorem 1 are replaced by

(2′) $\sum_{n=1}^{\infty} [\alpha(n)]^{\delta/(2+\delta)} < +\infty$ for some $\delta > 0$,

(3′) there exists a random variable Ψ such that $E|\Psi|^{2+\delta} < +\infty$ and

$$|x_{0,t} - x_{1,t}| \leqq \Psi \qquad \text{(a.e.)}$$

for all $t \geqq 1$,

(4′) $\sum_{n=1}^{\infty} \sup_{t\geqq 1} \|(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}|\mathscr{M}_0^n)\|_\theta < +\infty$,

where $\theta=(2+\delta)/(1+\delta)$.

Remark 2. The conclusion of Theorem 1 remains valid if the conditions (2)–(4) of Theorem 1 are replaced by

(2″) $\sum_{n=1}^{\infty} \alpha(n) < +\infty$,

(3″) there exists an essentially bounded random variable Ψ such that

$$|x_{0,t}-x_{1,t}| \leqq \Psi \qquad \text{(a.e.)}$$

for all $t \geqq 1$,

(4″) $\sum_{n=1}^{\infty} \sup_{t\geqq 1} \|(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}|\mathscr{M}_0^n)\|_1 < +\infty$.

§2. Proof of Theorem 1

Our result can be easily obtained by combining the following two theorems. The first theorem is proved by J. F. C. Kingman and D. L. Burkholder (cf. [3], [4]). The second is a central limit theorem for stationary sequences of weakly dependent case, which can be found in [2].

Theorem A. *There is a sequence $n_1<n_2<\cdots$ of positive integers and an integrable random variable y such that*

$$y=\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{n_j} \sum_{t=1}^{n_j} (x_{0,t}-x_{1,t}) \qquad \text{(a.e.)}. \tag{2·1}$$

Moreover we have $Ey=\gamma$ and

$$\sum_{k=s}^{t-1} y(T^k\omega) \leqq x_{s,t}(\omega). \tag{2·2}$$

Theorem B (*Theorem* 18·6·1 *in* [2]). *Let y be a random variable and T be a measure preserving transformation. Let $\{\mathscr{M}_a^b\}$ be a family of sub-σ-fields satisfying* $\mathbf{P}_1$ *and* $\mathbf{P}_2$. *Suppose that*

(1) $\sum_{n=1}^{\infty} [\phi(n)]^{\frac{1}{2}} < +\infty,$

(2) $E|y|^2 < +\infty,$

(3) $\sum_{n=1}^{\infty} \|y - E(y|\mathscr{M}_0^n)\|_2 < +\infty.$

Then there exists a nonnegative constant σ *and we have*

$$\lim_{n\to\infty} P\left\{\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1} y(T^k\omega) - nE(y)\right) < z\right\} = \Phi_\sigma(z)$$

at every continuity point of $\Phi_\sigma(z)$.

Proof of Theorem 1. By virtue of Theorem A, we can get

$$E\left|\frac{1}{\sqrt{t}}(x_{0,t}(\omega) - g_t) - \frac{1}{\sqrt{t}}\left(\sum_{k=0}^{t-1} y(T^k\omega) - t\gamma\right)\right|$$

$$\leqq \frac{1}{\sqrt{t}}(g_t - t\gamma) + E\left|\frac{1}{\sqrt{t}}(x_{0,t}(\omega) - \sum_{k=0}^{t-1} y(T^k\omega))\right| = \frac{2}{\sqrt{t}}(g_t - t\gamma).$$

On the other hand, from the condition (1) of Theorem 1, we have

$$\frac{2}{\sqrt{t}}(g_t - t\gamma) \longrightarrow 0$$

as $t\to\infty$. Hence our theorem can be deduced from the central limit theorem for the stationary sequence $\{y(T^k\omega);\ k=0, 1,\ldots\}$.

The condition (2) of Theorem 1 immediately implies (1) of Theorem B, so we only need to check that y satisfies the conditions (2) and (3) of Theorem B. Using the condition (3) of Theorem 1 and (2·1), we have

$$E|y|^2 \leqq E\left(\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \frac{1}{n_j}\sum_{t=1}^{n_j} |x_{0,t} - x_{1,t}|\right)^2 \leqq E|\Psi|^2 < +\infty,$$

which implies the condition (2) of Theorem B.

