Exact Solutions of Conformal Field Theory in Two Dimensions and Critical Phenomena

by

A.B.Zamolodchikov

L.D.Landau Institute for Theoretical Physics
Academy of Sciences, Kosygina 2, 117334 Moscow USSR

(preprint of Institute of Mathematical Physics, Academy of Sciences of Ukraina.87-65P, 1987, Kiev)

Translated from Russian by

Y.Kanie

Department of Mathematics, Faculty of Education
Mie University, Tsu 514 Japan

（1988年5月9日受理）

Abstract. Modern development of conformal field theory in two dimensions and its applications to critical phenomena are briefly reviewed. The specific properties of the renormalization group in two dimensions and the fundamentals of 2D conformal field theory are presented and the properties of degenerate representations of the Virasoro algebra and other infinite dimensional algebras, "minimal" models of conformal and superconformal field theory, "parafermionic" and other symmertries are discussed. We also investigate a perturbation theory around conformal solutions of field theory.

[Section 1]

Contents.

§1. Introduction.

In approaching to second order phase transition points, a characteristic length of fluctuations of order parameters — the correlation length R_c — grows unboundedly. These large-scale fluctuations, which lead to the appearance of singularities of thermodynamic functions, can be expressed in terms of effective field theory. Here details of microscopic structures of systems may be inessential, but interactions of fluctuations are defined only by the nature of order parameters and the quantity R_c themselves. These ideas developed by Kadanoff, Vaidomo, Patashinskii, Pokrovskii and others give a foundation of the scaling hypothesis and the universality of critical behaviours(c.f. [1] for example). At the critical point $T=T_c$, the correlation length is infinite, and the corresponding field theory is massless and is invariant under scale transformations

[Section 1]

$$x^{\mu} \longrightarrow \lambda\, x^{\mu} \qquad (1.1)$$

in its infrared asymptotic (where x^{μ} are coordinates of the space, $\mu=1,2,\cdots,\mathscr{D}$), under the condition that the field Φ_{ℓ} describing local fluctuations of thermodynamic characteristics of the systems, are transformed as

$$\Phi_{\ell} \longrightarrow \lambda^{-d_{\ell}}\, \Phi_{\ell} \quad , \qquad (1.2)$$

by (1.1), where the exponent d_{ℓ} is the anomalous scale dimension. Computation of the spectra $\{d_{\ell}\}$ of anomalous dimensions is the most important problem in the theory, since these quantities determine the characters of critical singularities of thermodynamic functions [1].

In these days, the universal and scaling properties are best understood in terms of renormalization groups(RG) (cf.[2]). In this approach, critical singularities are related to the existence of fixed points of RG in the "space of effective interactions" S [2]. A fixed point is essentially a field theory with the symmetry (1.1) at any scale. Critical behaviours are completely determined by characteristics of corresponding fixed points. Although it is quite difficult in experimental situations to find fixed points on the "unstable manifold" with sufficiently high dimension [2], defining so-called "multi-critical points", the investigation of all fixed points is of principal interest as a first step of general analysis of topological properties of RG.

In 1970's, Polyakov[3] presented a hypothesis that critical fluctuations have not only scale but conformal invariance. Conformal transformations are coordinate transformations

$$x^{\mu} \longrightarrow y^{\mu}(x) \ , \qquad (1.3)$$

which preserve angles of arbitrary two vectors at any given point (but may change their lengths). In other words,

$$dy_{\mu}\ dy^{\mu} = \frac{\partial y_{\lambda}}{\partial x^{\mu}} \frac{\partial y^{\lambda}}{\partial x^{\nu}}\ dx^{\mu}\ dx^{\nu} = \rho(x)\ dx_{\mu}\ dx^{\mu} \ . \qquad (1.4)$$

In fact, in homogeneous and isotropic systems, conformal invariance is derived from scaling invariance under the condition of the locality of interactions. Thus the classification of fixed points of RG is equivalent to the construction of all conformal invariant solutions of field theories.

Polyakov[4] proposed a "bootstrap" program of constructing such solutions, based on a hypothesis of algebras of operator expansions (or algebras of local fields). According to this hypothesis[5-7], in the field theory, there is a certain infinite "basic" set of local fields $A_j(x)$ (including local order parameters) such that any fluctuating quantity(e.g. product of components of order parameters placed at different points of the space) can be expanded by this basis. Thus the fields A_j form an algebra w.r.t. the operator expansion

$$A_i(x)\ A_j(o) = \sum_k C^k_{ij}(x)\ A_k(o), \qquad (1.5)$$

where $C^k_{ij}(x)$ is a numerical function. The expansion (1.5) should be understood as relations among correlation functions

$$\langle X \rangle = \langle A_{j_1}(x_1)\, A_{j_2}(x_2) \cdots\cdots A_{j_N}(x_N)\rangle \ . \qquad (1.6)$$

Assume that the series (1.5), for the product $A_{j_1}(x_1)\, A_{j_2}(x_2)$ in (1.6) for example, is convergent if $|x_1-x_2| < \min |x_k-x_2|$; $k=3,\cdots,N$. The set $\{A_j\}$ of fields can be considered as a basis of a infinite-dimensional vector space $\mathcal{A}$ ("the space of local fields") which plays a role of a conventional space of states in this approach. Clearly all informations on field theory are concentrated in the "structure functions" $C^k_{ij}(x)$. Basic "dynamical equations" in this approach are derived from the requirement of the associativity of the algebra of the operator expansions (1.5) which is equivalent to the condition of the crossing symmetry of correlations functions (1.6) [4,8]. Combining this condition with the requirement of conformal invariance of the operator algebra (1.5), Polyakov[4] obtained a system of "bootstrap" equations for anomalous dimensions d_j and the "structure constants" C^k_{ij}. However, in the case of dimension $\mathcal{D}>2$, the classification of fields by representations of the conformal group seems insufficient for a complete "decomposition" of bootstrap equations.

Whereas for $\mathcal{D}>2$ the conformal group(isomorphic to $O(\mathcal{D}+2)$) is of finite dimension, the conformal group of two dimensional space is of infinite dimension. To assure this fact, it is convenient to introduce the complex coordinates

$$z = x^1 + ix^2 \ ; \qquad \bar{z} = x^1 - ix^2 \ . \qquad (1.7)$$

[Section 1]

Then any substitutions of the type

$$z \longmapsto \xi(z) \quad ; \qquad \bar{z} \longmapsto \bar{\xi}(\bar{z}) \ , \qquad (1.8)$$

satisfy (1.4), where ξ and $\bar{\xi}$ are arbitrary functions. Here note that in the Euclidean space $\mathbb{R}^2$, the coordinates (1.7) are related by the relation $\bar{z} = z^*$, where the asterisk stands for the complex conjugation. However it is convenient to transfer to the complex space $\mathbb{C}^2$ (the correlation functions (1.6) are analytically continued to some domain in $\mathbb{C}^2$), where z and $\bar{z}$ are considered as independent complex variables($\mathbb{R}^2$ is some real section of $\mathbb{C}^2$). Here ξ and $\bar{\xi}$ in (1.8) are independent functions and the conformal group can be considered as the direct product $\Gamma\times\bar{\Gamma}$ of "right" and "left" groups of analytic substitutions of the variables z and $\bar{z}$. The infinite-dimensional symmetry (1.8) allows to advance in the research of 2-dimensional conformal field theory much further than in higher dimensional cases[8].

Generators of transformations of the symmetry (1.8) in conformal field theory are integrals of motion. The generators $L_n ; n=0,\pm1,\pm2,\cdots$ of the "right" group form the infinite dimensional Virasoro algebra $\mathcal{V}$ with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}(n^3-n)\ \delta_{n+m,0} \qquad (1.9)$$

(the generators $\bar{L}_n$ of the "left" symmetry form a similar algebra $\bar{\mathcal{V}}$ and commute with L_m, hence the algebra of symmetry of conformal field theory is $\mathcal{V}\times\bar{\mathcal{V}}$), where the number C is called the *central charge*; this number is the most important characteristic of the theory.

The Virasoro algebra (1.9) is well-known in the relativistic string theory(see e.g. [9,10]). In fact, the relativistic string theory is intensely developed in last two decades and presents a very interesting model of 2-dimensional conformal field theory. Recent development of 2-dimensional conformal field theory of many degrees is connected with achievements of the string theory [10]. We can say that the converse is also true: string theoretic problems, for example an attempt to reach the notion of "string physics" away from critical dimensions, are initiated in the works of Polyakov [11,12], or the problem of the compactification of "unnecessary" space dimensions in the superstring theory[13] is the most important stimulus for these developments.

Irreducible decompositions of representations of the algebra $\mathcal{V}\times\bar{\mathcal{V}}$ make possible a very detailed description of the structure of the space $\mathcal{A}$ of conformal theory and may give a series of explicit solutions of the conformal bootstrap equations [8]. In [8] an infinite series of exact solutions of 2-dimensional conformal field theory —"minimal models"— is constructed. The corresponding space $\mathcal{A}$ contains only a finite number of irreducble representations of $\mathcal{V}\times\bar{\mathcal{V}}$. The minimal models $\mathcal{M}(p/q)$ are parametrized by two mutually-prime natural numbers and correspond to the number

$$C(p/q) = 1 - \frac{6(p-q)^2}{pq} \quad . \tag{1.10}$$

The simplest minimal models $\mathcal{M}(3/4)$ and $\mathcal{M}(5/6)$ decribe the well-known critical points of the Ising model and the 3-state Potts model and $\mathcal{M}(4/5)$ and $\mathcal{M}(6/7)$ correspond to tricritical points [15].

[Section 1]

The role of the unitarity condition in 2-dimensional conformal field theory was clarified in [15]. A field theory is called unitary, if the corresponding space of states(the space $\mathcal{A}$ of fields) has a positive definite metric. The unitarity condition must be satisfied for statistical systems with "local" (interactions of nearest neighbors) and real hamiltonians bounded from below (more precisely, the existence of a self-adjoint transfer matrix is sufficient). In conformal field theory, the unitarity condition leads to a principle to select admissble representations of the Virasoro algebra. These representations must not contain any states of negative norms ("ghost"). In [15] it is shown that in the domain C<1, this condition chooses the following discrete series of values C (and corresponding anomalous dimensions)

$$C_p = 1 - \frac{1}{p(p+1)} , \tag{1.11}$$

which corresponds to the "minimal model" $\mathcal{M}(p/q)$ with $q=p+1, p=3,4,5, \cdots$. Hence the models $\mathcal{M}_p=\mathcal{M}(p/p+1)$ exhaust all unitary solutions of conformal field theory for C<1; other models $\mathcal{M}(p/q)$ with $q-p>1$ are not unitary. However we must remark that there are many interesting statistical systems*) which obviously does not satisfy the unitarity condition. Therefore nonunitary solutions of the conformal bootstrap must also be studied.

In some cases, the quantity C can be interpreted as a measure of

*) Known examples are the problem of nonself-intersecting polymer chains, magnetics with stochastic interactions etc.

[Section 1]

an effective number of degrees of freedom of fields, having large-scale fluctuations at a given fixed point. Hence it is natural to expect that fixed points of RG are the more stable, the less the corresponding value C is. The quantity C as a central charge in (1.9) is defined only at fixed points. However one can "extend" it to arbitrary points of the "space of effective interactions" S, i.e. introduce a function C(g), where g is a point of S, which coincides with the corresponding central charge at each fixed point g_{*A}: $C(g_{*A}) = C_A$. Under an action of RG transformations with a trajectory g(t) the quantity C(g) is of course a function of RG with a parameter t: C(g(t)). In unitary field theories, a function C(g) can be chosen such that C(g) decreases monotonically under an action of RG, i.e. $\frac{d}{dt}C(g(t)) \leq 0$, moreover the equality holds only at fixed points [17]. Thus a ranking of fixed points with the quantity C corresponds to a ranking of RG stability. This problem will be discussed in more detail in §2.

The value C=1 corresponds to the free massless bosonic fields, i.e. Gaussian fixed points. Such theories contain a parameter (which is interpreted as a "radius of compactification" of a field φ), and critical exponents depend continuously on this parameter in such a way that we deal with lines of fixed points. This line corresponds to a critical line of the Ashkin-Teller model (or the eight vertex model of Baxter)[18]. A one parameter family of solutions of the conformal bootstrap for spin correlations of the Ashkin-Teller model are constructed in [31]. For C≥2 there can be manifolds of fixed points of dimension > 1.

[Section 1]

Conformal field theory can have (and has in principle) infinite symmetries, higher than conformal. The investigation of such "higher" symmetry allows to construct new solutions of conformal field theory. So in [19-21] they investigate a 2-dimensional field theory with superconformal symmetries(its generators form the Neveu-Schwarz-Ramond algebra which contains the subalgebra $\mathcal{V}$) and construct corresponding superconformal "minimal models". Analogously as (1.11), there exists a unitary series of such "minimal models"[15,21] $\mathcal{SM}_p$; $p=3,4,5,\cdots$ with

$$C_p = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) . \tag{1.12}$$

Another example is a G×G invariant model of chiral fields with Wess-Zumino actions(WZ)[22-24]. At infrared fixed points [23,24], this model has a symmetry w.r.t. a current algebra $\hat{G}\times\hat{G}$ (i.e. a direct product of Kac-Moody algebras) and conformal invariance with

$$C(G,\kappa) = \frac{\kappa\mathcal{D}(G)}{\kappa+C_{\mathcal{V}}} , \tag{1.13}$$

where $C_{\mathcal{V}}$ is the quadratic Casimir operator for the adjoint representation, $\mathcal{D}(G)$ is the dimension of the (semisimple) group G and κ is the central charge of the current algebra[25]. In the above examples, "higher" symmetries are generated by local currents: "supercurrents" of spin 3/2 in superconformal theory and a current of spin 1 in WZ models. The symmetry generated by local currents of higher spins is considered in [26]. "Higher" symmetries generated by nonlocal ("parafermionic") currents are also possible. Such fields naturally appear in statistical systems with a discrete cyclic symmetry [27].

[Section 1]

The investigation of representations of algebras of such nonlocal currents allow to find new conformally invariant solutions of the field theory[28-30]. Some of these solutions are considered in the following paragraphs.

Thus in these days, some infinite series of exact solutions of 2-dimensional conformal field theory are known, and there exist methods of constructing new solutions and even a hope to find a complete classification of such solutions in this way, in other words, a complete classification of all fixed points of RG. However we should remark that an analysis of exhausting fixed points must include, besides the construction of corresponding conformal field theory, also calculations of corresponding class of universality, i.e. essentially the description of the structure of RG in some neighbourhood of this point. In a series of cases such calculations can be carried out in the frame of the perturbation theory. This kind of questions will also be discussed below.

[Section 1]

§2. Renormalization Group in Two Dimensions.

