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UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

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Contents

Introduction. The Virasoro algebra $\mathscr{L}$ is the Lie algebra over $\mathbf{C}$ of the following form:

$$\mathscr{L} = \sum_{n \in \mathbf{Z}} \mathbf{C} e_n \oplus \mathbf{C} e_0', \tag{1}$$

with the relations

$$[e_m, e_n] = (m-n)e_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}e_0'(m, n \in \mathbf{Z}); \tag{2}$$

$$e_0' \in \text{the center of the Lie algebra } \mathscr{L}.$$

The Lie algebra of this type was first appeared in the dual string model of elementary particle physics (cf. S. Mandelstam [12]). Quite recently the Virasoro algebra was used to analyze critical phenomena in the two dimensional statistical physics (cf. A. A. Belavin–A. M. Polyakov–A. B. Zamolodchikov [1]).

Introduce the triangular decomposition $\mathscr{L} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ of $\mathscr{L}$, where

$$\mathfrak{n}_\pm = \sum_{n \geq 1} \mathbf{C} e_{\pm n}; \qquad \mathfrak{h} = \mathbf{C} e_0 \oplus \mathbf{C} e_0'. \tag{3}$$

By V. G. Kac [8], for each $(h, c) \in \mathbf{C}^2$, there exists an irreducible $\mathscr{L}$-module $L(h, c)$, unique up to an isomorphism, with the following property. There exists a

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nonzero vector $v(h, c) \in L(h, c)$, called a vacuum element, such that

$$
\begin{aligned}
(4) \qquad & \mathfrak{n}_+ v(h,c) = 0; \\
& e_0 v(h,c) = h v(h,c), \quad e_0' v(h,c) = c v(h,c); \\
& U(\mathcal{L}) v(h,c) = L(h,c),
\end{aligned}
$$

where $U(\mathcal{L})$ denotes the universal enveloping algebra of $\mathcal{L}$.

If h and c are real numbers, there exists a nondegenerate hermitian bi-linear form $\{ , \}$ on $L(h, c)$, unique up to a constant factor, with the following properties:

$$
(5) \qquad \{u, e_m v\} = \{e_{-m} u, v\} (m \in \mathbb{Z}) \quad \text{and} \quad \{u, e_0' v\} = \{e_0' u, v\}.
$$

It should be remarked that this hermitian pairing is not necessarily positive-definite.

Then the problem of unitarity is stated as follows: When is the hermitian paring $\{ , \}$ positive-definite?

It is easy to show that if the hermitian pairing $\{ , \}$ is positive-definite, then we get (cf. Proposition 1-4)

$$
(6) \qquad h \geqslant 0 \quad \text{and} \quad c \geqslant 0.
$$

Using the so called Kac's determinant formula, we get easily the following (cf. Proposition 1-5):

PROPOSITION 1. *For $h \geqslant 0$ and $c \geqslant 1$, the hermitian pairing $\{ , \}$ on $L(h, c)$ is positive-definite.*

This proposition was known to many people, for example, V. G. Kac [9].

For the range $\{(h, c) \in \mathbb{R}^2; \; h \geqslant 0, 0 \leqslant c < 1\}$, D. Friedan-Z. Qiu-S. Shenker [4] asserted the following important results:

Assertion 2. Let $h \geqslant 0$ and $0 \leqslant c < 1$. If the hermitian pairing $\{ , \}$ on $L(h, c)$ is positive-definite, then (h, c) has the form

$$
(7) \qquad c = 1 - \frac{6}{(l+1)(l+2)} \quad \text{and} \quad h = \frac{[(l+2)p - (l+1)q]^2 - 1}{4(l+1)(l+2)}
$$

for some $l = 1, 2, \ldots$ and $1 \leqslant q \leqslant p \leqslant l$.