Moreover the condition (3) of Theorem 1 guarantees that

$$(2\cdot3) \qquad E(y|\mathscr{M}_0^n) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \frac{1}{n_j}\sum_{t=1}^{n_j} E(x_{0,t} - x_{1,t}|\mathscr{M}_0^n)$$

and

$$(2\cdot4)\qquad \left|\frac{1}{N}\sum_{j=1}^{N}\frac{1}{n_j}\sum_{t=1}^{n_j}\{(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}\,|\,\mathscr{M}_0^n)\}\right|$$

$$\leqq \Psi + E(\Psi\,|\,\mathscr{M}_0^n).$$

Therefore we obtain, from (2·3) and (2·4),

$$\|y-E(y|\mathscr{M}_0^n)\|_2$$

$$=\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{j=1}^{N}\frac{1}{n_j}\sum_{t=1}^{n_j}\{(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}\,|\,\mathscr{M}_0^n)\}\right\|_2$$

$$(2\cdot5)\qquad \leqq\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\frac{1}{n_j}\sum_{t=1}^{n_j}\|(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}\,|\,\mathscr{M}_0^n)\|_2$$

$$\leqq\sup_{t\geqq1}\|(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}|\mathscr{M}_0^n)\|_2.$$

Combining (2·5) and the condition (4) of Theorem 1, we get that the condition (3) of Theorem B holds for y. The proof is therefore complete.

Remark 3. Remarks 1 and 2 in §1 can be deduced from Theorems 18·6·2 and 18·6·3 in [2], instead of Theorem B, respectively. The proof is exactly similar to the proof of Theorem 1.

§3. Application to Products of Random Matrices

Let $Z^1, Z^2,\dots$ be a stationary sequence of random $k\times k$ matrices with strictly positive elements. By the stationarity there is a measure preserving transformation T such that

$$Z^{i+1}(\omega)=Z^i(T\omega)\qquad(i=1,2,\dots).$$

Let us define a family of sub-σ-fields as follows:

$$\mathscr{M}_a^b=\sigma\{Z^{a+1}, Z^{a+2},\dots, Z^{b+1}\},$$

where the notation $\sigma\{*\}$ stands for the σ-field generated by $*$. Clearly, this $\{\mathscr{M}_a^b\}$ satisfies $\mathbf{P}_1$ and $\mathbf{P}_2$ with respect to T. Associated with this family, let $\phi(n)$ and $\alpha(n)$ be the quantities defined by (1·1) and (1·2) respectively. We denote by $(A)_{i,j}$ the (i, j)-th element of a matrix A. We write

$$ {}^{t}Y^{s}=Z^{t}\cdot Z^{t-1}\cdots Z^{s}. $$

Under these notations, we can obtain the following

Theorem 2. *Suppose that* $(Z^{i};\, i=1, 2,\ldots)$ *is a stationary sequence of random matrices and the elements of these matrices are strictly positive. Suppose also that*

(1) *there exists a positive constant* C *such that*

$$ 1\leqq \max_{i,j}(Z^{1})_{i,j}/\min_{i,j}(Z^{1})_{i,j}\leqq C \qquad \text{(a.e.)}, $$

(2) $\sum_{n=1}^{\infty}[\phi(n)]^{\frac{1}{2}}<+\infty,$

and

(3) $E|\log(Z^{1})_{1,1}|^{2}<+\infty.$

Then there is a nonnegative constant σ *such that, for all* $1\leqq i, j\leqq k$,

$$ \lim_{t\to\infty} P\left\{\frac{1}{\sqrt{t}}[\log({}^{t}Y^{1})_{i,j}-E(\log({}^{t}Y^{1})_{i,j})]<z\right\}=\Phi_{\sigma}(z) $$

at every continuity point of $\Phi_{\sigma}(z)$.

Proof. Combining (1) and (3) of Theorem 2, we have

$$ E(|\log(Z^{1})_{i,j}|)<+\infty $$

for all $1\leqq i, j\leqq k$. This fact and the positivity of elements show that the family of random variables

$$ x_{s,t}=-\log({}^{t}Y^{s+1})_{1,1} \tag{3.1} $$

is a subadditive process (cf. [4], pp. 891–892).

First, we prove the asymptotic normality of the random variables

$$ \frac{1}{\sqrt{t}}\{x_{0,t}-E(x_{0,t})\}, $$

using Theorem 1.

