Invariant measures for a class of rational transformations and ergodic properties

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Abstract

This paper is concerned with giving explicitly the invariant density for a class of rational transformations from the real line $\mathbb{R}$ into itself. We proved that the invariant density can be written in terms of the fixed point z_0 in $\mathbb{C} \setminus \mathbb{R}$ or in terms of the periodic point z_0 in $\mathbb{C} \setminus \mathbb{R}$ with period 2. The explicit form of the density allows us to obtain the ergodic properties of the transformation R.

1 Introduction and main results

A various kind of 1-dimensional transformations have been found to have absolutely continuous invariant measures ([3]). However, there are not many transformations whose densities are explicitly known. The aim of this article is to prove that a rational transformation $R(x)$ on the real line $\mathbb{R}$, under some assumptions, has an invariant probability density $(1/\pi)\mathrm{Im}\,(1/(x-z_0))$, if there exists $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = z_0$ or with $R(z_0) = \overline{z_0}$. Precisely, we have the following theorems, which we shall prove in the second section by using the factor theorem.

2000 Mathematics Subject Classification: Primary 37A05; Secondary 37A50, 60F05
Keywords: Invariant measure, Rational transformation, Central limit theorem.
Abbreviated title (running head): Invariant measures for rational transformations.
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Theorem 1. *Assume that $R(x) = h(x)/g(x)$ is a rational transformation from $\mathbb{R}$ into itself with the following properties:*

(1) $g(x) = \prod_{k=1}^{n}(x - a_k)$ for some $a_1 < a_2 < \cdots < a_n$.

(2) $h(x)$ is a polynomial with real coefficients, $\deg(h(x)) \le n+1$ and $h(a_k) \ne 0$ for all k.

(3) The restriction R_j of R to the subinterval (a_j, a_{j+1}) is monotonic for each $j = 0, 1, \ldots, n$, where $a_0 = -\infty$ and $a_{n+1} = \infty$.

(4) There exists $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = z_0$, or with $R(z_0) = \overline{z_0}$.

Then

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x - z_0} f(R(x))dx = \int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x - z_0} f(x)dx \tag{1.1}$$

holds for any essentially bounded real-valued function $f(x)$. Hence the probability measure $d\mu = (1/\pi)\mathrm{Im}\,(1/(x - z_0))\,dx$ is invariant under R.

Theorem 1 can be rewritten as the following Theorems 2 and 3.

Theorem 2. *Suppose that for some $\alpha \ge 0, \beta \in \mathbb{R}, b_k > 0$ $(k = 1, \ldots, n)$*

$$R(x) = \alpha x + \beta - \sum_{k=1}^{n} \frac{b_k}{x - a_k}.$$

Suppose also that there exists $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = z_0$. Then

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x - z_0} f(R(x))dx = \int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x - z_0} f(x)dx$$

holds for any essentially bounded function $f(x)$.

Theorem 3. *Suppose that for some $\alpha \le 0, \beta \in \mathbb{R}, b_k < 0$ $(k = 1, \ldots, n)$*

$$R(x) = \alpha x + \beta - \sum_{k=1}^{n} \frac{b_k}{x - a_k}.$$

Suppose also that there exists $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = \overline{z_0}$. Then

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x - z_0} f(R(x))dx = \int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x - z_0} f(x)dx$$

holds for any essentially bounded function $f(x)$.

We will also use this result to study the ergodic properties of (R, μ) on $\mathbb{R}$, where μ is an absolutely continuous probability measure with a density $(1/\pi)\mathrm{Im}\,(1/(x-z_0))$. Note that we clearly have

$$\mathrm{Im}\frac{1}{x-z_0} = \frac{y_0}{(x-x_0)^2+{y_0}^2} = \frac{d}{dx}\arctan\left(\frac{x-x_0}{y_0}\right)$$

for $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C}\setminus\mathbb{R}$. Denote

$$\varphi(x) = \arctan\left(\frac{x-x_0}{y_0}\right).$$

Then we can prove that the transformation $T(t) := \varphi(R(\varphi^{-1}(t)))$ on $(-\pi/2, \pi/2)$ preserves the normalized Lebesgue measure λ and that (T, λ) is isomorphic to (R, μ) (see Lemma 2.1 in §2). Hence, the above results enable us to get the ergodic properties of the transformation R on $\mathbb{R}$ from those of T on $(-\pi/2, \pi/2)$.

As in Lemma 2.1 in §2, it is also clear that T is piecewise monotonic. The piecewise monotonic transformations on the finite interval have been widely investigated by many authors. In particular, if the piecewise monotonic transformations on the finite interval are uniformly expansive, then it has been shown that they have good ergodic properties ([4],[5],[6]). In Lemma 2.1 we give the relation (2.15)

$$T^{'}(t) = \frac{|x-z_0|^2}{|R(x)-z_0|^2}R^{'}(x).$$

Consequently, combining the relation (2.15) with the known results, we can easily prove the following Theorem 4, where $N(0,\sigma^2)(y)$ $(\sigma^2 > 0)$ stands for the distribution function of Gaussian measure with mean 0 and variance σ^2 and $N(0,0)(y)$ stands for that of Dirac measure. Examples that satisfy the assumptions of Theorem 4 will be found in Section 3.

Theorem 4. *(1) Suppose that $R(x)$ satisfies the assumptions in Theorem 1. Suppose also that the inequality*

$$\inf_{x\notin\{a_1,a_2,\ldots,a_n\}}\left|\frac{|x-z_0|^2}{|R(x)-z_0|^2}R^{'}(x)\right| > 1 \tag{1.2}$$

holds. Then for all μ-integrable functions f the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(R^k x) =: f^*(x) \tag{1.3}$$

exists μ-a.e. and the set $\{f^*(x) : x \in \mathbb{R}\}$ consists of M points for some $M \in \mathbb{N}$.