Space symmetries in the field theory guarantee the existence of local symmetric energy-momentum tensor (stress tensor in statistical physics) $T^{\mu\nu}(x)\in\mathscr{A}$, satisfying the equations

$$\partial_\mu T^{\mu\nu}(x) = 0 . \tag{2.1}$$

In a Lagrangean theory, the field $T^{\mu\nu}(x)$ describes a variation of (euclidean) action $H = \int \mathscr{H}(x) d^2x$

$$\frac{1}{2}\delta_\varepsilon H = \int d^2x\ \partial_\mu \varepsilon_\nu(x)\ T^{\mu\nu}(x) \tag{2.2}$$

under an infinitesimal coordinate transformation

$$x^\mu \longrightarrow x^\mu + \varepsilon^\mu(x) . \tag{2.3}$$

This statement is equivalent to the following relation for correlation functions (1.6)

$$\sum_{i=1}^{N} \langle A_1(x_1)\cdots\delta_\varepsilon A_i(x_i)\cdots A_N(x_N)\rangle$$

$$- 2\int d^2x\ \partial_\mu\varepsilon_\nu(x)\langle T^{\mu\nu}(x)A_1(x_1)\cdots A_N(x_N)\rangle = 0 \tag{2.4}$$

where $\delta_\varepsilon A(x)$ is the variation of the field $A(x)$ itself under the transformation (2.3). The variation $\delta_\varepsilon A(x)$ is also a local field, i.e. $\delta_\varepsilon A(x)\in\mathscr{A}$, and it depends linearly on the functions $\varepsilon^\mu(x)$ and their derivatives of finite order taken at a point x. If a meaningful lagrangean formulation of the theory is not known, one must postulate the relation (2.4). In this case it provides, in fact, a definition of the linear operator δ_ε, acting on $\mathcal{A}$. Matrix elements of this operator are easily expressed by coefficients $C^k_{ij}(x)$ of operator expansions (1.5), if it is noted that in the system (2.1), integrand in the second term of (2.4) is expressed in the divergence form. Therefore one can write

$$\frac{1}{2}\delta_\varepsilon A(x) = \int_{\partial\Lambda_x} dy^\lambda \varepsilon_{\mu\lambda}\varepsilon_\nu(y)T^{\mu\nu}(y)A(x) + \int_{\Lambda_x} d^2y\ \partial_\mu\varepsilon_\nu(y)T^{\mu\nu}(y)A(x), \quad (2.5)$$

where Λ_x is an arbitrary neighbourhood of a point x in $\mathbb{R}^2$, $\partial\Lambda_x$ is its boundary and $\varepsilon_{\mu\nu}$ is an antisymmetric tensor, $\varepsilon_{12}=1$. The right hand side of (2.5) does not depend on the choice Λ_x, in particular this domain may be taken arbitrarily small according to the condition of the local dependence of $\delta_\varepsilon A(x)$ on $\varepsilon^\mu(x)$.

In the simplest case of translations $\varepsilon^\mu(x) = \varepsilon^\mu$ and rotations $\varepsilon^\mu(x) = \omega^{\mu\nu}x_\nu$; $\omega^{\mu\nu} = \omega^{\nu\mu}$, the second term on the right hand side of (2.5) vanishes (this is a reflection of euclidean invariance of the theory) and the corresponding operator δ_ε reduces to the momentum operator

$$\delta_\varepsilon A(x) = \varepsilon^\mu\partial_\mu A(x) + \omega^{\mu\nu}(x_\mu\partial_\nu + \varepsilon_{\mu\nu}\Sigma)\ A(x)\ , \quad (2.6)$$

if $\varepsilon^\mu(x) = \varepsilon^\mu + \omega^{\mu\nu}x_\nu$, where Σ is a spin operator of the field A(x). It is convenient that the basis vectors A_j in $\mathcal{A}$ are selected such that $\Sigma A_j = S_j A_j$, where S_j are integers (for Bose fields) or half-integers (for Fermi fields). In the following we denote by $\mathcal{A}^{(0)} \subset \mathcal{A}$ the subspace consisting of spinless fields, $\Sigma\mathcal{A}^{(0)} = 0$.

Other important types of coordinate transformations (2.3) are uniform dilations. Denote by D the corresponding operator in $\mathcal{A}$, that is,

$$\delta_{\varepsilon} A(X) = \varepsilon \ (x^{\mu}\partial_{\mu} + D) \ A(x) \ , \tag{2.7}$$

if $\varepsilon^{\mu}(x) = \varepsilon \cdot x^{\mu}$. In this case, (2.4) takes the form

$$\sum_{i=1}^{N} \left\langle \left(x_i^{\mu}\frac{\partial}{\partial x_i^{\mu}} + D^{(i)} \right) A_1(x_1)\cdots A_N(x_N) \right\rangle + 2\int d^2x \langle \Theta(x) A_1(x_1)\cdots A_N(x_N)\rangle = 0 \tag{2.8}$$

where $D^{(i)}$ denotes the operator D applying to the fields $A_i(x_i)$ and $\Theta = -T^{\mu\mu}$ is the trace of the stress tensor. Note that the integral in (2.8) may diverge at $x \longrightarrow x_i$; in this case the corresponding matrix elements of the operator D contain the dependence on the cut-off parameter R_0. Behaviors of field theories under scale transformations are described by the renormalization group. Varieties of the methods of RG in the field theory and statistical physics are explained in many textbooks(for example [2]). Here we discuss specific properties of RG in 2-dimensional field theory.

The "space of local interactions" S is a basic notion of RG[2]. In the Lagrangean field theory, this is the manifold of action functionals $H[\varphi] = \int \mathcal{H}(\varphi(x), \partial_{\mu}\varphi(x))d^2x$ with the locality condition of interactions. It is usual to assume that the theory provides ultra violet cut-off; in addtion, the locality condition can be broken at the distance $\sim R_0$. Generally speaking, the "space of interactions" S is of infinite dimension. Nevertheless assume that one can deal with them as finite dimensional manifolds. Usually this is justified as only finite dimensional submanifolds are essential[2]. Let $\{g^a\}$ =

$\{g^1, g^2, \cdots\}$ be a certain system of coordinates in S (corresponding points of S will be denoted by g). This means that a density of action H(x) is a function of some (generally infinitely many) set of parameters g^a (the "coupling constants"), i.e. $\mathcal{H}(x) = \mathcal{H}_g(x)$. The derivative

$$\Phi_a(x) = \frac{\partial \mathcal{H}_g(x)}{\partial g^a} \tag{2.9}$$

is a local field, i.e. $\Phi_a \in \mathcal{A}_g$, where the index g indicates that the given space corresponds to a point $g \in S$. When we consider only homogeneous and isotropic interactions, we will regard that all fields are spinless, i.e. $\Phi_a \in \mathcal{A}_g^{(0)}$. Thus the space $\mathcal{A}_g^{(0)}$ can be considered as a tangent space of S at a point g. For the correlation functions (1.6), this corresponds to the relation

$$\frac{\partial}{\partial g^a}\langle A_1(x_1)\cdots A_N(x_N)\rangle_g = \sum_{i=1}^{N}\langle A_1(x_1)\cdots B_a A_i(x_i)\cdots A_N(x_N)\rangle_g - \int d^2x\ \langle \Phi_a(x) A_1(x_1)\cdots A_N(x_N)\rangle_g \ , \tag{2.10}$$

where the operator B_a indicates the possible explicit dependence of the field A on g: $B_a A = \frac{\partial}{\partial g^a} A$. The necessity of introducing such dependence is obvious in such a case that the integral in (2.10) diverges at $x \longrightarrow x_i$. Corresponding matrix elements of the operators B_a must depend on R_0 in order to compensate the divergent contribution of the integral, since we mean that (1.6) are "renormalized" correlation functions independent of R_0. The trace Θ of the stress tensor belongs to $\mathcal{A}_g^{(0)}$ and can be expanded by the basis vectors (2.9)

$$\Theta(x) = \sum_a \beta^a(g)\ \Phi_a(x) \ , \tag{2.11}$$

where the coefficients $\beta^a(g)$ are ovbiously components of vector fields on S, which are called β-functions. By combining (2.8) and (2.10), it is easy to obtain the equations of RG in the Callan-Symanzyk form

$$\sum_{i=1}^{N} \langle \left[\frac{1}{\lambda}\ x_i^\mu \frac{\partial}{\partial x_i^\mu} + \gamma^{(i)}(g)\right] A_1(x_1)\cdots A_N(x_N)\rangle$$

$$= \sum_a \beta^a(g)\ \frac{\partial}{\partial g^a}\ \langle A_1(x_1)\cdots A_N(x_N)\rangle \ , \tag{2.12}$$

where the linear operators $\gamma^{(i)}(g)$ defined by the formula

$$\gamma(g) = \frac{1}{\lambda}\ D - \beta^a(g)\ B_a \ , \tag{2.13}$$

act on fields $A_i(x_i)$ in (2.12). The operator $\gamma(g)$ is called a matrix of anomalous dimension. By verifying the compatibility of the equations (2.8) and (2.10), it is possible to show that the operator $\gamma(g)$ acts on the basis vectors (2.9) in the following way

$$\gamma(g)\ \Phi_a \equiv \gamma_a^b(g)\ \Phi_b = (\delta_a^b + \frac{\partial\beta^b}{\partial g^a})\ \Phi_b \ . \tag{2.14}$$

This relation is equivalent to the important statement on the absence of renormalization of components of the energy-momentum tensor, i.e.

$$\gamma(g)\ T^{\mu\nu} = T^{\mu\nu} \ . \tag{2.15}$$

In the renormalized theory, any β^a and any matrix elements of the operator γ do not depend on R_0. From (2.14) it is obvious that two field theories corresponding to $g(t_1)$ and $g(t_2)$ of one integral curve of the Gell-Mann-Low equation

$$dg^a = \beta^a(g)\ dt\ , \tag{2.16}$$

are distinguished only by the scale transformation $x_\mu \longrightarrow e^{\frac{t_1-t_2}{2}} x^\mu$. The scaling behavior of field theories depends, in this way, on the singularity and global topological properties of the vector field $\beta^a(g)$. The simplest (and most important) singularity of this vector field is fixed points g_{*a}: $\beta(g_{*A}) = 0$ (the index A enumerates fixed points). Fixed points may be isolated, but also may form a submanifold of S of dimension >0. Critical behaviors of statistical systems are directly related with fixed points of RG as explained for example in [2].

Let us assume that the considered field theory satisfies the positivity condition(cf.[16]). In particular, this means that any metric $G_{ab}(g,R)$ in $\mathcal{A}_g^{(0)}$ defined by the two-point function

$$G_{ab}(g,R) = \langle \Phi_a^\dagger(R)\ \Phi_b(0)\rangle_g\ , \tag{2.17}$$

is positive definite. In the following we will assume that the coordinates g^a are real and $\Phi_a^\dagger = \Phi_a$. It proves to be that the positivity condition reduces to the specific restriction on the property of RG in 2-dimensional theory.

Define the fields T and $\bar{T}$ by the components of the tensor $T^{\mu\nu}$ as follows

[Section 2]

$$T = T^{11} - T^{22} + 2i\,T^{12} \quad ; \quad \overline{T} = T^{11} - T^{22} - 2i\,T^{12} \,, \qquad (2.18)$$

satisfying the equations $\Sigma\, T = 2T$; $\Sigma\, \overline{T} = -2\overline{T}$. The equation of motion (2.1) in these notations are rewritten in the form

$$\partial_{\overline{z}}\, T = \partial_z\, \Theta \qquad ; \quad \partial_z\, \overline{T} = \partial_{\overline{z}}\, \Theta \,, \qquad (2.19)$$

where z and $\overline{z}$ are complex coordinates (1.7) and $-\,\Theta = T^{11} + T^{22}$. Consider the two-point functions

$$\langle T(z,\overline{z})T(0,0)\rangle = F(t)/z^4 \quad , \quad \langle T(z,\overline{z})\Theta(0,0)\rangle = H(t)/z^3\overline{z} \quad ;$$

$$\langle \Theta(z,\overline{z})\Theta(0,0)\rangle = G(t)/(z\overline{z})^2 \,, \qquad (2.20)$$

where $t = \log(z\overline{z})$. Equations (2.19) imply the following relations for the functions F,G,H:

$$\dot{F} = \dot{H} - 3\,H \quad ; \quad \dot{H} - H = \dot{G} - 2\,G \,, \qquad (2.21)$$

where the dot stands for the differentiation w.r.t. t. Introduce the quantity

$$C = 2F + 4H - 6G \,. \qquad (2.22)$$

The equation

$$\dot{C} = -12\,G \qquad (2.23)$$

is an immediate consequence of (2.21). Since $G(t) \geq 0$ because of the positive definiteness of the metric (2.17), equation (2.23) implies that C(t) is a monotonically decreasing function of t. The equality $G(t) = 0$ holds when $\Theta = \beta^a \Phi_a = 0$, i.e. in the theory corresponding to a fixed point; in this case C is constant.

[Section 2]

The function C(t) can be given a meaning of a measure of numbers of degrees of freedom, having, with an appreciable probability, fluctuations with space dimensions $e^{t/2}$, hence the conclusion on the decrease of this quantity seems natural. At a fixed point, the equality C(t) = Const. reflects scale invariance of fluctuations. If we fix t, for example, put t = 0, then the quantity C will depend only on the "coupling constants" g^a. In addition, from equations of RG (2.12) and (2.15) we easily derive

$$\beta^a(g) \frac{\partial}{\partial g^a} C(g) = -12\, G_{ab}(g)\, \beta^a(g)\, \beta^b(g) \leq 0 \;, \tag{2.24}$$

where $G_{ab}(g) = G_{ab}(g,1)$ is the positive definite metric (2.17). This relation shows that the "flow of RG" described by equation (2.16) implies the decrease of the fucntion C(g), moreover stationary points C(g) are fixed points of RG, i.e. $\partial C(g)/\partial g = 0 \Longrightarrow \beta(g) = 0$. After considering general properties of conformal field theories corresponding to fixed poins in the following paragraphs, we will prove the converse: every fixed point g_{*A} is stationary for C(g).

Thus every fixed point g_{*A} is characterized by the constant C_A which is the value of C(g) at this point

$$C_A = C(g_{*A}) \; . \tag{2.25}$$

At a fixed point the field Θ vanishes, in view of (2.11). In addition the functions G and H in (2.20) vanish, and the two-point function

$$\langle T(z,\bar{z})T(0,0)\rangle_{g_{*A}} = \frac{C_A}{2z^4} \tag{2.26}$$

is completely represented by the constant (2.25). This constant is

[Section 2]

an important characteristic of conformal field theory described by a fixed point g_{*A}; this coincides with a central charge of the Virasoro algebra (1.9).

The relations deduced above have two obvious corollaries:

a) If two fixed points g_{*1} and g_{*2} are connected by a trajectory $g(t)$ on RG such that $g(-\infty) = g_{*1}$; $g(\infty) = g_{*2}$, then the corresponding constants (2.25) are related by the inequality

$$C_2 < C_1 \; ; \qquad (2.27)$$

b) If there exists a continuous manifold of fixed points $\vartheta_* \subset S$, then all points of this manifold are characterized by the same value C.