P. Goddard–A. Kent–D. Olive [6] constructed operators of the Virasoro algebra $\mathcal{L}$, giving unitary representations of $\mathcal{L}$ corresponding to the cited central charge $c = 1 - 6/(l+1)(l+2)$, through the so-called coset space representations associated to the quarternion projective space HP^{l-1}. We apply their operators to the level 1 integrable highest weight modules $L(\Lambda)$ of the affine Lie algebra of type $C_l^{(1)}$, and decompose $L(\Lambda)$ as a module over the subalgebra of type $C_1^{(1)} + C_{l-1}^{(1)}$, then we get the main result of this paper.

THEOREM 3. *For every* (h, c) *of the form* (7), *the hermitian pairing* $\{\,,\}$ *is positive-definite.*

This paper is organized as follows. In §1, we summarize the known results about $\mathscr{L}$-modules $L(h, c)$ and their unitarity. In §2, we summarize the known facts about the affine Lie algebra of type $C_l^{(1)}$ and their integrable highest weight modules $L(\Lambda)$. In §3, we introduce a subalgebra $C_1^{(1)} + C_{l-1}^{(1)}$ and Segal operators w.r.t. the pair $(\mathfrak{g}, \mathfrak{k}) = (C_l^{(1)}, C_1^{(1)} + C_{l-1}^{(1)})$. We decompose the level 1 integrable highest weight module $L(\Lambda)$ of $\mathfrak{g}$ into the sum of integrable highest modules of $\mathfrak{k}$ as a $\mathfrak{k}$-module, and show that the Virasoro algebra $\mathscr{L}$ acts on the space $\mathscr{S}(\Lambda; \mathfrak{k})$ of $\mathfrak{k}$-singular vectors in $L(\Lambda)$, where a vector is called $\mathfrak{k}$-singular if it is annihilated by any element of $\mathfrak{n}_+ \cap \mathfrak{k}$. In §4, we show that $\mathscr{S}(\Lambda; \mathfrak{k})$ is decomposed into the direct sum of some $L(h, c)$'s corresponding to the values (7) with multiplicity free, and each $L(h, c)$ with the value (7) occurs in $\mathscr{S}(\Lambda; \mathfrak{k})$ for some dominant integral weight Λ of level 1. This proof is carried out by calculating the branching coefficients of the decomposition of the $\mathfrak{k}$-module $L(\Lambda)$, and by comparing the characters of the $\mathscr{L}$-module $L(h, c)$ corresponding to the values (7).

The authors express their hearty thanks to Professor M. Jimbo for explicit computations of the branching coefficients.

§1. Unitarizable highest weight representations of the Virasoro algebra

(1.1) *Verma modules and their irreducible quotients.* In this paragraph, we summarize the known facts about representations of the Virasoro algebra ((cf. V. G. Kac [8], F. L. Feigin–D. B. Fuks [2], A. Tsuchiya–Y. Kanie [13]).

The *Virasoro algebra* $\mathscr{L}$ is the Lie algebra over $\mathbb{C}$ of the following form*:

$$\mathscr{L} = \sum_{n \in \mathbb{Z}} \mathbb{C} e_n \oplus \mathbb{C} e_0', \tag{1-1}$$

with the relations

$$\begin{cases} [e_m, e_n] = (m-n) e_{m+n} + \dfrac{m^3 - m}{12} \delta_{m+n,0} e_0' \quad (m, n \in \mathbb{Z}); \\ e_0' \in \text{the center of the Lie algebra } \mathscr{L}. \end{cases} \tag{1-2}$$

The dual space $\mathfrak{h}^*$ of the abelian subalgebra $\mathfrak{h} = \mathfrak{h}(\mathscr{L}) = \mathbb{C} e_0 \oplus \mathbb{C} e_0'$ of $\mathscr{L}$ is identified with $\mathbb{C}^2$ by setting for $(h, c) \in \mathbb{C}^2$,

$$(h, c)(e_0) = h \quad \text{and} \quad (h, c)(e_0') = c.$$

For each $(h, c) \in \mathbb{C}^2$, the left and right $\mathscr{L}$-modules $M(h, c)$ and $M^\dagger(h, c)$ with cyclic vectors $|h, c\rangle \in M(h, c)$ and $\langle c, h| \in M^\dagger(h, c)$ are defined respectively by