(2) Moreover, if we assume further that $f(x)$ is a function of bounded variation on $\mathbb{R}$ and that ν is a probability measure on $\mathbb{R}$ with a density $d\nu/d\mu$ with respect to μ, then there exist $c_i \geq 0$ (i=1,2,...,M) with $\sum_{i=0}^{M} c_i = 1$ and ${\sigma_i}^2 \geq 0$ (i=1,2,...,M) for which

$$\lim_{n\to\infty} \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(R^k x) - f^*(x)) \leq y \right\} = \sum_{i=1}^{M} c_i N(0, {\sigma_i}^2)(y) \tag{1.4}$$

holds for all continuity points of the right hand side. If we assume further that ${\sigma_i}^2 > 0$ $(i = 1, 2, \ldots, M)$ and that $(1 + x^2)(d\nu/dx)$ is of bounded variation, then we have

$$\sup_{y\in\mathbb{R}} \left| \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(R^k x) - f^*(x)) \leq y \right\} - \sum_{i=1}^{M} c_i N(0, {\sigma_i}^2)(y) \right| \leq \frac{C}{\sqrt{n}} \tag{1.5}$$

for some $C > 0$.

(3) If $R(x)$ satisfies the above assumptions and if $\deg(h(x)) = n + 1$, then we have that (R, μ) is exact and $M = 1$. Hence, the central limit theorem holds for the transformation R: if $f(x)$ and ν satisfy the assumptions in (2), then the limit

$$\lim_{n\to\infty} \frac{1}{n} \int \left\{ \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \right\}^2 d\mu =: \sigma^2 \tag{1.6}$$

exists and

$$\lim_{n\to\infty} \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \leq y \right\} = N(0, \sigma^2)(y) \tag{1.7}$$

holds for all continuity points of $N(0, \sigma^2)(y)$. If we assume further that $\sigma^2 > 0$ and that $(1 + x^2)(d\nu/dx)$ is of bounded variation, then there exists a constant $C > 0$ such that

$$\sup_{y\in\mathbb{R}} \left| \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \leq y \right\} - N(0, \sigma^2)(y) \right| \leq \frac{C}{\sqrt{n}}. \tag{1.8}$$

holds for all $n \in \mathbb{N}$.

2 Proofs

In this section we prove Theorems 1, 2, 3 and 4. First we show that Theorem 1 is derived from Theorems 2 and 3.

2.1 Proof of Theorem 1

Because $g(x) = \prod_{k=1}^{n}(x - a_k)$ for some $a_1 < a_2 < \cdots < a_n$, $\deg(h(x)) \leq n+1$ and $h(a_k) \neq 0$ for all k, the rational function $R(x) = h(x)/g(x)$ can be rewritten as

$$R(x) = \alpha x + \beta - \sum_{k=1}^{n} \frac{b_k}{x - a_k}\,. \tag{2.1}$$

This shows that

$$\lim_{x \to a_k} |R(x)| = \infty$$

and

$$\lim_{x \uparrow a_k} R(x) = - \lim_{x \downarrow a_k} R(x)$$

for all $k = 1, 2, \ldots, n$. These properties and the assumption (3) in Theorem 1 imply that the restriction $R_j := R|_{(a_j, a_{j+1})}$ is increasing for all $j = 0, 1, \ldots, n$, or decreasing for all $j = 0, 1, \ldots, n$. Hence we have

$$\alpha \geq 0,\ b_k > 0 \ \ (k = 1, \ldots, n), \tag{2.2}$$

or

$$\alpha \leq 0,\ b_k < 0 \ \ (k = 1, \ldots, n). \tag{2.3}$$

Denote $\mathbb{C}_+ = \{\, z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0 \,\}$ and $\mathbb{C}_- = \{\, z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0 \,\}$. If the inequalities (2.2) are satisfied, then it is easy to see that $R(\mathbb{C}_+) \subset \mathbb{C}_+$ and $R(\mathbb{C}_-) \subset \mathbb{C}_-$. And hence $R(z_0) = \overline{z_0}$ implies $z_0 \in \mathbb{R}$. Similarly, the inequalities (2.3) show that $R(\mathbb{C}_+) \subset \mathbb{C}_-$ and $R(\mathbb{C}_-) \subset \mathbb{C}_+$ and that $R(z_0) = z_0$ implies $z_0 \in \mathbb{R}$. The above arguments ensure us to get Theorem 1 from combining Theorems 2 and 3.

2.2 Proof of Theorem 2

First we prove Theorem 2 in the case of $\alpha > 0, b_k > 0$ $(k = 1, \ldots, n)$. In this case it is also easy to see that R_j is increasing and $R_j((a_j, a_{j+1})) = (-\infty, \infty)$ for all $j = 0, 1, \ldots, n$. Hence there exist inverse functions R_j^{-1} such that $R(R_j^{-1}(y)) = y$ holds for all $j = 0, 1, \ldots, n$ and for all $y \in \mathbb{R}$. The equations $R(R_j^{-1}(y)) = y$ are rewritten as

$$y g(R_j^{-1}(y)) - h(R_j^{-1}(y)) = 0 \tag{2.4}$$

for all $j = 0, 1, \ldots, n$ and for all $y \in \mathbb{R}$. Note that in this case $yg(x) - h(x)$ is a polynomial in x of degree $n+1$. The factor theorem shows that

$$y g(x) - h(x) = -\alpha \prod_{j=0}^{n} (x - R_j^{-1}(y)) \tag{2.5}$$

holds for all $y \in \mathbb{R}$. Differentiating the equation (2.5) with respect to y, we get

$$g(x) = \alpha \sum_{i=0}^{n} (R_i^{-1})'(y) \prod_{j \neq i} (x - R_j^{-1}(y)) .$$

Dividing this by (2.5), we have

$$\frac{g(x)}{yg(x) - h(x)} = -\sum_{i=0}^{n} \frac{(R_i^{-1})'(y)}{x - R_i^{-1}(y)} .$$

Put $x = z_0$, and we get

$$\frac{g(z_0)}{yg(z_0) - h(z_0)} = \sum_{i=0}^{n} \frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0} . \tag{2.6}$$

Because $h(z_0) = z_0 g(z_0)$, the left hand side of (2.6) is equal to $1/(y - z_0)$. Thus we obtain the key equation

$$\mathrm{Im} \frac{1}{y - z_0} = \sum_{i=0}^{n} \mathrm{Im} \frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0} . \tag{2.7}$$

Note that the function $\mathrm{Im}(1/(x - z_0))$ is essentially bounded and integrable on $\mathbb{R}$, since z_0 is not a real number.