[Section 2]

§3. Energy-momentum Tensor in Conformal Field Theory

At a fixed point of RG, $\Theta = 0$ and equation (2.19) has the form

$$\partial_{\bar{z}} T = 0 \quad ; \quad \partial_z \bar{T} = 0 . \tag{3.1}$$

In view of (3.1), we will write $T = T(z)$; $\bar{T} = \bar{T}(\bar{z})$. Equation (3.1) means that, for example, the correlation function

$$\langle T(z) A_1(z_1, \bar{z}_1) \cdots A_N(z_N, \bar{z}_N) \rangle \tag{3.2}$$

is a single-valued analytic function with singularities —— with poles of finite order —— at points $z_1, z_2, \cdots, z_N$. The residues at these poles are determined by the variations $\delta_\varepsilon A_i$ of the field A_i by infinitesimal conformal transformations (1.8)

$$z \longrightarrow z + \varepsilon(z) \quad ; \quad \bar{z} \longrightarrow \bar{z} + \bar{\varepsilon}(\bar{z}) . \tag{3.3}$$

In fact, for such transformations the second term in (2.5) drops out (at $\Theta = 0$), and we get

$$\delta_\varepsilon A(z, \bar{z}) = \oint_{C_z} \frac{d\xi}{2\pi i} \varepsilon(\xi) \, T(\xi) \, A(z, \bar{z}) , \tag{3.4}$$

where the integration is taken along a contour around the point z. Thus the components T and $\bar{T}$ of the energy-momentum tensor are generators of "right" and "left" conformal transformations in the field theory. It is useful, by expanding the function $\varepsilon(\xi)$ in (3.4) in a Laurent series near the point z, to introduce the infinite basis of operators

$$L_n = \delta_{\varepsilon_n} \; ; \quad \varepsilon_n(\xi) = (\xi - z)^{n+1} \quad ; \; n = 0, \pm 1, \pm 2, \cdots ;$$

[Section 3]

these operators act in the space $\mathcal{A}$ of fields. An equivalent definition of these operators can be given by the operator expansion

$$T(z_1)\ A(z_2,\bar{z}_2) = \sum_{n=-\infty}^{\infty} (z_1-z_2)^{-n-2}\ L_n\ A(z_2,\bar{z}_2), \tag{3.5}$$

where A is an arbitrary field in $\mathcal{A}$. Analogously introduce the operators $\bar{L}_n$; $n=0,\pm1,\pm2,\cdots$. By comparing this definition with (2.6), (2.7) and (2.13), it is easily obtained that

$$L_{-1}A(z,\bar{z}) = \partial_z A(z,\bar{z}) \ ; \qquad \bar{L}_{-1}A(z,\bar{z}) = \partial_{\bar{z}} A(z,\bar{z}) \ ;$$

$$L_0 - \bar{L}_0 = \Sigma \ ; \qquad L_0 + \bar{L}_0 = D = 2\gamma(g_*). \tag{3.6}$$

A basis $\{A_j\}$ in $\mathcal{A}$ are suitably chosen such that

$$L_0\ A_j = \Delta_j\ A_j \ ; \qquad \bar{L}_0\ A_j = \bar{\Delta}_j\ A_j \ , \tag{3.7}$$

where the real numbers $(\Delta_j,\bar{\Delta}_j)$ are called "right" and "left" dimensions; obviously $s_j = \Delta_j - \bar{\Delta}_j$ is the spin and $d_j = \Delta_j + \bar{\Delta}_j$ is the anomalous scale dimension (1.2) of the field A_j .

Owing to (2.15) and (3.6), we can write the general expression for the variation $\delta_\varepsilon T$ for the transformation (3.3):

$$\delta_\varepsilon\ T(z) = \varepsilon(z)\ T'(z) + 2\ \varepsilon'(z)\ T(z) + \frac{C}{12}\ \varepsilon'''(z) \ ; \tag{3.8a}$$

$$\delta_{\bar{\varepsilon}}\ T(z) = 0 \ , \tag{3.8b}$$

where the prime means the differentiation, and the constant C is not fixed by the general requirement of symmetries and is a parameter of the theory. The expression (3.8a) is equivalent to the following form of the operator expansion:

$$T(z_1)T(z_2) = \frac{C}{2(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2} T(z_2) + \frac{1}{z_1-z_2} T'(z_2) + \text{reg}, \quad (3.9)$$

where reg means the contribution of the terms, regular at $z_1 \longrightarrow z_2$. The commutation relations (1.9) among the operators L_n are easily deduced from (3.9). Similar formulae as (3.8),(3.9) with an obvious modification, are valid for the field $\overline{T}(\overline{z})$ of course. In the following, we restrict ourselves to considering the theory which is invariant under the space reflections, where $\overline{C} = C$. The commutativity of the operators L_n and $\overline{L}_m$ follows from (3.8b). Remark that $\langle T \rangle = 0$ in the infinite system, and (3.9) implies the expression (2.26) for the two-point function $\langle T\ T \rangle$. Thus the "central charge" C in (1.9) coincides with the value $C = C(g_*)$ of the function C(g) introduced in §2.

From (1.9) it is obvious that the operator L_n decreases the "right" dimension by n units. Therefore, in the space $\mathcal{A}$ of an arbitrary conformal theory there must exist "primary" fields Φ_ℓ satisfying the equations

$$\begin{aligned} &L_n \Phi_\ell = \overline{L}_n \Phi_\ell = 0 \qquad \text{for } n>0 \ ; \\ &L_0 \Phi_\ell = \Delta_\ell \Phi_\ell \quad ; \qquad \overline{L}_0 \Phi_\ell = \overline{\Delta}_\ell \Phi_\ell \quad ; \end{aligned} \qquad (3.10)$$

otherwise the spectra of anomalous dimensions $\{d_j\}$ are not bounded from below. A nontrivial theory contains some (or even infinitely many) primary fields; the index ℓ is introduced for labeling them. The singular terms of the operator expansion (3.5) for primary fields are of the form:

$$T(z_1)\Phi_\ell(z_2,\bar{z}_2) = \frac{\Delta_\ell}{(z_1-z_2)^2}\Phi_\ell(z_2,\bar{z}_2) + \frac{1}{z_1-z_2}\frac{\partial}{\partial z_2}\Phi_\ell(z_2,\bar{z}_2) + \text{reg}, \quad (3.11)$$

which implies obviously that primary fields are transformed in a particularly simple way for the substitution (1.8):

$$\Phi_\ell(z,\bar{z}) = \left(\frac{d\xi}{dz}\right)^{-\Delta_\ell} \left(\frac{d\bar{\xi}}{d\bar{z}}\right)^{-\bar{\Delta}_\ell} \Phi_\ell(\xi,\bar{\xi}) \ . \quad (3.12)$$

Fields obtained by applying the operators $L_n, \bar{L}_n$ with $n<0$ to Φ_ℓ ("conformal descendants" of the field Φ_ℓ)

$$L_{-n_1} L_{-n_2} \cdots L_{-n_N} \bar{L}_{-m_1} \cdots \bar{L}_{-m_M} \Phi_\ell \quad (3.13)$$

consist of the space $[\Phi_\ell] \subset \mathcal{A}$ —— the "conformal family" of the field Φ_ℓ. Dimensions of the fields (3.13) differ from $(\Delta_\ell, \bar{\Delta}_\ell)$ by positive integers. Generally speaking, the space spanned by all the vectors of the form (3.13) is the base space of an irreducible representation of $\mathcal{V}\times\bar{\mathcal{V}}$. This statement is not valid, if Δ_ℓ or $\bar{\Delta}_\ell$ takes special values (see §4). In these cases, however, we can get irreducible representations by factoring the space (3.13) by the corresponding invariant subspace. In all cases, the "conformal family" $[\Phi_\ell]$ is called an irreducible representation. In view of the commutativity $[L_n, \bar{L}_m] = 0$, the conformal family can be considered as the direct product

$$[\Phi_\ell] = [\Delta_\ell] \otimes [\bar{\Delta}_\ell] \quad (3.14)$$

of the irreducible representations $[\Delta_\ell]$ and $[\bar{\Delta}_\ell]$ of the algebras $\mathcal{V}$

and $\overline{\mathcal{V}}$ respectively; each of these representations is completely characterized by the value of corresponding dimension Δ_ℓ or $\overline{\Delta}_\ell$. The space of fields of the conformal theory is the sum of the conformal families

$$\mathcal{A} = \bigoplus_{\ell} [\Phi_\ell] \tag{3.15}$$

In the space (3.15) of the conformal field theory, we can introduce the metric by setting (in an infinite system)

$$(A_1,A_2) = \lim_{z,\overline{z}\to\infty} \langle e^{zL_0}\, e^{\overline{z}\overline{L}_0}\, A_1^\dagger(z,\overline{z})\, A_2(0,0)\rangle , \tag{3.16}$$

which is obviously related with (2.17) by regular conformal transformations. In this metric

$$L_n^\dagger = L_{-n} \qquad \text{and} \qquad \overline{L}_n^\dagger = \overline{L}_{-n} , \tag{3.17}$$

and primary fields Φ_ℓ can be taken as

$$(\Phi_\ell,\ \Phi_{\ell'}) = \delta_{\ell,\ell'} . \tag{3.18}$$

In unitary theories, the metric (3.16) must be positive-definite.

The identity operator I is, of course, a primary field of dimension (0,0) (moreover $T,\overline{T}\in[I]$). Generally it can be shown (in unitary theories) that any field Ψ of null "left" dimension $\overline{\Delta} = 0$ satisfies the equation

$$\partial_{\overline{z}}\, \Psi = 0 \tag{3.19}$$

(for the field $\overline{\Psi}$ of $\Delta = 0$ a similar equation as (3.19) w.r.t. z is satisfied; (3.1) may be an example. Therefore any field of $\bar{\Delta} = 0$ generates an infinite set of integrals of motion. The property (3.19) is satisfied by all fields belonging to the subspace [0] of the conformal family [I], defined by the decomposition (3.14). The corresponding integrals of motion exist in any conformal field theory. The subspace [0] contains T(z) and any possible components of fields which can be obtained by the "fusion" of T(z). The simplest among these will be used later, hence we introduce special notations for them. The field

$$T_4 = (L_{-2}^2 - \frac{3}{5} L_{-4})\ I \tag{3.20}$$

is of dimension (4,0) (spin 4). This field is the first of the omitted regular terms in (3.9)

$$T(z_1)T(z_2) = \cdots\cdots + \frac{3}{10} T''(z_2) + T_4(z_2) + O(z_1-z_2), \tag{3.21}$$

where $O(z_1-z_2) \longrightarrow 0$ as $z_1 \longrightarrow z_2$. Analogously to (3.5), we can introduce the operators Λ_n, $n=0,\pm1,\pm2,\cdots$

$$T_4(z_1)\ A(z_2,\bar{z}_2) = \sum_{n=-\infty}^{\infty} (z_1-z_2)^{-n-4}\ \Lambda_n\ A(z_2,\bar{z}_2)\ , \tag{3.22}$$

so simple calculations show that

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k\ L_{n-k}: + \frac{1}{5}\kappa_n\ L_n\ , \tag{3.23}$$

where

$$\kappa_{2k} = (1-k^2) \quad ; \quad \kappa_{2k-1} = (1+k)(2-k)\ . \tag{3.24}$$

[Section 3]

We mention also the commutation relation

$$[L_n, \Lambda_m] = (3n-m)\, \Lambda_{n+m} + \frac{1}{6}\left(\frac{22}{5}+C\right)(n^3-n)\, L_{n+m}\,. \qquad (3.25)$$

The space of fields of spin 6 in [0] differing from the product of T and T_4 is of dimension 2; basis vectors can be chosen in the forms

$$T_6^{(1)} = [L_{-2}^3 - \frac{1}{3} L_{-3}^2 - \frac{19}{15} L_{-4}L_{-2} - \frac{2}{3} L_{-6}]\, I\ ; \qquad (3.26a)$$

$$T_6^{(2)} = \frac{1}{9}[-\frac{5}{2} L_{-3}^2 + 4\, L_{-4}L_{-2} + \frac{10}{7} L_{-6}]\, I\ . \qquad (3.26b)$$

The requirement of conformal invariance of the operator algebra (1.5) reduces to the following general form of the operator expansion of the product of the primary fields

$$\Phi_{\ell_1}(z,\bar{z})\, \Phi_{\ell_2}(0,0) = \sum_{\ell} \mathbb{C}_{\ell_1\ell_2}^{\ell}\, z^{\Delta_\ell-\Delta_{\ell_1}-\Delta_{\ell_2}}\, \bar{z}^{\bar{\Delta}_\ell-\bar{\Delta}_{\ell_1}-\bar{\Delta}_{\ell_2}}\, [\Phi_\ell(0,0)+ \cdots], \qquad (3.27)$$

where the dots in the square bracket denote the contribution of all fields from corresponding conformal families. This contribution represents a series of integral positive powers of z and $\bar{z}$. Moreover the coefficients in this series are completely determined by the requirement of conformal invariance (3.27) [8,31]. The numerical coefficients $\mathbb{C}_{\ell\ell_1\ell_2} \equiv \mathbb{C}_{\ell_1\ell_2}^{\ell}$ in (3.27) are called structure constants of the operator algebra. By the normalization (3.18) the quantities $\mathbb{C}_{\ell\ell_1\ell_2}$ are symmetric in all indices. The structure of the space $\mathcal{A}$ of dimension $(\Delta_\ell,\bar{\Delta}_\ell)$ of primary fields Φ_ℓ and the structure constants

$\mathbb{C}_{\ell\ell_1\ell_2}$ must be selected in order to guarantee the associativity of the algebra of the operator expansions (3.27) [8]. In the following we will often use the abbreviated notations of (3.27)

$$\Phi_{\ell_1} \Phi_{\ell_2} = \sum_{\ell} [\Phi_{\ell}] \; . \tag{3.28}$$

[Section 3]

§4. Degenerate Repersentations and Minimal Models

Let Φ_Δ be an arbitrary primary field of "right" dimension Δ (below we discuss representations of the "right" algebra $\mathcal{V}$, since representations of $\bar{\mathcal{V}}$ have the same properties). Denote by $\mathfrak{U}_\Delta$ the space spanned by all vectors of the form

$$L_{-n_1} L_{-n_2} \cdots L_{-n_N} \Phi_\Delta \ ; \quad 1 \le n_1 \le n_2 \le \cdots \le n_N \ . \tag{4.1}$$

Obviously $\mathfrak{U}_\Delta$ is a base of some representation of $\mathcal{V}$. However this representation is reducible, if $\mathfrak{U}_\Delta$ contains a vector $\chi_{\Delta+L}$ ("null-vector") satisfying the equations

$$L_n \chi_{\Delta+L} = 0 \ , \quad n>0 \quad ; \qquad L_0 \chi_{\Delta+L} = (\Delta+L)\, \chi_{\Delta+L} \tag{4.2}$$

with some integer L. In this case, the subspace $\mathfrak{U}_{\Delta+L} \subset \mathfrak{U}_\Delta$ generated by ones obtained by applying the operators L_n with $n<0$ to $\chi_{\Delta+L}$ is invariant w.r.t. the action of $\mathcal{V}$. In order to get the irreducible representation with the "highest weight vector" Φ_ℓ, we must factor $\mathfrak{U}_\Delta$ by the invariant subspace $\mathfrak{U}_{\Delta+L}$, i.e. set

$$\chi_{\Delta+L} = 0 \ . \tag{4.3}$$

If $\mathfrak{U}_\Delta$ contains some independent null-vectors, one must set each of them to be zero. The factor space $[\Delta] = \mathfrak{U}_\Delta / \mathfrak{U}_{\Delta+L}$ is called a "degenerate" irreducible representation of $\mathcal{V}$ and the number L is called a level of the degeneracy. In such case we say also that the corresponding conformal family $[\Phi_\Delta]$ (3.14) and the primary field Φ_Δ itself are "degenerate". If $\mathfrak{U}_\Delta$ does't contain null-vectors, then $[\Delta] = \mathfrak{U}_\Delta$ and the field Φ_Δ is "nondegenerate".