(a) Under the condition (1) of Theorem 2, it is already shown in [1] that

$$g_{t+1}-g_t=\gamma+O((1-C^{-3})^t)$$

([1], p. 467), where $g_t=E(x_{0,t})$ and $\gamma=\lim_{t\to\infty} g_t/t$ as in § 1. This fact allows us to deduce the condition (1) of Theorem 1.

(b) The condition (2) of Theorem 2 immediately implies (2) of Theorem 1.

(c) The obvious equality

$$\frac{({}^tY^1)_{1,1}}{({}^tY^2)_{1,1}}=\frac{\sum_{i=1}^{k}\sum_{j=1}^{k}({}^tY^3)_{1,i}(Z^2)_{i,j}(Z^1)_{j,1}}{\sum_{i=1}^{k}({}^tY^3)_{1,i}(Z^2)_{i,1}}$$

and the condition (1) of Theorem 2 show that the following evaluations

$$C^{-1}\sum_{j=1}^{k}(Z^1)_{j,1}\leqq\frac{({}^tY^1)_{1,1}}{({}^tY^2)_{1,1}}\leqq C\sum_{j=1}^{k}(Z^1)_{j,1}.$$

Hence we have

$$|x_{0,t}-x_{1,t}|\leqq|\log(Z^1)_{1,1}|+\log kC^2.$$

This inequality and the condition (3) of Theorem 2 allow us to derive (3) of Theorem 1.

(d) If A, B and C are $k\times k$ matrices and if the notation $(A)_{*,j}$ stands for $\sum_{i=1}^{k}(A)_{i,j}$, then the equality

$$(3\cdot2)\qquad \frac{(ABC)_{1,1}}{(AB)_{1,1}}=\sum_{i=1}^{k}\frac{(B)_{*,i}}{(B)_{*,1}}(C)_{i,1}$$

$$+\frac{\sum_{i=1}^{k}\sum_{j=1}^{k}(A)_{1,j}\left[\frac{(B)_{j,i}}{(B)_{*,i}}-\frac{(B)_{j,1}}{(B)_{*,1}}\right](B)_{*,i}(C)_{i,1}}{\sum_{j=1}^{k}(A)_{1,j}(B)_{j,1}}$$

can be easily shown. Now take $A={}^{m+n}Y^{n+1}$, $B={}^{n}Y^2$ and $C=Z^1$. With these substitutions we have

$$(3\cdot3)\qquad \frac{({}^{m+n}Y^1)_{1,1}}{({}^{m+n}Y^2)_{1,1}}=\sum_{i=1}^{k}\frac{({}^nY^2)_{*,i}}{({}^nY^2)_{*,1}}(Z^1)_{1,1}$$

$$+\frac{\sum_{i=1}^{k}\sum_{j=1}^{k}({}^{m+n}Y^{n+1})_{1,j}\left[\frac{({}^nY^2)_{j,i}}{({}^nY^2)_{*,i}}-\frac{({}^nY^2)_{j,1}}{({}^nY^2)_{*,1}}\right]({}^nY^2)_{*,i}(Z^1)_{i,1}}{\sum_{j=1}^{k}({}^{m+n}Y^{n+1})_{1,j}\,({}^nY^2)_{j,1}}.$$

We denote by α the first term in the right hand side of $(3\cdot3)$ and by β the second term.

The obvious identity

$$\frac{({}^{m+n}Y^m)_{i_1,j_1}}{({}^{m+n}Y^m)_{i_2,j_2}}=\frac{\sum_{r,s}(Z^{m+n})_{i_1,r}({}^{m+n-1}Y^{m+1})_{r,s}(Z^m)_{s,j_1}}{\sum_{r,s}(Z^{m+n})_{i_2,r}({}^{m+n-1}Y^{m+1})_{r,s}(Z^m)_{s,j_2}}$$

and the condition (1) of Theorem 2 allow us to derive

$$(3\cdot4)\qquad C^{-2}\leqq({}^{m+n}Y^m)_{i_1,j_1}/({}^{m+n}Y^m)_{i_2,j_2}\leqq C^2,$$

as was pointed out in [1] (cf. [1], Lemma 2). From $(3\cdot4)$ it clearly follows that