Now we have

$$\int_{-\infty}^{\infty} \mathrm{Im} \frac{1}{x - z_0} f(R(x)) dx = \sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} \mathrm{Im} \frac{1}{x - z_0} f(R(x)) dx$$

for any essentially bounded function $f(x)$. Since $R_i(a_i + 0) = -\infty$ and $R_i(a_{i+1} - 0) = \infty$, we get

$$\sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} \mathrm{Im} \frac{1}{x - z_0} f(R(x)) dx = \int_{-\infty}^{\infty} \sum_{i=0}^{n} \mathrm{Im} \frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0} f(y) dy . \tag{2.8}$$

The key equation (2.7) enables us to obtain

$$\int_{-\infty}^{\infty} \sum_{i=0}^{n} \mathrm{Im} \frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0} f(y) dy = \int_{-\infty}^{\infty} \mathrm{Im} \frac{1}{y - z_0} f(y) dy, \tag{2.9}$$

which completes the proof of Theorem 2 in the case $\alpha > 0$, $b_k > 0$ $(k = 1, \dots, n)$.

If $\alpha = 0$, $b_k > 0$ $(k = 1, \dots, n)$, the same result can be proved by modifying the above argument. In this case $R_0((a_0, a_1)) = (\beta, \infty)$ and $R_n((a_n, a_{n+1})) = (-\infty, \beta)$, however we have $R_j((a_j, a_{j+1})) = (-\infty, \infty)$ for all $j = 1, 2, \dots, n-1$ as before. If $y \in (\beta, \infty)$,the equation (2.4)

$$yg(R_j^{-1}(y)) - h(R_j^{-1}(y)) = 0$$

holds for any $j = 0, 1, \ldots, n-1$. On the other hand, if $y \in (-\infty, \beta)$, the equation (2.4) holds for any $j = 1, 2, \ldots, n$. Note that $yg(x) - h(x)$ is a polynomial in x of degree n, since $\alpha = 0$. Instead of (2.5) we have

$$yg(x) - h(x) = \begin{cases} (y-\beta)\prod_{j=0}^{n-1}(x - R_j^{-1}(y)), & y > \beta \\ (y-\beta)\prod_{j=1}^{n}(x - R_j^{-1}(y)), & y < \beta\,. \end{cases} \tag{2.10}$$

Differentiate the equation (2.10) with respect to y. Then we get the equation

$$g(x) = \begin{cases} \prod_{j=0}^{n-1}(x - R_j^{-1}(y)) + (\beta - y)\sum_{i=0}^{n-1}(R_i^{-1})'(y)\prod_{j\neq i}(x - R_j^{-1}(y)), & y > \beta \\ \prod_{j=1}^{n}(x - R_j^{-1}(y)) + (\beta - y)\sum_{i=1}^{n}(R_i^{-1})'(y)\prod_{j\neq i}(x - R_j^{-1}(y)), & y < \beta\,. \end{cases}$$

As before, divide this by (2.10). Then we have

$$\frac{g(x)}{yg(x) - h(x)} = \begin{cases} 1/(y-\beta) + \sum_{i=0}^{n-1}(R_i^{-1})'(y)/\left(R_i^{-1}(y) - x\right), & y > \beta \\ 1/(y-\beta) + \sum_{i=1}^{n}(R_i^{-1})'(y)/\left(R_i^{-1}(y) - x\right), & y < \beta. \end{cases}$$

Putting $x = z_0$, we obtain, as before,

$$\frac{1}{y - z_0} = \begin{cases} 1/(y-\beta) + \sum_{i=0}^{n-1}(R_i^{-1})'(y)/(R_i^{-1}(y) - z_0), & y > \beta \\ 1/(y-\beta) + \sum_{i=1}^{n}(R_i^{-1})'(y)/(R_i^{-1}(y) - z_0), & y < \beta. \end{cases}$$

Therefore, the equation, corresponding to (2.7),

$$\mathrm{Im}\frac{1}{y - z_0} = \begin{cases} \sum_{i=0}^{n-1}\mathrm{Im}\left((R_i^{-1})'(y)/\left(R_i^{-1}(y) - z_0\right)\right), & y > \beta \\ \sum_{i=1}^{n}\mathrm{Im}\left((R_i^{-1})'(y)/\left(R_i^{-1}(y) - z_0\right)\right), & y < \beta \end{cases} \tag{2.11}$$

has been proved.

We have

$$\int_{-\infty}^{\infty}\mathrm{Im}\frac{1}{x - z_0}f(R(x))dx = \sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}\mathrm{Im}\frac{1}{x - z_0}f(R(x))dx$$

for any essentially bounded function $f(x)$. Remark that in this case $R_0(-\infty) = \beta$, $R_0(a_1 - 0) = \infty$, $R_n(a_n + 0) = -\infty$, and $R_n(\infty) = \beta$, however $R_i(a_i + 0) = -\infty$ and $R_i(a_{i+1} - 0) = \infty$ for $i = 1, 2, \ldots, n-1$. Then we have

$$\begin{aligned} &\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}\mathrm{Im}\frac{1}{x - z_0}f(R(x))dx \\ &= \sum_{i=0}^{n-1}\int_{\beta}^{\infty}\mathrm{Im}\frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0}f(y)dy + \sum_{i=1}^{n}\int_{-\infty}^{\beta}\mathrm{Im}\frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0}f(y)dy. \end{aligned}$$

The equation (2.11) shows that the right hand side is rewritten as

$$\int_{\beta}^{\infty}\mathrm{Im}\frac{1}{y - z_0}f(y)dy + \int_{-\infty}^{\beta}\mathrm{Im}\frac{1}{y - z_0}f(y)dy = \int_{-\infty}^{\infty}\mathrm{Im}\frac{1}{y - z_0}f(y)dy.$$

This enables us to have the result in question

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx = \int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x-z_0} f(x)dx.$$

2.3 Proof of Theorem 3

The proof of Theorem 3 is similar to that of Theorem 2. Hence, we sketch only the difference.