[Section 4]

The simplest case of the degeneracy is $\Delta = 0$; in addition L = 1, $\chi_1 = L_{-1}\Phi_0$. Equation (3.19) (more precisely an analogous equation w.r.t. z), appearing in this case, is an example of the degeneration. The field Φ_Δ is degenerate with L=2, if Δ takes any of the following two values:

$$\Delta_{(1,2)} = \Delta_0 + \frac{1}{4}(\alpha_+ + 2\alpha_-)^2 \ ; \qquad \Delta_{(2,1)} = \Delta_0 + \frac{1}{4}(2\alpha_+ + \alpha_-)^2 \ , \qquad (4.4)$$

where

$$\Delta_0 = \frac{C-1}{24} \ ; \quad \alpha_\pm = (24)^{-1/2}(\sqrt{1-C} \pm \sqrt{25-C}) \ ; \ \alpha_+\alpha_- = -1 \ . \qquad (4.5)$$

In addition the null-vector has the form

$$\chi_{\Delta+2} = (L_{-2} - \frac{3}{2(2\Delta+1)} L_{-1}^2)\ \Phi_\Delta \ . \qquad (4.6)$$

All cases of degenerate representations of $\mathcal{V}$ are enumerated by Kac's formula [32,33]

$$\Delta_{(n,m)} = \Delta_0 + \frac{1}{4}(n\alpha_+ + m\alpha_-)^2 \ ; \ L = nm \ , \qquad (4.7)$$

where n,m are any natural numbers; the right part in (4.7) determines the corresponding level of the degeneracy. Denote temporarily by $\Phi_{(n,m)}$ the degenerate primary field of the "right" dimension $\Delta_{(n,m)}$. Degenerate fields have a series of important properties:

a) The correlation functions containing degenerate fields satisfy linear differential equations. The simplest example is the equation $\partial_z\Phi_{(1,1)} = 0$ ($\Delta_{(1,1)}=0$, see above). In general case, these equations can be obtained (if the expression is known for corresponding null-vectors) with the aid of the local operator expansion

(3.11) and the boundary condition imposed on the field T(z). For example, for the field $\Phi_{(1,2)}$(or $\Phi_{(2,1)}$) in an infinite system case, we obtained [8]

$$\left\{\frac{3}{2(2\Delta+1)}\frac{\partial^2}{\partial z^2}-\sum_{i=1}^{N}\frac{\Delta_i}{(z-z_i)^2}-\sum_{i=1}^{N}\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\} \times \langle\Phi_{(1,2)}(z)\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle = 0 , \qquad (4.8)$$

where Φ_i is any primary field of dimension Δ_i; in (4.8) inessential terms with the argument $\bar{z}$ are omitted. The generalization of (4.8) to the case of systems with boundaries and the topology of tori can be found in [34,35].

b) Degenerate fields(precisely speaking, degenerate conformal families) make up a closed algebra w.r.t. the operator expansion (1.5) This statement can be obtained by studying differential equations (4.8) which impose, of course, a stringent constraint on the structure of the operator expansion (3.27). The operator expansion of degenerate fields has the following structure [8]

$$\Phi_{(n_1,m_1)}\ \Phi_{(n_2,m_2)} = \sum_{\ell=0}^{\ell_1}\sum_{k=0}^{k_1}[\Phi_{(n_0+2\ell,m_0+2k)}] , \qquad (4.9)$$

where $n_0 = |n_1-n_2|+1$; $m_0 = |m_1-m_2|+1$; $\ell_1 = \min(n_1,n_2)-1$; $k_1 = \min(m_1,m_2)-1$; here we use the abbreviated notations (3.28).

The operator algebra of the degenerate fields (4.9) can be tried to interprete as a conformal field theory. In addition, of course, we must restrict ourselves to the case C<1, since in the other range of this parameter the spectrum of dimension (4.7) is neither real

[Section 4]

nor bounded from below. Acceptable models in a "physical" point of view appear, if the quantity $\rho = -\alpha_-/\alpha_+$ takes a rational value

$$\rho = -\alpha_-/\alpha_+ = p/q \ , \tag{4.10}$$

where p and q are mutually prime natural numbers. The corresponding values C are given by the formula (1.10). It turns out that at these values (4.10) the "complete" operator algebra of the degenerate fields (4.9) contains the subalgebra

$$\mathcal{A}(p/q) = \frac{1}{2} \bigoplus_{n=1}^{p-1} \bigoplus_{m=1}^{q-1} [\Phi_{(n,m)}] \ , \tag{4.11}$$

consisting of a finite number of conformal fields. We remark that in the case (4.10) the spectrum (4.7) has a form

$$\Delta_{(n,m)} = \frac{(qn-pm)^2-(q-p)^2}{4pq} \ ; \quad 1 \le n \le p-1 \ ; \ 1 \le m \le q-1 \ , \tag{4.12}$$

and satisfy the relation $\Delta_{(p-n,q-m)} = \Delta_{(n,m)}$ so that

$$\Phi_{(p-n,q-m)} = \Phi_{(n,m)} \ . \tag{4.13}$$

Although in view of (4.13) each conformal family appears formally twice in the sum (4.11), we mean that the space $\mathcal{A}(p/q)$ contains (p-1)(q-1)/2 different conformal families; in this sense the "factor" $\frac{1}{2}$ is set in (4.11). The structure of the operator algebra (4.11) is described by the same formula (4.9), where however

$$\begin{aligned} \ell_1 &= \min_{i=1,2} (n_i-1, \ p-n_i-1) \ ; \\ k_1 &= \min_{i=1,2} (m_i-1, \ q-m_i-1) \ . \end{aligned} \tag{4.14}$$

[Section 4]

The operator algebra (4.11) is called a "minimal model" $\mathcal{M}(p/q)$.

The equality (4.13) means that each primary field in (4.11) is "degenerate twice", i.e. the corresponding space $\mathfrak{U}_{(n,m)}$ contains two independent null-vectors on the levels L=nm and L'=(p-n)(q-m). In particular, the identity operator $I = \Phi_{(1,1)} = \Phi_{(p-1,q-1)}$ has, besides the already mentioned degeneracy $L_{-1}I = 0$, an additional degenerary at the level (p-1)(q-1). This means that some field of spin (p-1)(q-1), "composed" from T(z), vanishes in the model $\mathcal{M}(p/q)$. For example, in the model $\mathcal{M}(2/5)$ (C=-22/5), the field (3.20) of spin 4 vanishes:

$$\mathcal{M}(2/5): \qquad T_4 = 0 . \tag{4.15}$$

Equation (4.15) completely determines all properties of the model $\mathcal{M}(2/5)$. In fact, consider the operator expansion (3.22) with an arbitrary primary field $A = \Phi_\Delta$. The singular terms in this expansion are easily deduced from (3.23)

$$\begin{aligned} T_4(z)\Phi_\Delta(0,0) = {} & (\Delta+\tfrac{1}{5})\left(\frac{\Delta}{z^4} + \frac{2}{z^3}L_{-1} + \frac{5}{2\Delta+1}\frac{1}{z^2}L_{-1}^2 + \frac{5}{(2\Delta+1)(\Delta+1)}\frac{1}{z}L_{-1}^3\right)\Phi_\Delta \\ & + 2\Delta\left(\frac{1}{z^2} + \frac{3}{\Delta+1}\frac{1}{z}L_{-1}\right)\left(L_{-2} - \frac{3}{2(2\Delta+1)}L_{-1}\right)\Phi_\Delta \\ & + \frac{2(\Delta-1)}{z}\left(L_{-3} - \frac{2}{\Delta+1}L_{-1}L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)}L_{-1}^3\right)\Phi_\Delta + \text{reg} . \end{aligned} \tag{4.16}$$

Therefore the equality (4.15) fixes a spectrum of dimensions of all primary fields of the model $\mathcal{M}(2/5)$: $\Delta_{(1,1)} = \Delta_{(1,4)} = 0$; $\Delta_{(1,2)} = \Delta_{(1,3)} = -1/5$, and it also shows that the field $\Phi_{(1,2)} = \Phi_{(1,3)}$ satisfies equation (4.6), and also the equation

$$\left(L_{-3} - \frac{2}{\Delta+1} L_{-1}L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)} L_{-1}^{3}\right) \Phi_{(1,3)} = 0 , \qquad (4.17)$$

corresponding to the level 3 degeneracy, where $\Delta = \Delta_{(1,3)}$. Here we remark that "physical" contents of the model $\mathcal{M}(2/5)$ are studied by Cardy[36] who showed that this model describes the critical singularity of Lie and Yang, which appears, for example in the Ising model with $T > T_c$, at a specific complex value of the magnetic field $ih(T)$.

So far we have not discussed the locality condition of fields and the associativity of the operator algebras of the "minimal models". Remind that each primary field Φ_ℓ is characterized by two dimensions $(\Delta_\ell, \bar{\Delta}_\ell)$, where for a local field Σ_ℓ of spin $s_\ell = \Delta_\ell - \bar{\Delta}_\ell$, the condition $s_\ell \in \frac{1}{2}\mathbb{Z}$ must be satisfied. In the minimal model $\mathcal{M}(p/q)$, one can always construct $(p-1)(q-1)/2$ spinless primary fields; in the following, unless otherwise stated, $\Phi_{(n,m)}$ denotes the spinless field of dimension $(\Delta_{(n,m)}, \Delta_{(n,m)})$. In fact, in the model $\mathcal{M}(p/q)$ with $pq \in 2\mathbb{Z}$ one can construct local primary fields of nonzero spin. We will not discuss here this interesting question; some examples will be mentioned below. However let us make some comments on the associativity of the operator algebra (4.11). It turns out that for all $\mathcal{M}(p/q)$ one can obtain the associativity of the operator algebra (4.9), (4.14) by a suitable choice of the structure constants $\mathbb{C}_{(n_1,m_1)(n_2,m_2)(n_3,m_3)}$ (see (3.27)). The values of the constants $\mathbb{C}$, which guarantee the associativity, can be calculated by solving equation (4.8) or using Feigin-Fuks representation [33] for correlation functions, as it is done in [38,39]. General expressions for these constants, found in [39], are cumbersome; we refer only some special cases.

$$\mathbb{C}_{(1,2)(n,m)(n,m+1)} = \left[\frac{\gamma(2-2\rho)}{\gamma(1-\rho)}\;\frac{\gamma(n-m\rho)}{\gamma(1+n-(1+m)\rho)}\right]^{1/2} \;; \qquad (4.18a)$$

$$\mathbb{C}_{(1,3)(n,m)(n,m)} = \frac{\Gamma(2-2\rho)}{\Gamma(2\rho)}\left[\frac{\gamma^{3}(\rho)}{\gamma(3\rho-1)}\right]^{1/2}\frac{\gamma(n+(1-m)\rho)}{\gamma(1+n-(1+m)\rho)}; \qquad (4.18b)$$

$$\mathbb{C}_{(1,3)(n,m)(n,m+2)} = \frac{2\rho-1}{(m+1)\rho-n}\left[\frac{\gamma(2-3\rho)}{\gamma(1-\rho)}\;\frac{\gamma(n-m\rho)}{\gamma(1+n-(1+m)\rho)}\right]^{1/2}, \qquad (4.18c)$$

where ρ is given in (4.10) and $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

As is already mentioned in the introduction, the unitarity condition imposes a very strong restriction on the choice of admissible representations of $\mathcal{V}\times\overline{\mathcal{V}}$ just in the region C<1 [15]. In [15] it is shown that in this region the unitarity condition can be satisfied only if C takes the value (1.11), where only representations $[\Delta_{(n,m)}]$ of dimension $\Delta_{(n,m)}$ given by the Kac's formula (4.12) with q=p-1 are unitary at given value ρ. All these correspond exactly to the series $\mathcal{M}_p \equiv \mathcal{M}(\frac{p}{p+1})$; p=3,4,5,$\cdots$ of minimal models (4.11) which we will call the "unitary series". Below we will see some simple models of unitary series.

Primary fields appearing in $\mathcal{M}_p$ are suitably organized in a rectangular table of size p×(p-1) (in view of (4.13) one field $\Phi_{(n,m)}$ corresponds to two cells of such table).

1/2	1/16	0
0	1/16	1/2

figure 1.

An example of such table for the models $\mathcal{M}_3$ (C=1/2) is drawn in figure 1, where the values coresponding to dimensions (4.12) are placed in cells. First of all we remark that the field $\Phi_{(2,1)} = \Phi_{(1,3)}$ is of dimension $\frac{1}{2}$. The locality condition allows the existence of the Fermi fields ψ and $\overline{\psi}$ of dimensions (1/2,0) and (0,1/2) respectively. These fields satisfy equation (3.19), i.e. $\partial_{\bar{z}}\psi = \partial_z\bar{\psi} = 0$ and the operator expansion (4.9), (4.14): $\psi(z)\psi(0) = z^{-1} + \text{reg}$, $\bar{\psi}(\bar{z})\bar{\psi}(0) = \bar{z}^{-1} + \text{reg}$, $\psi(z)\bar{\psi}(0) = i[\varepsilon(0,0)+\cdots]$, where ε is the primary Bose field of dimension $(\frac{1}{2},\frac{1}{2})$. Thus the fields $\psi(z)$ and $\bar{\psi}(\bar{z})$ are, in fact, the "right" and "left" components of the free massless Majorana Fermi field. The energy-momentum tensor $T(z) = -\frac{1}{2} :\psi(z)\partial_z\psi(z):$ satisfies (3.9) with $C = \frac{1}{2}$. The spinless primary field $\sigma = \Phi_{(1,2)} = \Phi_{(2,2)}$ of dimension $(\frac{1}{16},\frac{1}{16})$ is local w.r.t. ε, but in view of (4.9),

$$\psi(z)\sigma(0,0) = \sum_{n=0}^{\infty} z^{n-1/2}\, a_{-n}\, \sigma(0,0) \ , \tag{4.19}$$

i.e. the field σ cannot be local w.r.t. ψ and $\bar{\psi}$. From (4.19) it is obvious that the product $\psi(z)\sigma(0,0)$ changes a sign by the analytic continuation on z along a closed contour around the point 0. We will say in this and similar cases that the field ψ is "$\frac{1}{2}$-local" w.r.t. σ. The operators a_n defined by the expansion of the form (4.19) form a Clliford algebra $\{a_n, a_m\} = \delta_{n+m,0}$, where { , } is the anticommutator. From the properties mentioned above, it is clear that the model $\mathcal{M}_3$ describes a critical point of the Ising model [40], where the fields σ and ε correspond to the local magnetization and the density of the energy respectively. The field $\mu = a_0\sigma$ has also dimension $(\frac{1}{16},\frac{1}{16})$ and $\frac{1}{2}$-local w.r.t. σ; this field describes "disorder parameters"[40]. One can find in [8,41] calculations of correlation functions of the fields σ and μ in the critical Ising model on the basis of differential equations (4.8).

Now we consider the model $\mathcal{M}_4$ ($C = \frac{7}{10}$); the corresponding table of primary fields is drawn in figure 2.