$$C^{-2}\leqq({}^nY^2)_{*,i}/({}^nY^2)_{*,1}\leqq C^2.$$

Therefore, using the condition (1) of Theorem 2, we have

$$(3\cdot5)\qquad kC^{-3}(Z^1)_{1,1}\leqq\alpha\leqq kC^3(Z^1)_{1,1}.$$

To deal with β, we derive

$$(3\cdot6)\qquad \left|\frac{({}^nY^2)_{j,i}}{({}^nY^2)_{*,i}}-\frac{({}^nY^2)_{j,1}}{({}^nY^2)_{*,1}}\right|\leqq(1-C^{-3})^{n-2}$$

from $(3\cdot4)$ and the condition (1) of Theorem 2. The proof of $(3\cdot6)$ is all the same as the proof of Lemma 3 in [1], p. 463, so we omit it. Inequality $(3\cdot4)$ guarantees that

$$(3\cdot7)\qquad \max_i({}^nY^2)_{*,i}/\min_j({}^nY^2)_{j,1}\leqq kC^2.$$

Combining $(3\cdot6)$ and $(3\cdot7)$ allows us to deduce that

$$(3\cdot8)\qquad \beta\leqq kC^2(1-C^{-3})^{n-2}\sum_{i=1}^{k}(Z^1)_{i,1}\leqq k^2C^3(1-C^{-3})^{n-2}(Z^1)_{1,1}.$$

Using (3·5) and (3·8), we can get

$$\beta/\alpha \leqq kC^6(1-C^{-3})^{n-2}.$$

Hence, from the identity

$$\log(\alpha+\beta)=\log\alpha+\log(1+(\beta/\alpha))=\log\alpha+O(\beta/\alpha)$$

as $\beta/\alpha\to 0$, we have

$$(3\cdot 9) \qquad x_{0,m+n}-x_{1,m+n}=-\log[({}^{m+n}Y^1)_{1,1}/({}^{m+n}Y^2)_{1,1}]$$
$$=-\log\alpha+O((1-C^{-3})^n)$$

uniformaly in m and ω.

Since $\log\alpha$ is $\mathscr{M}_0^n$-measurable, we can easily obtain

$$|(x_{0,t}-x_{1,t})-E(x_{0,t}-x_{1,t}|\mathscr{M}_0^n)|$$
$$=\begin{cases} 0, & (t\leqq n), \\ O((1-C^{-3})^n), & (t>n), \end{cases}$$

uniformly in t and ω. This shows that the condition (4) of Theorem 1 holds. Therefore we can apply Theorem 1 to the random variables $x_{0,t}$ defined by (3·1). Consequently, we get the result of Theorem 2 for $i=j=1$.

To deal with the other values of (i, j), note first that the inequality

$$|\log({}^tY^1)_{i,j}-\log({}^tY^1)_{1,1}|\leqq 2\log C$$

follows from (3·4), as was pointed out in [1]. Therefore random variables

$$\frac{1}{\sqrt{t}}\{\log({}^tY^1)_{1,1}-E(\log({}^tY^1)_{1,1})\}$$

and

$$\frac{1}{\sqrt{t}}\{\log({}^tY^1)_{i,j}-E(\log({}^tY^1)_{i,j})\}, \qquad (1\leqq i, j\leqq k),$$

have an asymptotically same distribution. This completes the proof.

By using Remarks 1 and 2 in §1 instead of Theorem 1, we can get the following Remarks 4 and 5 respectively. Their proofs can be copied from the previous one.

Remark 4. The conclusion of Theorem 2 remains valid if the conditions (2) and (3) of Theorem 2 are replaced by

(2′) $\sum_{n=1}^{\infty}[\alpha(n)]^{\frac{\delta}{2+\delta}}<+\infty$ for some $\delta>0$,

(3′) $E|\log(Z^1)_{1,1}|^{2+\delta}<+\infty$.

Remark 5. The conclusion of Theorem 2 remains valid again if the conditions (2) and (3) of Theorem 2 are replaced by

(2″) $\sum_{n=1}^{\infty}\alpha(n)<+\infty$,

(3″) $\log(Z^1)_{1,1}$ is an essentially bounded random variable.

Remark 6. It is easy to see that the assumption AII in [1], p. 464, is stronger than our condition (2) of Theorem 2, so that our results cover Furstenberg and Kesten's central limit theorem in [1].

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