First we consider the case $\alpha < 0,\, b_k < 0$ $(k = 1, \ldots, n)$. As in the first case of the proof of Theorem 2, however R_j is decreasing, $R_j((a_j, a_{j+1})) = (-\infty, \infty)$ for all $j = 0, 1, \ldots, n$ and $R(R_j^{-1}(y)) = y$ holds for all $j = 0, 1, \ldots, n$ and all $y \in \mathbb{R}$. Therefore, we also have (2.4)

$$yg(R_j^{-1}(y)) - h(R_j^{-1}(y)) = 0$$

for all $j = 0, 1, \ldots, n$ and all $y \in \mathbb{R}$. The same argument as the first half of the proof of Theorem 2 allows us to have the equation (2.6)

$$\frac{g(z_0)}{yg(z_0) - h(z_0)} = \sum_{i=0}^{n} \frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0}.$$

holds for all $y \in \mathbb{R}$. However, we have $h(z_0) = \overline{z_0} g(z_0)$ in this case. Therefore, the left hand side of (2.6) is now equal to $1/(y - \overline{z_0})$. Remarking that

$$\mathrm{Im}\frac{1}{y - \overline{z_0}} = -\mathrm{Im}\frac{1}{y - z_0},$$

we obtain the analogous equation to (2.7) ,

$$\mathrm{Im}\frac{1}{y - z_0} = \sum_{i=0}^{n} \mathrm{Im}\frac{-(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0}. \tag{2.12}$$

Now we have

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx = \sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx$$

for any essentially bounded function $f(x)$. Since $R_i(a_i + 0) = \infty$ and $R_i(a_{i+1} - 0) = -\infty$, we get the equation, corresponding to (2.8),

$$\sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx = \int_{\infty}^{-\infty} \sum_{i=0}^{n} \mathrm{Im}\frac{(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0} f(y)dy.$$

The key equation (2.12) enables us to obtain

$$\int_{-\infty}^{\infty} \sum_{i=0}^{n} \mathrm{Im} \frac{-(R_i^{-1})'(y)}{R_i^{-1}(y) - z_0} f(y)dy = \int_{-\infty}^{\infty} \mathrm{Im} \frac{1}{y - z_0} f(y)dy,$$

which completes the proof of Theorem 3 in the case $\alpha < 0$, $b_k < 0$ $(k = 1, \ldots, n)$.

Second, we consider the case of $\alpha = 0$, $b_k < 0$ $(k = 1, \ldots, n)$. As in the second case in the proof of Theorem 2, $R_0((a_0, a_1)) = (-\infty, \beta)$ and $R_n((a_n, a_{n+1})) = (\beta, \infty)$, however we have $R_j((a_j, a_{j+1})) = (-\infty, \infty)$ for all $j = 1, 2, \ldots, n-1$. When $y \in (\beta, \infty)$, the equation (2.4)

$$yg(R_j^{-1}(y)) - h(R_j^{-1}(y)) = 0$$

holds for any $j = 1, 2, \ldots, n$. On the other hand, if $y \in (-\infty, \beta)$, the equation (2.4) holds for any $j = 0, 1, \ldots, n-1$. Note that the polynomial $yg(x) - h(x)$ in x is of degree n, since $\alpha = 0$. Instead of (2.10) we have

$$yg(x) - h(x) = \begin{cases} (y - \beta) \prod_{j=1}^{n} (x - R_j^{-1}(y)), & y > \beta \\ (y - \beta) \prod_{j=0}^{n-1} (x - R_j^{-1}(y)), & y < \beta\,. \end{cases} \tag{2.13}$$

This shows

$$g(x) = \begin{cases} \prod_{j=1}^{n} (x - R_j^{-1}(y)) + (\beta - y) \sum_{i=1}^{n} (R_j^{-1})'(y) \prod_{j \neq i} (x - R_j^{-1}(y)), & y > \beta \\ \prod_{j=0}^{n-1} (x - R_j^{-1}(y)) + (\beta - y) \sum_{i=0}^{n-1} (R_j^{-1})'(y) \prod_{j \neq i} (x - R_j^{-1}(y)), & y < \beta\,. \end{cases}$$

As before, divide this by (2.13). Then we have

$$\frac{g(x)}{yg(x) - h(x)} = \begin{cases} 1/(y - \beta) + \sum_{i=1}^{n} (R_i^{-1})'(y) / \left(R_i^{-1}(y) - x\right), & y > \beta \\ 1/(y - \beta) + \sum_{i=0}^{n-1} (R_i^{-1})'(y) / \left(R_i^{-1}(y) - x\right), & y < \beta. \end{cases}$$

Put $x = z_0$ and note that $h(z_0) = \overline{z_0} g(z_0)$.