[Section 4]

3/2	3/5	1/10	0
7/16	3/80	3/80	7/16
0	1/10	3/5	3/2

figure 2

The field $\Phi_{(3,1)} = \Phi_{(1,4)}$ is of dimension $\frac{3}{2}$, hence one can construct local Fermi fields $S(z)$ and $\overline{S}(\overline{z})$ of dimension $(\frac{3}{2},0)$ and $(0,\frac{3}{2})$ respectively. By studying the operator expansion (4.9) in this case, one can show that the fields $S(z)$ and $\overline{S}(\overline{z})$ are generators of the superconformal symmetry of the model $\mathcal{M}_4$ (see §5). The existence of the "higher" symmetry in the model $\mathcal{M}_4$ allows the classification by representations of the Neveu-Schwarz-Ramond algebra NSR (§5). So the four fields $\Phi = \Phi_{\left(\frac{1}{10},\frac{1}{10}\right)}$, $\Psi = \Phi_{\left(\frac{3}{5},\frac{1}{10}\right)}$, $\overline{\Psi} = \Phi_{\left(\frac{1}{10},\frac{3}{5}\right)}$, $\tilde{\Phi} = \Phi_{\left(\frac{3}{5},\frac{3}{5}\right)}$ are the components of the superfield $\underline{\Phi}_{\left(\frac{1}{10},\frac{1}{10}\right)}(z,\theta;\overline{z},\overline{\theta})$, and the corresponding conformal families are combined into the "superconformal family" $[\underline{\Phi}_{\left(\frac{1}{10},\frac{1}{10}\right)}]_{NS}$. Analogously the fields S and $\overline{S}$ are in a representation of the "superconformal family" $[I]_{NS}$. The fields $\sigma = u_{\left(\frac{3}{80},\frac{3}{80}\right)}$ and $\sigma' = \Phi_{\left(\frac{7}{16},\frac{7}{16}\right)}$ are $\frac{1}{2}$-local w.r.t. S and $\overline{S}$ and generate a representation of the Ramond algebra [21]. In [15,21] it is shown that the model $\mathcal{M}_4$ describes the tricritical behavior of statistical systems with the Ising symmetry, where the dimensions of the fields σ and σ' describe the exponents of the magnetic spin operators and so do the dimensions of the fields Φ and $\tilde{\Phi}$ the "thermal" exponents at tricritical points.

3	13/8	2/3	1/8	0
7/5	21/40	1/15	1/10	2/5
2/5	1/10	1/15	21/40	7/5
0	1/8	2/3	13/8	3

figure 3

The table of fields of the model $\mathcal{M}_5$ is drawn in figure 3. Here there exist Bose fields $W(z)$ and $\overline{W}(\overline{z})$ of dimension

[Section 4]

(3,0) and (0,3). These fields also represent some "higher" symmetry ("W-algebra", §6) in the model $\mathcal{M}_5$. It is shown in [15] that $\mathcal{M}_5$ describes the critical point of the 3-state Potts model, where the fields $\sigma = \Phi_{\left(\frac{1}{15},\frac{1}{15}\right)}$ and $\varepsilon = \Phi_{\left(\frac{2}{5},\frac{2}{5}\right)}$ correspond to order parameters and the energy density. The structure of the operator algebra $\mathcal{M}_6$ will be understood further in the light of the "higher" symmetry in this model, discussed in §6. Moreover we mention that the model $\mathcal{M}_6$ describes the tricritical behavior of the 3-state Potts model [15].

We remark that the "higher" symmetry is a general property of the model $\mathcal{M}_p$. In fact, the field $\Phi_{(p-1,1)} = \Phi_{(1,p)}$ is of dimension $s_p = (p-1)(p-2)/4 \in \frac{1}{2}\mathbb{Z}$, hence one can construct the local "currents" $\Psi_p(z)$ and $\bar{\Psi}_p(\bar{z})$ of dimension $(s_p,0)$ and $(0,s_p)$. The corresponding algebras for $p \geq 6$ are not yet studied.

Finally we mention that the models $\mathcal{M}_p$, $p=4,5,6,\cdots$ describe multicritical points of statistical systems with a scalar order parameter φ and the Ising symmetry $\varphi \longrightarrow -\varphi$. In the Lagrangean field theory, such "p-1-critical" points are described by effective actions of the form*)

$$\mathcal{H}_p = \int [\frac{1}{2}\, \partial_\mu\varphi\, \partial_\mu\varphi + g\, \varphi^{2p-2}]\, d^2x\ . \tag{4.20}$$

In addition, the field $\varphi = \Phi_{(2,2)}$ of $\mathcal{M}_p$ is identified with the "fundamental" field φ in (4.20); remaining fields $\Phi_{(n,m)}$ in $\mathcal{M}_p$ correspond (by a suitable choice of the regularization [43]) to components of fields. For example, $\Phi_{(n,n)} = \,:\varphi^{n-1}:$, $1 \leq n \leq p-1$; $\Phi_{(n+1,n)} = \,:\varphi^{p-4+n}:$, $1 \leq n \leq p-2$ [43].

[Section 4]

*) In order to avoid the misunderstanding we emphasize that the field theory (4.20) has conformaly invariant solutions only at special coupling constants $g = g_*$ and counterterms φ^{2n}, $n=1,2,\cdots,p-2$ (depending on the regularization). Of course in a perturbation domain $g \ll p-2$, the theory (4.20) has no conformal solutions.

[Section 4]

§5. Superconformal Field Theory

The superconformal symmetry in 2-dimensional field theory[19-21] is generated by the "right" and "left" supercurrents $S(z)$ and $\bar{S}(\bar{z})$ which are Fermi fields of dimensions $(\frac{3}{2},0)$ and $(0,\frac{3}{2})$ respectively. The algebra of generators is defined, analogously as (3.9), by the singular terms of the operator expansion

$$S(z_1)S(z_2) = \frac{2C}{3(z_1-z_2)^3} + \frac{2}{z_1-z_2} T(z_2) + \text{reg}, \tag{5.1}$$

and the same expansion for $\bar{S}$. The field S(or $\bar{S}$) is a primary field and satisfies (3.11). The operator expansion (5.1) and (3.11) has an obvious symmetry $T \longrightarrow T$, $S \longrightarrow -S$, hence fields of two types can exist in the superconformal theory: the Neveu-Schwarz field A_{NS} is local w.r.t. S and $\bar{S}$ and the Ramond field A_R is $\frac{1}{2}$-local (see §4) w.r.t. these currents. We denote by $\mathcal{A}_{NS}$(or $\mathcal{A}_R$) the subspace of Neveu-Schwarz (or Ramond) fields in $\mathcal{A}$: $\mathcal{A} = \mathcal{A}_{NS} \oplus \mathcal{A}_R$. The operator expansion

$$S(z_1)A_*(z_2,\bar{z}_2) = \sum_k (z_1-z_2)^{-k-3/2}\, S_k\, A_*(z_2,\bar{z}_2) \tag{5.2}$$

defines the operators S_k, where $k \in \mathbb{Z}+\frac{1}{2}$ if $*$=NS, and $k \in \mathbb{Z}$ if $*$=R. It is obvious that the subspaces $\mathcal{A}_{NS}$ and $\mathcal{A}_R$ are invariant under the actions of S_k. These operators, together with L_n(3.5), form the Neveu-Schwarz-Ramond algebra NSR with (anti)commutation relations

$$\begin{aligned} [L_n,\, S_k] &= \tfrac{1}{2}(n-2k)\, S_{n+k}\ ; \\ \{S_k,\, S_\ell\} &= 2L_{\ell+k} + \tfrac{C}{3}(k^2-\tfrac{1}{4})\, \delta_{k+\ell,0}\ , \end{aligned} \tag{5.3}$$

which are easily obtained from (5.1) and (3.11). The analogously

[Section 5]

defined operators $\bar{S}_k$ satisfy the same relations as (5.3) and anti-commutate with S_k. Thus the complete symmetry of the superconformal theory is $NSR \times \overline{NSR}$.

The arguments, analogous to those given in §3, show that the spaces $\mathcal{A}_{NS}$ and $\mathcal{A}_R$ are decomposed into "superconformal families": $\mathcal{A}_{NS} = \bigoplus_{\ell} [\Phi_\ell]_{NS}$, $\mathcal{A}_R = \bigoplus_{\lambda} [\Phi_\lambda]_R$, where the "primary superconformal" (which is called simply "primary" in this paragraph) fields Φ_ℓ, Φ_λ satisfy (3.10) and

$$S_k \, \Phi = \bar{S}_k \, \Phi = 0 \qquad \text{for } k>0 \, , \tag{5.4}$$

and the spaces $[\Phi_\ell]_{NS}$ and $[\Phi_\lambda]_R$ are constructed analogously as (3.16) by means of the operators $L_n, \bar{L}_n, S_k, \bar{S}_k$ with $n,k<0$ and realize irreducible representations of $NSR \times \overline{NSR}$.

As for the conformal families in §3, superconformal families are completely determined by the dimensions $(\Delta, \bar{\Delta})$ of their own primary feilds.

One must remark that the operators

$$\delta_\varepsilon = \oint \frac{dz}{2\pi i} \, \varepsilon(z) \, T(z) \quad ; \qquad \delta_\omega = \oint \frac{dz}{2\pi i} \, \omega(z) \, S(z) \tag{5.5}$$

can be interpreted as generators of infinitesimal transformations of the coordinates $(Z;\bar{Z}) = (z,\theta;\bar{z},\bar{\theta})$ of the 2+2-dimensional superspace

$$z \longrightarrow z + \varepsilon(z) - \omega(z)\theta \quad ; \quad \theta \longrightarrow \theta + \frac{1}{2} \, \varepsilon'(z) + \omega(z) \, , \tag{5.6}$$

where θ and $\bar{\theta}$ are odd coordinates and ε(or ω) is an even(or odd) infinitesimal analytic function. The distinctive property of (5.6) isthat it is a conformal transformation of the 1-form $dz+\theta d\theta$. The fields $S(z)$ and $T(z)$ can be considered as the components of the "super stress tensor":

$$S(z,\theta) = S(z) + 2\theta T(z) \ , \qquad (5.7)$$

and each primary Neveu-Schwarz field Φ_ℓ is a component of the superfield

$$\mathbf{\Phi}_\ell(Z,\overline{Z}) = \Phi_\ell(z,\overline{z}) + \theta\Psi_\ell(z,\overline{z}) + \overline{\theta}\overline{\Psi}_\ell(z,\overline{z}) + i\theta\overline{\theta}\,\tilde{\Phi}_\ell(z,\overline{z}) \qquad (5.8)$$

where $\Psi_\ell = S_{-1/2}\Phi_\ell$, $\overline{\Psi}_\ell = \overline{S}_{-1/2}\Phi_\ell$, $\tilde{\Phi}_\ell = -i\ S_{-1/2}\ \overline{S}_{-1/2}\Phi_\ell$. The superfield (5.8) satisfies the equations

$$S_{-1/2}\ \mathbf{\Phi}_\ell = -2\Delta_\ell\theta\ \mathbf{\Phi}_\ell\ ; \qquad L_0\ \mathbf{\Phi}_\ell = (\Delta_\ell + \frac{1}{2}\theta\frac{\partial}{\partial\theta})\ \mathbf{\Phi}_\ell ;$$
$$S_{-1/2}\ \mathbf{\Phi}_\ell = (\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})\ \mathbf{\Phi}_\ell , \qquad (5.9)$$

where Δ_ℓ is the "right" dimension of the field Φ_ℓ.

Since the field $S_k A_R$ can't be local w.r.t. A_R, the space $\mathcal{A}_R$ is naturally decomposed into two classes of the locality: $\mathcal{A}_R = \mathcal{A}_R^{(+)} \oplus \mathcal{A}_R^{(-)}$, where all fields are mutually local in each classes and any field $A_R^{(+)}$ is $\frac{1}{2}$-local w.r.t. $A_R^{(-)}$. The operators S_k act in $\mathcal{A}_R$ as S_k: $\mathcal{A}_R^{(\varepsilon)} \longrightarrow \mathcal{A}_R^{(-\varepsilon)}$; $\varepsilon=\pm$. In particular, primary Ramond fields are actually represented by "doublets" of fields $\Phi_\lambda^{(\varepsilon)} \in \mathcal{A}_R^{(\varepsilon)}$, and the operators S_0 and $\overline{S}_0$ act on them as 2×2-matrices. For example, for a spinless field Φ_λ of dimension $(\Delta_\lambda, \overline{\Delta}_\lambda)$

$$S_0\Phi_\lambda^{(\varepsilon)} = 2^{-3/2}(1+\varepsilon i)\beta_\lambda\Phi_\lambda^{(-\varepsilon)}; \quad \overline{S}_0\Phi_\lambda^{(\varepsilon)} = 2^{-3/2}(1-\varepsilon i)\beta_\lambda\Phi_\lambda^{(-\varepsilon)}, \qquad (5.10)$$

where the parameter β is related to Δ_λ by the equality $\Delta_\lambda - \frac{\hat{C}}{16} = \frac{1}{4}\beta_\lambda^2$;

we use the notation $\hat{C} = 2C/3$ here and below. Only one possible exception is the Ramond field $\Phi_{(0)}$ of dimension $\Delta_{(0)} = \hat{C}/16$ (the "Ramond vacuum"), if it appears in the theory. For it, $S_0\Phi_{(0)} = \bar{S}_0\Phi_{(0)} = 0$ and the second component is not necessary to exist (for definiteness one can consider that $\Phi_{(0)} \in \mathcal{A}_R^{(+)}$).

Degenerate irreducible representations of the NSR algebra are determined in the same way as for the Virasoro algebra (see §4). In this case, all cases of degeneracies are enumerated by the following formula very much similar to (4.9):

$$\Delta_{(n,m)} = \Delta_0 + \frac{1}{4}(n\beta_+ + m\beta_-)^2 + \frac{1}{32}(1-(-1)^{n+m}) \ , \qquad (5.11)$$

where $\Delta_0 = (\hat{C}-1)/16$, and

$$\beta_\pm = \frac{1}{4}(\sqrt{1-\hat{C}} \pm \sqrt{9-\hat{C}}) \ ; \ \beta_+\beta_- = -\frac{1}{2} \ . \qquad (5.12)$$

Here n and m are natural numbers, where (5.11) is related to Neveu-Schwarz (Ramond) representations, if $n+m \in 2\mathbb{Z}$ ($n+m \in 2\mathbb{Z}+1$ resp.). The formula (5.11) is discovered by Kac [32] as for (4.7). The degenerate primary fields $\Phi_{(n,m)}$ ($\Phi_{(n,m)} \in \mathcal{A}_{NS}$ if $n+m \in 2\mathbb{Z}$; $\Phi_{(n,m)} \in \mathcal{A}_R$ if $n+m \in 2\mathbb{Z}+1$) in a superconformal theory have fundamental properties similar to degenerate conformal fields in §4 [19-21]. Namely correlation functions containing degenerate superconformal fields satisfy linear differential equations(whose examples can be found in [19,21,44]), and degenerate superconformal families form a closed operator algebra with the same structure as (4.9).