Then we obtain, as before,

$$\frac{1}{y - \overline{z_0}} = \begin{cases} 1/(y - \beta) + \sum_{i=1}^{n} (R_i^{-1})'(y) / (R_i^{-1}(y) - z_0), & y > \beta \\ 1/(y - \beta) + \sum_{i=0}^{n-1} (R_i^{-1})'(y) / (R_i^{-1}(y) - z_0), & y < \beta. \end{cases}$$

Recall that

$$\mathrm{Im} \frac{1}{y - \overline{z_0}} = -\mathrm{Im} \frac{1}{y - z_0}\,.$$

Therefore, the equation, corresponding to (2.11),

$$\mathrm{Im} \frac{1}{y - z_0} = \begin{cases} \sum_{i=1}^{n} \mathrm{Im} \left((R_i^{-1})'(y) / \left(z_0 - R_i^{-1}(y)\right)\right), & y > \beta \\ \sum_{i=0}^{n-1} \mathrm{Im} \left((R_i^{-1})'(y) / \left(z_0 - R_i^{-1}(y)\right)\right), & y < \beta \end{cases} \tag{2.14}$$

has been proved.

As before we have

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx = \sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx$$

for any essentially bounded function $f(x)$. Remark that in this case $R_0(-\infty) = \beta$, $R_0(a_1-0) = -\infty$, however $R_i(a_i+0) = \infty$ and $R_i(a_{i+1}-0) = -\infty$ for $i = 1, 2, \ldots, n-1$, $R_n(a_n+0) = \infty$, and $R_n(\infty) = \beta$. Then the equation (2.14) ensures us to have

$$\begin{aligned}
&\sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx \\
= &\sum_{i=1}^{n} \int_{\infty}^{\beta} \mathrm{Im}\frac{(R_i^{-1})'(y)}{R_i^{-1}(y)-z_0} f(y)dy + \sum_{i=0}^{n-1} \int_{\beta}^{-\infty} \mathrm{Im}\frac{(R_i^{-1})'(y)}{R_i^{-1}(y)-z_0} f(y)dy \\
= &\int_{\beta}^{\infty} \mathrm{Im}\frac{1}{y-z_0} f(y)dy + \int_{-\infty}^{\beta} \mathrm{Im}\frac{1}{y-z_0} f(y)dy \\
= &\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{y-z_0} f(y)dy.
\end{aligned}$$

This shows the result in question

$$\int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x-z_0} f(R(x))dx = \int_{-\infty}^{\infty} \mathrm{Im}\frac{1}{x-z_0} f(x)dx.$$

2.4 Proof of Theorem 4

Recall that $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C}\backslash\mathbb{R}$ satisfies the relation $R(z_0) = z_0$ or $R(z_0) = \overline{z_0}$. Theorem 1 shows that $d\mu := (1/\pi)\varphi'(x)dx$ is an invariant probability for the transformation R where $\varphi(x) := \arctan\{(x-x_0)/y_0\}$. Define the transformation T on the interval $(-\pi/2, \pi/2)$ by $T(t) := \varphi(R(\varphi^{-1}(t)))$. Then we can get the following Lemma, which is a key to the proof of Theorem 4.

Lemma 2.1. *Assume that the conditions on R in Theorem 1 are satisfied. Then (R, μ) is measure theoretically isomorphic to (T, λ), where λ denotes the normalized Lebesgue measure on the interval $(-\pi/2, \pi/2)$. Moreover, T has the following properties:*

(1) T preserves the normalized Lebesgue measure λ.

(2) The restrictions $T|_{(\varphi(a_i),\varphi(a_{i+1}))}$ (i=0,1,. . . ,n) are monotonic.

(3) $T|_{(\varphi(a_i),\varphi(a_{i+1}))}$ (i=0,1,. . . ,n) are smooth and

$$T'(t) = \frac{|x-z_0|^2}{|R(x)-z_0|^2} R'(x) \tag{2.15}$$

holds for all $t \notin \{\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\}$, where $x = \varphi^{-1}(t)$.

Proof. Recall that we have

$$\mathrm{Im}\frac{1}{x-z_0} = \frac{y_0}{(x-x_0)^2+{y_0}^2} = \frac{d}{dx}\arctan\left(\frac{x-x_0}{y_0}\right) = \varphi^{'}(x)\,.$$

This is followed by

$$\lambda(A) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I_A(t)dt = \frac{1}{\pi}\int_{-\infty}^{\infty} I_A(\varphi(x))\varphi^{'}(x)dx = \mu(\varphi^{-1}(A)).$$

Hence, we have that $\lambda(A) = \mu(\varphi^{-1}(A))$ and (R,μ) is measure theoretically isomorphic to (T,λ). This immediately shows the assertion (1), since R preserves μ.

Because $R|_{(a_i,a_{i+1})}$ $(i = 0,1,\ldots,n)$ are monotonic and φ is increasing, $T|_{(\varphi(a_i),\varphi(a_{i+1}))}$ $(i = 0,1,\ldots,n)$ are also monotonic. Recall that $\varphi(x) := \arctan\{(x-x_0)/y_0\}$ and hence $\varphi^{-1}(t) = x_0 + y_0\tan t$. Then we easily have

$$\begin{aligned} T^{'}(t) &= \varphi^{'}(R(\varphi^{-1}(t)))R^{'}(\varphi^{-1}(t))(\varphi^{-1})^{'}(t) \\ &= \frac{y_0}{(R(\varphi^{-1}(t))-x_0)^2+{y_0}^2}R^{'}(\varphi^{-1}(t))y_0(1+\tan^2 t) \\ &= \frac{(x-x_0)^2+{y_0}^2}{(R(x)-x_0)^2+{y_0}^2}R^{'}(x) \\ &= \frac{|x-z_0|^2}{|R(x)-z_0|^2}R^{'}(x), \end{aligned}$$

where $x = \varphi^{-1}(t)$. This completes the proof. □

Lemma 2.1 shows that the dynamical system (R,μ) on the real line $\mathbb{R}$ is isomorphic to (T,λ) on the finite interval $(-\pi/2,\pi/2)$ and that (T,λ) is piecewise smooth and piecewise monotonic. The relation (2.15) implies that if the assumption (1.2) is satisfied, then the transformation T is piecewise expanding and smooth enough.

On the other hand it is already known that such T has a finite number of absolutely continuous ergodic invariant measures $\lambda_1,\lambda_2,\ldots,\lambda_M$ and the other absolutely continuous invariant measures are convex combinations of them (cf. [4], [5] and [9]). Birkhoff's ergodic theorem shows that if $\tilde{f} \in L^1(\lambda_i)$ then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\tilde{f}(T^k t) = \int \tilde{f}d\lambda_i \qquad (\lambda_i \text{ a.e.})$$

holds. Note that the supports of ergodic measures are mutually disjoint and that the normalized Lebesgue measure λ is also a convex combination of $\lambda_1,\lambda_2,\ldots,\lambda_M$.

Hence if $\tilde{f}$ is a λ-integrable function, then $\tilde{f}$ is λ_i-integrable for all $i = 1,2,\ldots,M$. This observation shows that for a λ-integrable function $\tilde{f}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\tilde{f}(T^k t) = \tilde{f}^*(t) \qquad (\lambda \text{ a.e.}) \tag{2.16}$$

holds and $\tilde{f}^*(t) = \int \tilde{f} d\lambda_i$ for λ a.e.$t \in \text{supp}\,\{\lambda_i\}$ $(i = 1, 2, \ldots, M)$. Now, let f be a μ-integrable function on $\mathbb{R}$. Then $f \circ \varphi^{-1}$ is λ-integrable function on the interval $(-\pi/2, \pi/2)$. Hence, replacing $\tilde{f}$ by $f \circ \varphi^{-1}$ and t by $\varphi(x)$ in (2.16), we can get the relation (1.3). This shows the first part of Theorem 4.

In order to prove the second part, we apply Theorem 1 in [6] to the transformation in question (see also [5]). Hence, if $\tilde{f}$ is a function of bounded variation defined on the interval $(-\pi/2, \pi/2)$ and if $\tilde{\nu}$ is an absolutely continuous probability measure, then there exist nonnegative constants $c_1, c_2, \ldots, c_M$ with $\sum_{i=1}^{M} c_i = 1$ and ${\sigma_i}^2 \geq 0$ $(i = 1, 2, \ldots, M)$ for which

$$\lim_{n\to\infty} \tilde{\nu} \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\tilde{f}(T^k t) - \tilde{f}^*(t)) \leq y \right\} = \sum_{i=1}^{M} c_i N(0, {\sigma_i}^2)(y) \tag{2.17}$$

holds for all continuity points of the right hand side. If we assume further that ${\sigma_i}^2 > 0$ for all $i = 1, 2, \ldots, M$, and that $d\tilde{\nu}/d\lambda$ is of bounded variation, then

$$\sup_{y\in\mathbb{R}} \left| \tilde{\nu} \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\tilde{f}(T^k t) - \tilde{f}^*(t)) \leq y \right\} - \sum_{i=1}^{M} c_i N(0, {\sigma_i}^2)(y) \right| \leq \frac{C}{\sqrt{n}} \tag{2.18}$$

for some $C > 0$.

Let $f(x)$ be a function of bounded variation on $\mathbb{R}$. Then $\tilde{f}(t) := (f \circ \varphi^{-1})(t)$ is also a function of bounded variation, because $\varphi^{-1}(t)$ is strictly increasing. Suppose that ν is a probability measure on $\mathbb{R}$ which is absolutely continuous with respect to μ. Then it is clear that the probability measure $\tilde{\nu}(A) := \nu(\varphi^{-1}A)$ is absolutely continuous. Note that we have

$$\begin{aligned} &\tilde{\nu} \left\{ t \in (-\pi/2, \pi/2)\,;\ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\tilde{f}(T^k t) - \tilde{f}^*(t) \right) \leq y \right\} \\ &= \left(\nu \circ \varphi^{-1}\right) \left\{ t \in (-\pi/2, \pi/2)\,;\ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\tilde{f}(T^k t) - \tilde{f}^*(t) \right) \leq y \right\} \\ &= \nu \left\{ x \in \mathbb{R}\,;\ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\left(f \circ \varphi^{-1}\right) \left(T^k \varphi(x)\right) - \left(f \circ \varphi^{-1}\right)^* \left(\varphi(x)\right) \right) \leq y \right\} \\ &= \nu \left\{ x \in \mathbb{R}\,;\ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(f(R^k x) - f^*(x) \right) \leq y \right\}. \end{aligned} \tag{2.19}$$

Therefore we get the relation (1.4), combining (2.17) and (2.19).

On the other hand we have

$$\begin{aligned}\tilde{\nu}(A) &= \int_{\varphi^{-1}A} \frac{d\nu}{dx}(x)\,dx \\ &= \int_A \frac{d\nu}{dx}(\varphi^{-1}(t))(\varphi^{-1})'(t)\,dt \\ &= \int_A \frac{d\nu}{dx}(\varphi^{-1}(t))y_0(1+\tan^2 t)\,dt \\ &= \int_A \frac{d\nu}{dx}(\varphi^{-1}(t))\,y_0\left(1+\left(\frac{\varphi^{-1}(t)-x_0}{y_0}\right)^2\right)dt.\end{aligned}$$

This shows that the total variation of $d\tilde{\nu}/d\lambda$ is equal to the one of

$$y_0\left(1+\left(\frac{x-x_0}{y_0}\right)^2\right)\frac{d\nu}{dx}(x).$$

Therefore, if $(x^2+1)(d\nu/dx)$ is of bounded variation, so is $d\tilde{\nu}/d\lambda$. This and (2.18) show the inequality (1.5) of Theorem 4.