By choosing the parameter C such that the quantity $\rho = -\beta_-/\beta_+$ takes a rational value, one can get a closed operator algebra containing a finite set of superconformal families of the form (4.11) —— "minimal superconformal models" [19-21]. Basic interest is the "unitary series" $\mathcal{SM}_p; p=3,4,5,\cdots$ of minimal models [21] corresponding to the choice

$$\rho = \frac{p}{p+2} \quad ; \tag{5.13}$$

moreover C takes the value (1.12). The space $\mathcal{A}$ of fields of the model $\mathcal{SM}_p$ contains $[p^2/2]$ primary fields $\Phi_{(n,m)}$; $n=1,2,\cdots,p-1$; $m=1,2,\cdots,p+1$ ([ ] stands for the integral part), where $\Phi_{(p-n,p+2-m)} = \Phi_{(n,,m)}$ and is of dimension

$$\Delta_{(n,m)} = \frac{((p+2)n-pm)^2-4}{8p(p+2)} + \frac{1}{32}(1-(-1)^{n+m}) \ . \tag{5.14}$$

In [44] we calculated the values of the structure constants (in the formula analogous to (3.27) guaranteed by the associativity of the operator algebra $\mathcal{SM}_p$. We remark that the operator algebra $\mathcal{SM}_p$ has various symmetries depending on the parity of the number p. So for $p\in 2\mathbb{Z}+1$, the spaces $\mathcal{A}_R^{(+)}$ and $\mathcal{A}_R^{(-)}$ are isomorphic to each other, and the model $\mathcal{SM}_p$ is symmetric w.r.t.the duality transformation" (similar to Kramers-Wannier symmetry of the Ising model[40]); $\mathcal{A}_R^{(+)} \longleftrightarrow \mathcal{A}_R^{(-)}$. For $p\in 2\mathbb{Z}$, the model $\mathcal{SM}_p$ contains the "Ramond vacuum" $\Phi_{(p/2,p/2+1)}\in\mathcal{A}_R$ [21] and does not have this symmetry. At the same time the model $\mathcal{SM}_p$ with even p has $Z_2\times Z_2$-symmetry of the following form

$$\Phi_{(n,m)} \longrightarrow (\varepsilon_1)^{n+1}\ (\varepsilon_2)^{m+1}\ \Phi_{(n,m)} \ ;$$

ε_1 and ε_2 are arbitrary signs: $\varepsilon_1,\varepsilon_2 = \pm 1$.

It is easily shown that $\mathcal{SM}_3 = \mathcal{M}_4$ (§4); this model describes a tricritical point of the Ising model. The model $\mathcal{SM}_3$ (C=1) corresponds to a special point of the Gaussian model (see the introduction). We remark that $\mathcal{SM}_4$ has actually N=2 extended supersymmetry*) [21] (other models $\mathcal{SM}_p$ with $p\in 2\mathbb{Z}$ also have "higher symmetries"). The problem of the "physical" interpretation of fixed points of $\mathcal{SM}_p$ with $p>4$ remains open basically (but see [46]). The connection of the model $\mathcal{SM}_p$ with Lagrangean field theories are treated in [43].

*) Models of conformal field theory with N=2 extended supersymmetry is studied in [28], where there are references on earlier works.

[Section 5]

§ 6. "Parafermionic" and other Symmetries.

In §3, a conformal field theory is formulated as a closed associative algebra $\mathcal{A}$, where the space $\mathcal{A}$ contains mutually-local fields. In practice it is useful to decompose this space, by introducing special non-local fields(actually this is already done in §5 in the consideration of superconformal field theory). We will say that a field A(x) is γ-local w.r.t. B(x), if the product $A(x_1)B(x_2)$ gains the phase multiplier $\exp(2\pi i\gamma)$ by continuation, for example, in the variable x_1 along a closed contour around the point x_2 (counter-clockwise, figure 4). Relations of the operator algebra (1.5) are transcribed for such fields without essential changes; basic difference is that the coefficients $C^k_{ij}(x)$ are no longer single-valued functions of $x \in \mathbb{R}^2$.

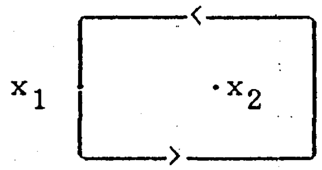

figure 4

Naturally γ-local fields appear in statistical systems with Z_N-symmetry [27]. In such system (in a continuous limit), there are N-1 components σ_k, k=1,2,$\cdots$,N-1 of the "order parameter" correspon-ding to N-1 representaions of Z_N;

$$\Omega\ \sigma_k = \omega^k\ \sigma_k\ , \tag{6.1}$$

where Ω is the generator of the group Z_N: $\Omega^N = E$ and $\omega = \exp(2\pi i/N)$. Besides there are N-1-component fields μ_ℓ, ℓ=1,2,$\cdots$,N-1, which are "disorder parameters" [27]; the fields μ_ℓ are mutually-local, but μ_ℓ are $\frac{k\ell}{N}$-local relative to σ_k. The "dual" group $\tilde{Z}_N$ acts on the fields μ_ℓ:

[Section 6]

$$\tilde{\Omega}\ \mu_\ell = \omega^\ell\ \mu_\ell\ , \qquad (6.2)$$

where $\tilde{\Omega}$ is the generator of $\tilde{Z}_N$. Thus the group $Z_N \times \tilde{Z}_N$ appears actually in a Z_N-symmetric field theory. We will say that the field $A_{(k,\ell)}$ is of $Z_N \times \tilde{Z}_N$-charge (k,ℓ), if $\Omega\ A_{(k,\ell)} = \omega^k A_{(k,\ell)}$ and $\tilde{\Omega}\ A_{(k,\ell)} = \omega^\ell A_{(k,\ell)}$; clearly the numbers k and ℓ are defined modulo N. The fields σ_k and μ_ℓ have charge $(k,0)$ and $(0,\ell)$ respectively. It is not difficult to show that the field $A_{(k,\ell)}$ is γ-local w.r.t. $A_{(k',\ell')}$ with $\gamma = (k\ell'+k'\ell)/N$. By taking "fusions" of the fields σ_k and μ_ℓ in all possible ways one can construct the space of γ-local fields

$$\mathcal{A} = \bigoplus_{k=0}^{N=1} \bigoplus_{\ell=0}^{N=1} \mathcal{A}_{(k,\ell)}\ , \qquad (6.3)$$

generating an algebra closed w.r.t. the operator expansions, where

$$A_{(k,\ell)} A_{(k',\ell')} \in \mathcal{A}_{(k+k',\ell+\ell')}\ . \qquad (6.4)$$

For each $N=2,3,4,\cdots$, a $Z_N \times \tilde{Z}_N$-model of a conformal field theory is constructed in [28], which is characterized by the presence of special "parafermionic symmetries". Namely it is assumed that the space (6.3) contains the fields $\psi_k \in A_{(k,k)}$ and $\bar{\psi}_k \in A_{(k,-k)}$, $k=1,2,\cdots,N-1$ (where $\psi_k^\dagger = \psi_{N-k}$ and $\bar{\psi}_k^\dagger = \bar{\psi}_{N-k}$) which satisfy the equations

$$\partial_{\bar{z}}\ \psi_k = \partial_z\ \bar{\psi}_k = 0\ ,$$

i.e. $\psi_k = \psi_k(z)$ and $\bar{\psi}_k = \bar{\psi}_k(\bar{z})$. The fields ψ_k and $\bar{\psi}_k$ are of dimensions $(\Delta_k, 0)$ and $(0, \Delta_k)$ respectively, where

$$\Delta_k = \frac{k(N-k)}{N} \qquad (6.5)$$

(spins of non-local fields can be fractional). The fields $\psi_k(z)$, the "parafermionic currents", generate a closed algebra, defined by the operator expansions

$$\psi_{k_1}(z_1)\psi_{k_2}(z_2) = C_{k_1,k_2}(z_{12})^{-2k_1k_2/N}\{\psi_{k_1+k_2}(z_2)+\cdots\};\ k_1+k_2\neq 0 \qquad (6.6a)$$

$$\psi_k(z_1)\psi_k^{\dagger}(z_2) = (z_{12})^{-2\Delta_k}\ \{I+\frac{2\Delta_k}{C}(z_{12})^2\ T(z_2)\ +\cdots\}\ , \qquad (6.6b)$$

where T(z) is the "right" component of the energy momentum tensor, which generates the Virasoro algebra (1.9) with the central charge C, and $C_{k_1k_2}$ are some numerical coefficients. It can be shown [28] that the requirement of the associativity of the operator algebra (6.6) determines completely the parameters C_{k_1,k_2}:

$$C_{k_1,k_2} = \frac{(k_1+k_2)!(N-k_1)!(N-k_2)!}{(N-k_1-k_2)!\ k_1!\ k_2!} \qquad (6.7)$$

and the values of the central charges $C=C_N$, where

$$C_N = \frac{2(N-1)}{N+2}\ . \qquad (6.8)$$

The fields consistuting the space (6.4) of such model can be classified by representations of the "algebra of parafermionic currents" (6.6) (and the similar algebra generated by $\bar{\psi}_k(\bar{z})$). By omitting the details (which can be found in [28]), we remark that the algebra uniquely determines the structure of the space (6.4) (This phenomenon is similar to the case in which the equation of type (4.15) determines the structure of the space $\mathcal{A}(p/q)$). Namely the space (6.3) is decomposed into N "parafermionic families"*)

$$\mathcal{A} = \bigoplus_{k=0}^{N-1} [\sigma_k]_\psi , \tag{6.9}$$

where $\sigma_k \in \mathcal{A}_{(k,0)}$ is a spinless "primary field" of dimension (d_k, d_k),

$$d_k = \frac{k(N-k)}{N(N+2)} , \tag{6.10}$$

and the subspace $[\sigma_k]_\psi$ contains all fields obtained by "fusions" of any currents ψ_ℓ, $\bar{\psi}_\ell$ with σ_k. The fields σ_k of $Z_N \times Z_N$-charge (k,0) are identified with the components of the "order parameter". In particular, each subspace $[\sigma_k]_\psi$ contains a series of the fields $\Psi_k^{(m)} \in A_{(k-m,-m)}$ of dimension $(d_k^{(m)}, d_k)$, where

$$d_k^{(m)} = d_k + \frac{m(k-m)}{N} . \tag{6.11}$$

Clearly by this formula, the field $\mu_k^\dagger = \mu_{N-k} = \Psi_k^{(k)} \in \mathcal{A}_{(0,N-k)}$ is of spin 0 and the same dimension (d_k, d_k) as for σ_k. The fields μ_k, k=1, 2,···,N-1 are the components of the "disorder parameter". In [28] it is shown that the space (6.9) forms an associative operator algebra, and the corresponding structure constants and some correlation

*) In mathematical points of view, representations of the "algebra of parafermionic currents" (6.6) are closely related with the representations of the Kac-Moody algebra $\hat{su}(2)$ [28].

[Section 6]

functions are calculated. Therefore for each N=2,3,4,⋯ the space (6.9) represents a conformal field theory which we denote by $\mathfrak{Z}_N$. The operator algebra has a $Z_N \times \tilde{Z}_N$-symmetry and the "selfduality", i.e. the invariance w.r.t. the exchange $\sigma_k \longleftrightarrow \mu_k$. It can be shown that all models Z_N are unitary [28].

One can say that the model $\mathfrak{Z}_2$ coincides with $\mathcal{M}_3$ and $\mathfrak{Z}_3 = \mathcal{M}_5$. These models describe critical points of the Ising model and the 3-state Potts model respectively. The model $\mathfrak{Z}_4$(C=1) is a special case of the Gaussian field theory and the model $\mathfrak{Z}_6$ has a superconformal symmetry (generated by the supercurrents $S=\psi_3$ and $\bar{S}=\bar{\psi}_3$; $\Delta_3 = 3/2$) and coincides with the model $\mathcal{SM}_6$ (see §5).

The models $\mathfrak{Z}_N$ with N≥5 describe "critical points of bifurcation" of Z_N-Ising models. In order to explain the problem, we consider a system of "spin" $\sigma_{\vec{r}}$ placed at nodes $\vec{r}$ of a 2-dimensional rectangular lattice. The "spins" $\sigma_{\vec{r}}$ take values in the group Z_N, i.e. $\sigma_{\vec{r}} = \omega^{(\vec{r})}$; $n(\vec{r})=0,1,2,\cdots,N-1$. The "$Z_N$-Ising model" corresponds to the choice of the partition fucntion of the form

$$P\{\sigma_{\vec{r}}\} = \exp\{-\sum_{\vec{r},\alpha=1,2} \mathcal{H}(\sigma_{\vec{r}},\sigma_{\vec{r}+\vec{e}_\alpha})\} = \prod_{\vec{r},\alpha} W(\sigma_{\vec{r}},\ \sigma_{\vec{r}+\vec{e}_\alpha}) \qquad (6.12)$$

($\vec{e}_\alpha$ are the basis vectors of the lattice), i.e. the Hamiltonian $\mathcal{H}$ takes into acount interactions only of the nearest neighbours. The function $W(\sigma,\sigma')$ is represented in the form

$$W(\sigma,\sigma') = \sum_{k=0}^{N-1} w_k\ (\sigma^* \sigma')^k \qquad (6.13)$$

with positive coefficients $w_k = w_{N-k}$, i.e. the partition function

(6.12) depends on $[\frac{N}{2}]$ parameters w_k, $k=1,2,\cdots\leq\frac{N}{2}$ $(w_0=1)$. Depending on the values of these parameters, the system can be divided into the three phases*): I) $\langle\sigma\rangle \neq 0$, $\langle\mu\rangle = 0$; II) $\langle\sigma\rangle = 0$, $\langle\mu\rangle \neq 0$; III) $\langle\sigma\rangle = \langle\mu\rangle = 0$ (see figure 5 where the phase diagram of the Z_5-models is drawn). All three phases meet together at the bifurcation points (b and b^* in figure 5) which are critical and are described by the model $\mathfrak{Z}_N$ [28].

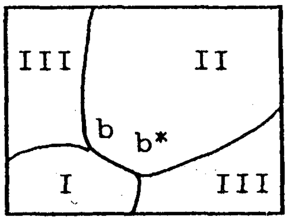

figure 5

The algebra (6.6) of the fileds ψ_k of spin (6.5) is the simplest example of the "algebra of parafermionic currents" associated with the group Z_N. In [29] a more complicated example is studied: the algebra of the "parafermionic currents" $\psi(z)$ and $\psi^{\dagger}(z)$ of spin $\frac{4}{3}$. This algebra is defined by the operator expansions similar to (6.6), however in this case, the associativity condition does not completely determine parameters of the theory and there can be many conformal theories with this symmetry. The corresponding "unitary minimal models" $\mathscr{S}_3\mathscr{M}_p$, p=3,4,5··· are constructed in [29]. These models have $Z_3\times\tilde{Z}_3$-symmetry and the "selfduality", and correspond to the values

$$C_p = 2\left[1 - \frac{12}{p(p+4)}\right] \tag{6.14}$$

of the central charge of the Virasoro algebra. The spectrum of dimen-

*) Here we simplify the situations somewhat. If N is not a prime number, phase diagrams of Z_N-Ising models are more complicated.

[Section 6]

sions of "primary" (in the sense of this algebra of "parafermionic currents") fields in the model $\mathcal{S}_3\mathcal{M}_p$ has the following form

$$\Delta_{(n,m)} = \frac{((p+4)n-pm)^2-16}{16p(p+4)} + \frac{1}{12}\left[1 - \cos^4\left(\frac{\pi(n-m)}{4}\right)\right] ; \qquad (6.15)$$

$n=1,2,\cdots,p-1$; $m=1,2,\cdots,p+3$. The simplest model $\mathcal{S}_3\mathcal{M}_3$ coincides with $\mathcal{M}_6$ and describes a tricritical point of the 3-state Potts model[15]. Besides $\mathcal{S}_3\mathcal{M}_4 = \mathcal{Z}_6 = \mathcal{S}\mathcal{M}_6$. There remains the "physical meaning" of other fixed points of $\mathcal{S}_3\mathcal{M}_p$ to be understood.