In order to prove the last part of Theorem 4 we remark that if $\deg(h(x)) = n+1$, then $\alpha \neq 0$ and hence $R((a_i, a_{i+1})) = (-\infty, \infty)$ for all $i = 0, 1, \ldots, n$. This implies that $T((\varphi(a_i), \varphi(a_{i+1}))) = (-\pi/2, \pi/2)$ for all $i = 0, 1, \ldots, n$. Thus the transformation T from the interval $(-\pi/2, \pi/2)$ into itself is piecewise C^2, piecewise expanding and piecewise onto. Then it follows by the Folklore Theorem (Theorem 6.1.1 in [3]) that such a transformation is exact (see also [1] and [10]). Therefore the number M of absolutely continuous ergodic measures for T is equal to 1. Hence the third part of Theorem 4 is proved.

3 Examples

We consider examples and applications in this section. First we prove the following proposition, which gives a sufficient condition for the existence of $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = z_0$.

Proposition 3.1. *Let*

$$R(x) = \alpha x + \beta - \sum_{k=1}^{n} \frac{b_k}{x - a_k} \tag{3.1}$$

and $0 \leq \alpha < 1$, $b_k > 0\,(k = 1, \ldots, n)$, $a_1 < a_2 < \cdots < a_n$. *Assume further that* $a_1 \leq (\beta/(1-\alpha))$ *and* $a_n \geq (\beta/(1-\alpha))$ *and that*

$$a_{i+1} - a_i < \sqrt{\frac{\{b_i^{1/3} + b_{i+1}^{1/3}\}^3}{1-\alpha}} \tag{3.2}$$

holds for $i = 1, 2, \ldots, n-1$. *Then there exists* $z_0 \in \mathbb{C} \setminus \mathbb{R}$ *with* $R(z_0) = z_0$.

Proof. Putting $\psi(x) = x + (\beta/(1-\alpha))$, we easily have

$$\psi^{-1}(R(\psi(x))) = \alpha x - \sum_{k=1}^{n} \frac{b_k}{x - (a_k - (\beta/(1-\alpha)))}.$$

Remark also that $R(z_0) = z_0$ if and only if

$$\psi^{-1}(R(\psi(z_0 - (\beta/(1-\alpha))))) = z_0 - (\beta/(1-\alpha)).$$

Hence we can assume that $\beta = 0$ without loss of generality.

First we prove that the equation $x = \alpha x - \sum_{k=1}^{n} (b_k/(x - a_k))$ has $n-1$ real solutions. In fact we have for $a_i < x < a_{i+1}$

$$\begin{aligned}
(R(x) - x)' &= \sum_{k=1}^{n} \frac{b_k}{(x - a_k)^2} - (1-\alpha) \\
&\geq \frac{b_i}{(x - a_i)^2} + \frac{b_{i+1}}{(x - a_{i+1})^2} - (1-\alpha) \\
&\geq \frac{\{b_i^{1/3} + b_{i+1}^{1/3}\}^3}{(a_{i+1} - a_i)^2} - (1-\alpha).
\end{aligned}$$

The assumption (3.2) shows that the right hand side is greater than 0. Hence, $G(x) := R(x) - x$ is strictly increasing in (a_i, a_{i+1}). On the other hand, as $b_k > 0\,(k = 1, 2, \ldots, n)$, we clearly have $R(a_k+0) = -\infty$ and $R(a_{k+1}-0) = \infty$. This implies that $G(a_i+0) = -\infty$ and $G(a_{i+1} - 0) = \infty$. Therefore, $R(x) - x = 0$ has a unique real solution in (a_i, a_{i+1}) for each $i = 1, 2, \ldots, n-1$.

The assumptions $a_1 \leq 0$ and $0 \leq \alpha < 1$ ensure us to have $x \leq \alpha x < R(x)$ for all $x \in (-\infty, a_1)$, and hence $R(x) - x > 0$ in $(-\infty, a_1)$. Therefore, $R(x) - x = 0$ has no real solution in $(-\infty, a_1)$. Similarly, we can get that there is no real solution in (a_n, ∞).

From the above arguments we have obtained that the equation $R(x) - x = 0$ has $n-1$ real solutions. On the other hand the equation $R(x) - x = 0$ clearly has $n+1$ solutions. Therefore there is $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = z_0$. This completes the proof. □

Remark 3.1. *The condition (3.2) is the best possible in the following sense: Consider the transformation $R(x) = \alpha x - (x-a)^{-1} - (x+a)^{-1}$, where $0 \leq \alpha < 1$ and $a > 0$. Then it can be easily proved that $R(x)$ has $z_0 \in \mathbb{C} \setminus \mathbb{R}$ with $R(z_0) = z_0$ if and only if the condition (3.2) is satisfied.*

We consider some examples using the above proposition.

Example 1. *Let us consider the transformation $R(x) = \alpha x - bx^{-1}$ with $0 \leq \alpha < 1$ and $b > 0$. Putting $\psi(x) = \sqrt{b}x$, we get $\psi^{-1}(R(\psi(x))) = \alpha x - x^{-1}$. Hence, we can assume $b = 1$ without loss of generality. However this transformation $R(x) = \alpha x - x^{-1}$ satisfies* *the assumptions of Proposition 3.1, we directly get that the fixed point* z_0 *of* R *in* $\mathbb{C}$ *is* $z_0 = \mathrm{i}y_0 = \mathrm{i}\sqrt{1/(1-\alpha)}$ *in this case. Theorem 2 shows that* $d\mu = \pi^{-1}\mathrm{Im}\,(1/(x - \mathrm{i}y_0))\,dx$ *is an invariant probability for the transformation* R.

Let us consider the transformation $T(t) := \varphi(R(\varphi^{-1}(t)))$, *where* $\varphi(x) := \arctan(x/y_0)$. *Using Lemma 2.1 we have*

$$\begin{aligned} T^{'}(t) &= \frac{|x - z_0|^2}{|R(x) - z_0|^2} R^{'}(x) \\ &= \frac{|x - z_0|^2}{|R(x) - R(z_0)|^2} R^{'}(x) \\ &= \frac{|x - z_0|^2}{|\alpha(x - z_0) + (x - z_0)/(xz_0)|^2} R^{'}(x) \\ &= \frac{\alpha + 1/x^2}{|\alpha + (1/(xz_0))|^2} \\ &= \frac{\alpha x^2 + 1}{\alpha^2 x^2 + 1 - \alpha}. \end{aligned} \tag{3.3}$$

For $0 < \alpha \leq 1/2$ *we have the estimation*

$$\frac{\alpha x^2 + 1}{\alpha^2 x^2 + 1 - \alpha} \geq \frac{\alpha x^2 + 1}{\alpha(1-\alpha)x^2 + 1 - \alpha} = \frac{1}{1-\alpha},$$

since $\alpha \leq 1 - \alpha$. *If* $1/2 \leq \alpha < 1$, *then we have*

$$\frac{\alpha x^2 + 1}{\alpha^2 x^2 + 1 - \alpha} \geq \frac{\alpha x^2 + 1}{\alpha^2 x^2 + \alpha} = \frac{1}{\alpha}.$$

Hence, the transformation T *on* $(-\pi/2, \pi/2)$ *is uniformly expansive. Precisely, we have*

$$T^{'}(t) \geq \min\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right) > 1$$

for all $t \neq 0$. *Therefore,* $R(x) = \alpha x - x^{-1}$ $(0 < \alpha < 1)$ *satisfies the assumption of Theorem 4 and the conclusions of Theorem 4 are valid for* (R, μ).

In the case $\alpha = 0$, *we have*

$$T(t) = \begin{cases} t + \pi/2, & (-\pi/2 < x < 0), \\ t - \pi/2, & (0 < x < \pi/2). \end{cases}$$

Put $A = (-\pi/2, -\pi/4) \cup (0, \pi/4)$. *Then we have* $T^{-1}A = A$ *and* $\lambda(A) = 1/2$. *Hence neither* (T, λ) *nor* (R, μ) *is ergodic.*

Note that the relation $R(\mathrm{i}y_0) = \mathrm{i}y_0$ is regarded as $-R(\mathrm{i}y_0) = \overline{\mathrm{i}y_0}$. Thus we get the following example from Example 1.

Example 2. *Let us consider the transformation*

$$R(x) = -\alpha x + \frac{b}{x}$$

with $0 \le \alpha < 1$ *and* $b > 0$*. As is in Example 1, we can also assume* $b = 1$ *without loss of generality. Remark that the transformation* $-R(x)$ *is the one in Example 1. This fact shows that* $R(z_0) = \overline{z_0}$ *holds for* $z_0 = \mathrm{i}y_0 = \mathrm{i}\sqrt{1/(1-\alpha)}$*. Thus, Theorem 3 can be applied and hence*

$$d\mu = \frac{1}{\pi}\mathrm{Im}\frac{1}{x - \mathrm{i}y_0}\,dx$$

is an invariant probability for the transformation R*. The analogous argument allows us to have*

$$-T^{'}(t) \ge \min\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right) > 1,$$

and hence the same results as those of Example 1 hold.

If the number n of poles is more than 2, it is generally not easy to get the desired estimation of $|T^{'}(t)|$. However, there are some examples that satisfy the assumption of Theorem 4.

Example 3. *Let us consider the transformation*

$$R(x) = \alpha x - \frac{1}{x-1} - \frac{1}{x+1}$$

with $0 \le \alpha < 1$*. We can easily get that* $R(\mathrm{i}y_0) = \mathrm{i}y_0$*, where* $y_0 = \sqrt{(1+\alpha)/(1-\alpha)}$*. As in Example 1, we obtain*

$$T^{'}(t) = \frac{\alpha(x^2-1)^2 + 2x^2 + 2}{\alpha^2(x^2-1)^2 + (1-\alpha)^2x^2 + (1-\alpha)(1+\alpha)}. \tag{3.4}$$

The right hand side of (3.4) is not smaller than

$$\frac{\alpha(x^2-1)^2 + 2x^2 + 2}{(1-\alpha)\left\{\alpha(x^2-1)^2 + (1-\alpha)x^2 + 1 + \alpha\right\}} \ge \frac{1}{1-\alpha}$$

for $0 < \alpha \le 1 - \alpha$*. For* $0 < 1 - \alpha \le \alpha$ *the right hand side of (3.4) is greater than or equal to*

$$\frac{\alpha(x^2-1)^2 + 2x^2 + 2}{\alpha\left\{\alpha(x^2-1)^2 + (1-\alpha)x^2 + 1 + \alpha\right\}} \ge \frac{1}{\alpha}.$$

Therefore, if $0 < \alpha < 1$*, we also get the inequality*

$$T^{'}(t) \ge \min\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right) > 1$$

for all $t \notin \{-\pi/2, \varphi(-1), \varphi(1), \pi/2\}$*. If* $\alpha = 0$*, then it is clear that the right hand side of (3.4) is equal to 2 and*

$$T(t) = \begin{cases} 2t + \pi & (-\pi/2 < t < -\pi/4), \\ 2t & (-\pi/4 < t < \pi/4), \\ 2t - \pi & (\pi/4 < t < \pi/2). \end{cases}$$

Consequently, Theorem 4 can be also applied for these transformations.

Example 4. *Consider the transformations*

$$R(x) = -\alpha x + \frac{1}{x-1} + \frac{1}{x+1} \tag{3.5}$$

with $0 \le \alpha < 1$*. Because the transformations* $-R(x)$ *are those in Example 3 and have the fixed point* $\mathrm{i}y_0$ $(y_0 \neq 0)$*, we have* $R(\mathrm{i}y_0) = \overline{\mathrm{i}y_0}$*, where* $y_0 = \sqrt{(1+\alpha)/(1-\alpha)}$*. Hence, Theorem 3 shows that* R *has the invariant probability density* $y_0/\pi(x^2 + {y_0}^2)$*. Similar arguments allows us to have the parallel results to those of Example 3.*

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