One can assume that the unitary series $\mathcal{M}_p$, $\mathcal{S}\mathcal{M}_p$ and $\mathcal{S}_3\mathcal{M}_p$ are representations of the "family of series" $\mathcal{M}_p^{(q)}$ enumerated by the number $q=1,2,3\cdots$ and corresponding to the values

$$C_p^{(q)} = \frac{3q}{q+2}\left[1- \frac{2(q+2)}{p(p+q)}\right] ; \ p=3,4,5,\cdots \qquad (6.16)$$

of the central charge in (1.9) such that $\mathcal{M}_p \equiv \mathcal{M}_p^{(1)}$, $\mathcal{S}\mathcal{M}_p = \mathcal{M}_p^{(2)}$ and $\mathcal{S}_3\mathcal{M}_p = \mathcal{M}_p^{(4)}$. Actually the Goddard-Kent-Olive constrution [47] (of which we will not here go into detailed discussion) allows to construct some "stock" for spaces of fields of the models $\mathcal{M}_p^{(q)}$. However the corresponding associative operator algebras are not yet constructed; besides there remains an open problem of the symmetry of the models $\mathcal{M}_p^{(q)}$.

In conclusion, we refer one more kind of "higher" symmetries of a conformal field theory ——the "W-algebra" [26,48]. These symmetries are generated by local currents of higher spin $s\geq 3$. The simplest example of the "W-algebras" is generated by the currents $T(z)$ and $W(z)$, where $T(z)$ is the "right" component of the energy-momentum tensor of spin 2, and W is an additional current of dimension (3,0) considered in [26]. This algebra is defined by singular terms of the operator expansion (3.9) and

$$T(z_1)W(z_2) = \frac{3}{z_{12}^2} W(z_2) + \frac{1}{z_{12}} W'(z_2) + \text{reg} ; \qquad (6.17a)$$

$$W(z_1)W(z_2) = \frac{C}{3z_{12}^6} + \frac{2}{z_{12}^4} T(z_2) + \frac{1}{z_{12}^3} T'(z_2) + \frac{3}{10}\frac{1}{z_{12}^2} T''(z_2)$$

$$+ \frac{1}{15}\frac{1}{z_{12}} T'''(z_2) + \frac{2b}{z_{12}^2} T_4(z_2) + \frac{b}{z_{12}} T_4'(z_2) + \text{reg} ; \qquad (6.17b)$$

where $b = \frac{16}{22+5C}$, prime means the differentiation and the field $T_4(z)$ is defined in §3,(3.20). By introducing the operators W_n; $n=0, \pm1, \pm2, \cdots$ similarly to (3.5),(3.22) (5.2), one can express (6.17) in the commutation relations

$$[L_n, W_m] = (2n-m) W_{n+m} ; \qquad (6.18a)$$

$$[W_n, W_m] = (n-m) \left\{\frac{(n+m+2)(n+m+3)}{15} - \frac{(n+1)(m+2)}{6}\right\} L_{n+m} +$$

$$+ b(n-m) \Lambda_{n+m} + \frac{C}{3\cdot 5!} (n^2-4)(n^2-1)n\, \delta_{n+m,0} , \qquad (6.18b)$$

where Λ_n is the operator (3.23). It should be emphasized that the operators L_n and W_m are not generators of any Lie algebra, since (6.18b) includes the operators Λ_n. Rather (6.18) should be regarded as an example of associative algebras with quadratic defining relations. Similar family of algebras ("Yang-Baxter algebras") play a central role in the quantum method of the inverse problem(see [49]

[Section 6]

for example). Furthermore one should understand the "geometric meaning" of "symmetries" similar to (6.18) (the problem seems principally important for me).

As is already remarked in §4, the space of fields of the model $\mathcal{M}_5$ represents a "W-algebra" (6.18). For example, the sum $[\Phi_{2/5}] \oplus [\Phi_{7/5}]$ of conformal families(see figure 3) is an irreducible representation (6.18). Actually the model $\mathcal{M}_5$ is the first representative (p=4) of a series of unitary "minimal models" $W_3\mathcal{M}_p$, p=4,5,6,··· corresponding to the values

$$C_p = 2\left(1 - \frac{12}{p(p+1)}\right)$$

of the parameter C in (1.9), (6.18). These models are constructed in [48].

[Section 6]

§7. Perturbation Theory and RG near Fixed Points.

If the solution $\mathscr{A}_{g_*}$ of a conformal field theory corresponding to some fixed point $g_* \in S$ (see §2) is known, one can try to study behaviors of RG near this point.

Let $\{g^a\}$ be a certain coordinate system in S such that the origin of the coordinates coincides with the fixed point g_*, i.e. $g^a_* = 0$. Then $\beta^a(0) = 0$, where $\beta^a(g)$ are coefficients in (2.11). We assume that the functions $\beta^a(g)$ have Taylor series expansions in powers of g^a. It is clear that the linear term of this expansion is completely determined by the spectrum of anamalous dimensions of spinless fields g=0 in a conformal field theory. We denote $\Phi^0_a = \Phi_a|_{g=0}$, $\Phi^0_a \in \mathscr{A}^{(0)}_{g=0}$, where the fields Φ_a are defined as the derivatives (2.9). Choose suitably the coordinate system $\{g^a\}$ such that the fields Φ^0 have the definite dimension (Δ_a, Δ_a) and orthonormal w.r.t. the metric (2.17), i.e. $G_{ab}(0) = \delta_{ab}$. In addition the operator $\gamma(0)$ (2.13) has a diagonal form $\gamma^b_a(0) = \Delta_a \delta_{ab}$, and we get the well-known expression

$$\beta^a(g) = \varepsilon_a\, g^a + O(g^2) \ , \qquad (7.1)$$

where $\varepsilon_a = 1 - \Delta_a$. The higher terms of the expansion (7.1) can be, in prinple, caluculated by the perturbation theory. Of course, in general, some first terms of this expansion do not allow to predetermine global topological properties of RG. However we consider the case where the dimension of the fields Φ_a is near 1, i.e. $\varepsilon_a - \varepsilon \ll 1$. Moreover one can expect that the nonlinear terms in (7.1) become compatible with the linear term at $g^a \sim \varepsilon$. Thus in this case, RG has nontrivial behaviors (for example, may have other fixed points) in the domain $g^a \lesssim \varepsilon$, where the perturbation theory is applicable: characteistic of RG in this domain can be calculated as a power series in small parameter ε. This corresponds exactly to a basic idea of the ε-epansions [2].

Assume that Φ_a^0 is a primary conformal field, and let us compute the following terms of the expansion (7.1), in the case $\varepsilon_a - \varepsilon \ll 1$. By (2.10) we get

$$\frac{\partial}{\partial g^a} \langle \Phi_b(x) \Phi_c(0) \rangle \Big|_{g=0} = \langle (B_a^0 \Phi_b^0)(x) \Phi_c^0(0) \rangle + \langle \Phi_b^0(x) (B_a^0 \Phi_c^0)(0) \rangle + \int d^2y \ \langle \Phi_a^0(y) \Phi_b^0(x) \Phi_c^0(0) \rangle \ , \tag{7.2}$$

where $B_a^0 = B_a|_{g=0}$. The three-point function in the right hand side of (7.2) is expressed by the structure constants $\mathbb{C}_{abc}$ of the operator expansions (3.27):

$$\langle \Phi_{a_1}^0(x_1) \Phi_{a_2}^0(x_2) \Phi_{a_3}^0(x_3) \rangle = \mathbb{C}_{a_1 a_2 a_3} \ |x_{12}|^{2\delta_{12}} \ |x_{13}|^{2\delta_{13}} \ |x_{23}|^{2\delta_{23}} , \tag{7.3}$$

where $x_{12} = x_1 - x_2$, $\delta_{12} = \Delta_3 - \Delta_1 - \Delta_2$, and so on. The integral in (7.2) is, of course, obtained explicitly; it becomes $\mathbb{C}_{abc} I_{bc}^a (x^2)^{1-\Delta_a-\Delta_b-\Delta_c}$, where

$$I_{bc}^a = 2\pi \ \frac{\Gamma(1+\Delta_b-\Delta_c-\Delta_a)}{\Gamma(\Delta_a+\Delta_c-\Delta_b)} \ \frac{\Gamma(1+\Delta_c-\Delta_b-\Delta_a)}{\Gamma(\Delta_a+\Delta_b-\Delta_c)} \ \frac{\Gamma(2\Delta_a-1)}{\Gamma(2-2\Delta_a)} \ , \tag{7.4}$$

and it is taken for granted that possible divergence of this integral compensates the contribution of the operator B_a^0 in (7.2). In general,

[Section 7]

the operators B_a^0 are suitably chosen as the derivative (7.2) vanishes at the "normalization point", for example, at x=1. This corresponds to a special choice of a coordinate system in S, such that

$$G_{ab}(g) = G_{ab}(g,1) = \delta_{ab} + O(g^2) \ , \qquad (7.5)$$

where $G_{ab}(g)$ is the metric (2.17). With this choice of the coordinate system, we get after not complicated calculations:

$$\gamma_a^b(g) = \Delta_a \ \delta_{ab} + \sum_c C_{bc}^a \ g^c + O(g^2) \ ; \qquad (7.6a)$$

$$\beta^a(g) = \varepsilon_a g^a - \frac{1}{2} \sum_{b,c} C_{bc}^a \ g^b g^c + O(g^3) \ , \qquad (7.6b)$$

where

$$C_{bc}^a = (\varepsilon_b+\varepsilon_c-\varepsilon_a)\mathbb{C}_{abc} \ \tilde{I}_{bc}^a = 2\pi\mathbb{C}_{abc} + O(\varepsilon^3) \ , \qquad (7.7)$$

$$\tilde{I}_{bc}^a = \frac{1}{2} \ (I_{ac}^b + I_{ab}^c - I_{bc}^a) = 2\pi(\varepsilon_b+\varepsilon_c-\varepsilon_a)^{-1} + O(\varepsilon^2) \ . \qquad (7.8)$$

The first expression (7.7) for the coefficients C_{bc}^a in (7.6) is exact. For $\varepsilon \ll 1$ these coefficients $C_{abc} = C_{bc}^a$ are symmetric on its own indices and the β-function(7.6b), to within terms $\sim g^3$, is expressed in the form

$$\beta^a(g) = \frac{-1}{12} \sum_b G^{ab}(g) \ \frac{\partial C(g)}{\partial g^b} \ , \qquad (7.9)$$

where

$$C(g) = C_0 - 6 \sum_a \varepsilon_a g^a g^a + 2 \sum_{abc} C_{abc} g^a g^b g^c + O(g^4). \qquad (7.10)$$

[Section 7]

One can immediatly verify that if C_0 is the central charge of a conformal field g=0, then (7.10) coincides with the expansion of the function C(g), defined in §2.

As an example, we consider a neighborhood of fixed points of $\mathscr{M}_p$ with p>>1 (see §4), and restrict to perturbations of the form

$$\mathscr{H}_\lambda = \mathscr{H}^{(p)} + \lambda \int \Phi^{(p)}(x)\, d^2x \;, \tag{7.11}$$

where $\mathscr{H}^{(p)}$ is the action of the conformal theory $\mathscr{M}_p$ and $\Phi^{(p)} \equiv \Phi_{(1,3)}$. This perturbation is chosen such that the field $\Phi_{(1,3)}$ of $\mathscr{M}_p$ belongs to the subalgebra $\mathscr{A}_{(1,*)} = \bigoplus_{n=1}^{p} [\Phi_{(1,n)}]$ (see (4.9)) and for p>>1 this subalgebra contains the unique field of the following dimension near I:

$$\Delta_{(1,3)} = 1 - \varepsilon \quad ; \qquad \varepsilon = \frac{2}{p+1} \;. \tag{7.12}$$

Hence the corresponding RG can be considered as single-charged. We introduce the field $\Phi_g(x)$ (different from $\Phi^{(p)}$ in (7.11) by a normalization) and a new coupling constant $g = g(\lambda)$ such that the condition

$$G(g) \equiv \langle \Phi_g(x)\Phi_g(0)\rangle \big|_{|x|=1} = 1 \quad ; \quad \Phi_g = \frac{\partial \mathscr{H}_g}{\partial g} \tag{7.13}$$

is satisfied, where $\mathscr{H}_g \equiv \mathscr{H}_{g(\lambda)}$ is the action (7.11), parametrized by the new coupling constant g. Then the field $\Theta_g = -T^{\mu\mu}$ in the perturbation theory (7.11) is expressed in the form $\Theta_g = \beta(g)\Phi_g$, where $\beta(g)$ is given in (7.6b):

$$\beta(g) = \varepsilon g - \frac{2}{\sqrt{3}}\left(1 - \frac{3\varepsilon}{2}\right) g^2 - \frac{4}{3} g^3 + \cdots . \qquad (7.14)$$

Here we do not take into account the value of the structure constant $\mathbb{C}_{(1,3)(1,3)(1,3)} = \frac{4}{\sqrt{3}}\left(1 - \frac{3\varepsilon}{2} + O(\varepsilon^2)\right)$, which is easily obtained from (4.18b), and write the term $\sim g^3$ which needs more complicated calculations. From (7.14) it is clear that there is a new fixed point $g_{*1} = (\sqrt{3}\varepsilon/2)\cdot(1+\varepsilon/2+O(\varepsilon^2))$. Taking into account the unitarity condition valid in $\mathcal{M}_p$ which can not (for real g) be broken in the perturbation theory and the assertion in §2 on the decrease of the function C(g), one can claim in advance that the fixed point g_{*1} corresponds to a certain model $\mathcal{M}_{p'}$ with p'<p. The calculation of the central charge $C_1 = C(g_{*1})$ by the formula (7.10) at the specified accuracy gives

$$C_1 = C_p - \frac{3}{2}\varepsilon^2 - \frac{9}{4}\varepsilon^4 + \cdots = C_{p-1} , \qquad (7.15)$$

i.e. the fixed point g_{*1} describes the model $\mathcal{M}_{p-1}$. The calculation of anomalous dimension at g_{*1} by the formula (7.6a) also confirms this conclusion [17]. Thus the field theory (7.11), with an untraviolet asymptotic $\mathcal{M}_p$, for $\lambda>0$, has also a conformally invariant infrared asymptotic describing the model $\mathcal{M}_{p-1}$.

At an "intermediate" scale, conformal invariance in the perturbation theory (7.11) is certainly broken, and the operators L_n introduced in §3 are no longer integrals of motion (an exception is constituted by the generators (2.6) of euclidean symmetry which remains clearly in (7.11)). However we show that the theory with $\lambda\neq0$ has "higher" integrals of motion and is possibly completely integrable. For this it is more convenient to use expansions in power series of the coupling constant φ (not $g=g(\lambda)$ as happened above), since λ (as the field $\Phi \equiv \Phi^{(p)}$) has definite scale dimension $\lambda \sim R^{-2\varepsilon}$ and is the unique measuring parameter of the theory (7.11) (since $\Delta = \Delta_{(1,3)}<1$, the ultraviolet divergence in (7.11) disappears). In particular, the expression

$$\Theta = \varepsilon\,\lambda\,\Phi \tag{7.16}$$

for $\Theta = -T^{\mu\mu}$ in the theory (7.11) is correct according to the condition (2.15) on RG invariance of the field $T^{\mu\nu}$. Equation (2.19) with $\Theta = \varepsilon\lambda\Phi$ is not difficult, of course, to obtain by analysing the standard series of the perturbation theory (7.11)

$$Z_\lambda \langle X\rangle_\lambda = \sum_{n=0}^{\infty} \int\cdots\int \frac{-\lambda^n}{n!} \langle X\,\Phi(y_1)\cdots\Phi(y_n)\rangle_0\, d^2y_1\cdots d^2y_n\ , \tag{7.17}$$

where X is a product of any fields of the form (1.6), Z_λ is the "state sum" given by the series (7.17) with X=1 and the average in the right hand side is calculated in the unperturbed theory $\lambda=0$, i.e. in $\mathcal{M}_p$. For this, one should substitute $X = T(z,\bar{z})X'$ in (7.17) and use in the right hand side, the operator expansions (3.11) valid at $\lambda=0$.

We consider the field T_4, defined (for $\lambda=0$) by the expression (3.20). For $\lambda=0$, this field satisfies equation $\partial_{\bar{z}}T_4 = 0$. If $\lambda\neq 0$, the derivative $\partial_{\bar{z}}T_4 = F = \lambda F_1 + \lambda^2 F_2 + \cdots$ is a local field of spin 3, where the coefficients $F_n \sim (R^{-1})^{5-2n\varepsilon}$, i.e. they have dimensions $(4-n\varepsilon, 4-n\varepsilon)$. From the structure of the series (7.17), it is seen that these coefficients are constructed from fields belonging to the subalgebra $\mathcal{A}_{(1,**)} = \bigoplus_{\ell=0}^{[(p-1)/2]} [\Phi_{(1,2\ell+1)}]$ of the model $\mathcal{M}_p$. So the field F_1 is of dimension $(4-\varepsilon, 4-\varepsilon)$, hence it is a linear combination of the fields $L_{-3}\Phi$, $L_{-1}L_{-2}\Phi$, $L_{-1}^3\Phi$. However the field $\Phi = \Phi_{(1,3)}$ satisfies equation (4.17) such that $L_{-3}\Phi$ is expressed by the remaining two. Therefore by the remark (3.6), one can write

$$\partial_{\bar{z}}T_4 = \partial_z G_2 + O(\lambda^2) \quad ; \quad G_2 = \varepsilon\lambda(a_1 L_{-2}\Phi + a_2 \partial_z^2 \Phi) \ . \qquad (7.18)$$

The dimensionless coefficients a_1 and a_2 in (7.18) can be calculated by substituting $X = T_4(z)X'$ in the right hand side of (7.17) and using the operator expansions (4.16); in conclusion, one gets

$$a_1 = \frac{2\Delta}{\Delta+2} \ ; \ a_1 = \frac{\Delta}{6} + \frac{1}{30} - \frac{(5\Delta+1)(2\Delta-1)}{(2\Delta+1)(\Delta+1)} - \frac{3\Delta}{(2\Delta+1)(\Delta+2)} \ , \qquad (7.19)$$

where $\Delta = \Delta_{(1,3)} = 1-\varepsilon$. Turning to higher order in λ, we remark that the conformal families $[\Phi_{(1,2\ell+1)}]$ with $\ell>0$ cannot introduce any contribution in the coefficients F_n with $n>1$, since they do not contain fields of suitable dimensions. In general the correction in (7.18) can arise only for even p at the degree $\lambda^{\frac{p+1}{2}}$ from the field $\partial_z T$ of exactly necessary dimension. Finally we get

$$\partial_{\bar{z}} T_4 = \partial_z Q_2 \ , \qquad (7.20)$$

where $Q_2 = \varepsilon\lambda G_2$ for $p\in 2\mathbb{Z}$ and $Q_2 = \varepsilon\lambda G_2 + \lambda^{1/\varepsilon}\alpha(\varepsilon)T$ for $p\in 2\mathbb{Z}+1$; exact values of the dimensionless coefficient $\alpha(\varepsilon)$ at the given level of the analysis are not essential. Further we consider the derivative $\partial_{\bar{z}}T_6^{(1)}$ and $\partial_{\bar{z}}T_6^{(2)}$, where $T_6^{(1)}$ and $T_6^{(2)}$ are defined in (3.26). Analogous arguments show that to first order in λ these derivatives are linear combinations of the following fields of spin 5: $L_{-5}\Phi$,

$L_{-1}L_{-4}\Phi$, $L_{-1}^3 L_{-2}\Phi$, $L_{-1}^5\Phi$ (accounted the condition (4.17)). From these there are derivatives on z other than the first, hence there exists such combination $T_6 = T_6^{(1)} + hT_6^{(2)}$ that

$$\partial_{\bar{z}} T_6 = \partial_z Q_4 \ , \tag{7.21}$$

where

$$Q_4 = \varepsilon\lambda(b_1 L_{-4}\Phi + b_2 L_{-1}^2 L_{-2}\Phi + b_3 L_{-1}^4\Phi) + O(\lambda^2) \ ; \tag{7.22}$$

the values of the coefficients b_1, b_2, b_3 are not important now. More detailed calculations show that

$$T_6 = T_6^{(1)} + \frac{1}{6}\left(\frac{28}{5} + C_p\right) T_6^{(2)} \ , \tag{7.23}$$

where C_p is given in (1.11), and equation (7.21), as (7.20), is exact, where the corrections of higher approximations to (7.22) appear only for $p \in 2\mathbb{Z}+1$ to the order $\lambda^{1/\varepsilon}$: in this case in (7.22) contributions of the fields T_4 and $\partial_z^2 T$ are introduced. Equations (7.20) and (7.21) show that in the theory (7.11) there are "higher" integrals of motion

$$P_3 = \int (T_4 dz - Q_2 d\bar{z}) \ ; \qquad P_5 = \int (T_6 dz - Q_4 d\bar{z}) \tag{7.24}$$

and analogous integrals $\bar{P}_3$ and $\bar{P}_5$, constructed from the fields $\bar{T}_4$ and $\bar{T}_6$. Direct calculations prove that all operators $P_3, P_5, \bar{P}_3, \bar{P}_5$ are commuative with each other and with components of the momentum $P = P_1 = \oint (T dz - \Theta d\bar{z})$ and $\bar{P} = \bar{P}_1 = \oint (\bar{T} d\bar{z} - \Theta dz)$. One can assume that in the theory (7.11) there is a complete set of commutative integrals of motion P_{2k+1} and $\bar{P}_{2k+1}$, $k=0,1,2,\cdots$ (where the operators P_{2k+1} and $\bar{P}_{2k+1}$ are spin 2k+1 and -2k-1 respectively), although these fields with $k>2$ are not yet constructed.

We should give two remarks. First the conclusions above of equations (7.20) and (7.21) do not depend on the smallness of the quantity ε. Therefore the "perturbed" model $\mathcal{M}_p$ (7.11) has "higher" integrals for all $p=3,4,5,\cdots$. Second the existence of the integrals (7.24) does not depend also on the sign of λ in (7.11). For $\lambda<0$, i.e. for $g<0$, there is no particular reasons to expect the existence of some kind of nontrivial zeroes of the β-functions (7.14). If they do not exist, the field theory (7.11) with $\lambda<0$ has a finite correlation radius $R_c \sim \lambda^{-1/2\varepsilon}$. In this case, the integrals of motion (7.24) reduce to the conservation of a set of momenta of particles for scattering, the factorization of S-matrices and so on.

Finally, we remark that the assertions, analogous to above, allow to establish the existence of "higher" integrals of motion in the theories $\mathcal{M}_p$ with "perturbed" operators of the form $\mu\int\Phi_{(1,2)}(x)dx$ and in a series of other models.

Translator's acknowledgement. Thanks are due to Drs. M.Jimbo and S.-K. Yang for reading the manuscript and useful comments.

[Section 7]

References

1. Pitashinskii A.Z., Pokrovskii V.I. Fluctuation Theory of Phase Transitions (Pergamon Press, London, 1979).
2. Wilson K.G., Kogut J. The Renormalization Group and the ε-expansion, Phys. Reports **12C**(1974)75.
3. Polyakov A.M. Conformal symmetry of critical fluctuations, Pis'ma Zh.Eksp.Teor.Fiz.**12**(1970)538[JETP Lett.**12**(1970)381].
4. Polyakov A.M. Non-hamiltonian approach to conformal quantum field theory, Zh.Eksp.Teor.Fiz.**66**(1974)23[Sov.Phys.JETP **39**(1974)10].
5. Wilson K.G. Non-lagrangian models of current algebra, Phys.Rev. **179**(1969),1499.
6. Kadanoff L.P. Scaling, Universality and Operator Algebras, in "Phase Transitions and Critical Phenomena", Vol 5A, edited by C.Domb and M.S.Green (Academic Press, London, 1976).
7. Polyakov A.M. Properties of long and short range correlations in the critical region, Zh.Eksp.Teor.Fiz.**57**(1969)271[Sov.Phys.JETP **30**(1970)151].
8. Belavin, A.A., Polyakov A.M., Zamolodchikov A.B. Infinite conformal symmetry in two dimensional quantum field theory, Nucl. Phys., **B241** (1984)333.
9. Mandelstam S. Dual Resonance Models, Phys.Repots, **13C**(1974)259.
10. Schwarz J.H. Superstring Theory, Phys.Reports **89**(1982)223.
11. Polyakov A.M. Quantum geometry of bosonic strings, Phys.Lett.**B103** (1981)207; Quantum geometry of fermionic strings, Phys.Lett.**B103** (1981)211.
12. Polyakov A.M. Fine structure of strings, Nucl.Phys.**B268**(1986)406.
13. Candelas P., Horowitz G., Strominger A., Witten E. Vacuum

configurations for superstrings, Nucl.Phys.**B258**(1985)46.

14.Dotzenko Vl.S. Critical behaviour and associated conformal algebra of Z_3-potts model, J.Stat.Phys.**B258**(1985),46.

15.Friedan D., Qiu Z., Shenker S. Conformal invariance, Unitarity and two-dimensional critical exponents, Phys.Rev.Lett.**52**(1984)1575.

16.Simon B. $P(\varphi)_2$ Euclidean (Quantum) Field Theory (Princeton Univ. Press, Princeton, NJ, 1974).

17.Zamolodchikov A.B. Pis'ma Zh.Eksp.Teor.Fiz.**43**(1986)565[JETP Lett. **43**(1986)730].

18.Baxter R.J. Exactly Solved Models in Statistical Mechanics, (Academic Press, London,1982).

19.Eichenherr H. Minimal operator algebras in superconformal quantum field theory, Phys.Lett.**B151**(1985)26.

20.Bershadski M., Knizhnik V., Teitelman M. Superconformal symmetry in two dimensions, Phys.Lett.**B151**(1985)31.

21.Friedan D., Qiu Z., Shenker S. Phys.Lett.**B151**(1985)37.

22.Novikov S.P. Hamiltonian formalism and multivalued analogue of Morse theory, Russ.Math.Surveys **37**(1982)3.

23.Witten E. Non-Abelian bozonization in two dimensions, Comm.Math. Phys. **92**(1984)355.

24.Polyakov A.M., Wiegman P.B. Goldstone fields in two dimensions with multivalued action, Phys.Lett.**B141**(1984)223.

25.Knizhnik V.G., Zamolodchikov A.B. Current algebra and Wess-Zumino model in two dimensions, Nucl.Phys. **B247**(1984)83.

26.Zamolodchikov A.B. Infinite additional symmetries in two-dimensional conformal field theory, Teor.Mat.Fiz.**65**(1985)347 [TMP **65**(1985)1205].

[References]

27. Fradkin E., Kadanoff L.P. Nucl.Phys. **B170**[FS1](1980)1.

28. Fateev V.A., Zamolodchikov A.B. Nonlocal("parafermion") currents in two-dimensional conformal quantum field theory and self-dual critical points in $\mathbb{Z}_N$-symmetric statistical systems, Zh.Eksp.Teor.Fiz. **89**(1985)380[Sov.Phys.JETP **62**(1985)215]; Disorder Fields in two-dimensional conformal quantum-field theory and N=2 extended supersymmetry, Zh.Eksp.Teor.Fiz.**90**(1986) 1553[Sov.Phys.JETP **63**(1986)913].

29. Fateev V.A., Zamolodchikov A.B. Representations of the algebra of "parafermion currents" of spin 4/3 in two-dimensional conformal field theory: minimal models and the tricritical Potts Z_3 model, Teor.Mat.Fiz.**71**(1987)163[TMP**71**(1987)451].

30. Gepner D. New conformal field theories associated with Lie algebras and their partition functions, Nucl.Phys. **B270**(1987)10.

31. Zamolodchikov Al.B. Conformal symmetry in two dimensions;An explicit recurrence formula for the conformal partial wave amplitudes, Comm.Math.Phys. **96**(1984)419.

32. Kac V. Lecture Notes in Phys. **94**(1979)441.

33. Feigin B.L., Fuks D.B. Func.Anal.& its Appl.**16**(1982)47.

34. Cardy J.L. Conformal invariance and surface critical behaviour, Nucl.Phys. **B240**[FS12](1984)514.

35. Eguchi T., Ooguri H. Conformal and current algebras on general Riemann surfaces, Nucl.Phys. **B282**(1987)308.

36. Cardy J.L. Conformal invariance and Yang-Lee edge singularity in two dimensions, Phys.Rev.Lett.**54**(1985)1354.

37. Fisher M.E. Yang-Lee Edge singularity and φ^3 field theory, Phys. Rev.Lett. **40**(1978)1610.

[References]

38. Dotsenko Vl.S., Fateev V.A., Conformal algebra and multi-point correlation functions in 2D statistical models, Nucl.Phys. B240 [FS12](1984)312.

39. Dotsenko Vl.S., Fateev V.A., Operator algebra of the two-dimensional conformal theories with the central charge $C \leq 1$, Phys.Lett. B154(1985)291.

40. McCoy B., Wu T.T. The two-dimensional Ising model (Harvard Univ.Press, Cambridge, MA, 1973).

41. Burkhart T.M., Guim I. Temple Univ. Preprint 1986.

42. Huse D.A. On the exact multicritical points in the restricted SOS models, Phys.Rev. B30(1984)3908.

43. Zamolodchikov A.B. Conformal symmetry and multicritical points in two-dimensional quantum field theory, Yad.Fiz.44(1986)821 [Sov.J.Nucl.Phys.44(1986)529].

44. Zamolodchikov A.B., Pogosyan R.G. Operator algebra in two-dimensional superconformal field theory, preprint ITP, Chernogolovka, 1987.

45. Zamolodchikov Al.B. Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model, Zh.Eksp.Teor.Fiz.90(1986)1808[Sov.Phys.JETP 63(1986)1061].

46. Date E., Jimbo M., Kuniba A., Miwa T., Okado M. Exactly solvable SOS models; Local height probabilities and theta function identities, Nucl.Phys. B290[FS20](1987)231.

47. Goddard P., Kent A., Olive D. Phys.Lett. B152(1985)85.

48. Fateev V.A, Zamolodchikov A.B., Nucl.Phys. B280[FS18](1987)644.

[References]

From Editors

It is a pleasure to publish in our journal Dr. A.B. Zamolodchikov's lecture note which is originally written in Russian and is now translated by Dr. Y. Kanie into English. We express our deep gratitude to Dr. Y. Kanie for his translation, patient preparation for the manuscript and permitting us to publish his translation. We are also grateful to Dr. A.B. Zamolodchikov who kindly agrees with our plan of publishing his lecture note, and to Drs. T. Inami, M. Jimbo and P.B. Wiegmann for their generous help and encouragement.

K.-I. Aoki — Kyoto

S.-K. Yang — May, 1988