*The sign of the commutation relation is opposite from the one in A. Tsuchiya–Y. Kanie [13].

the following defining relations:

$$(1\text{-}3)\quad e_n|h,c\rangle = 0\ (n \geqslant 1), \qquad e_0|h,c\rangle = h|h,c\rangle, \qquad e_0'|h,c\rangle = c|h,c\rangle;$$
$$\langle c,h|e_{-n} = 0\ (n \geqslant 1), \qquad \langle c,h|e_0 = h\langle c,h|, \qquad \langle c,h|e_0' = c\langle c,h|.$$

They are called *Verma modules* with highest weight $(h, c) \in \mathfrak{h}^*$.

The bilinear pairing called the *vacuum expectations*

$$(1\text{-}4)\qquad \langle\,|\,\rangle : M^\dagger(h,c) \times M(h,c) \to \mathbf{C}$$

is uniquely defined by the following relations:

$$(1\text{-}5)\quad \langle c,h|h,c\rangle = 1;$$
$$\langle ue|v\rangle = \langle u|ev\rangle \qquad \big(e \in \mathscr{L},\, u \in M^\dagger(h,c),\, v \in M^\dagger(h,c)\big).$$

For each multi-index $M = (m_1, m_2, \ldots)$ of nonnegative integers with $\|M\| = \Sigma_{j\geqslant 1} jm_j < \infty$, we define the elements $e_+(M)$ and $e_-(M)$ of $U(\mathscr{L})$ by

$$(1\text{-}6)\qquad e_+(M) = e_1^{m_1} e_2^{m_2} \ldots \quad \text{and} \quad e_-(M) = \ldots e_{-2}^{m_2} e_{-1}^{m_1}.$$

Then the Verma modules $M(h,c)$ and $M^\dagger(h,c)$ have the following bases over $\mathbf{C}$:

$$(1\text{-}7)\qquad \big\{|N,h,c\rangle = e_+(N)|h,c\rangle;\ \|N\| < \infty\big\} \subset M(h,c);$$
$$\big\{\langle c,h,M| = \langle c,h|e_+(M);\ \|M\| < \infty\big\} \subset M^\dagger(h,c).$$

The modules $M(h,c)$ and $M^\dagger(h,c)$ have $\mathbf{Z}$-grading of the forms:

$$(1\text{-}8)\qquad M(h,c) = \sum_{d\geqslant 0} M_d(h,c) \quad \text{and} \quad M^\dagger(h,c) = \sum_{d\geqslant 0} M_d^\dagger(h,c),$$

where

$$M_d(h,c) = \sum_{\|N\|=d} \mathbf{C}|N,h,c\rangle \quad \text{and} \quad M_d^\dagger(h,c) = \sum_{\|M\|=d} \mathbf{C}\langle c,h,M|.$$

Then

$$\dim M_d(h,c) = \dim M_d^\dagger(h,c) = p(d),$$

where $p(d)$ is the number of the partitions of the integer d. Moreover the decompositions (1-8) are also the weight space decompositions of the $\mathscr{L}$-modules $M(h,c)$ and $M^\dagger(h,c)$ with respect to $\mathfrak{h}$. The subspaces $M_d(h,c)$ and $M_d^\dagger(h,c)$ belong to the same weight $(h + d, c) \in \mathbf{C}^2 = \mathfrak{h}^*$.

With respect to this grading, the vacuum expectation $\langle | \rangle$ is homogeneous in the sense that $\langle u|v\rangle = 0$ unless $\deg u = \deg v$.

For each integer $d \geqslant 0$, consider the $p(d) \times p(d)$-square matrix $A_d(h,c) = (A_d(h,c)^M_N)$ defined by

$$A_d(h,c)^M_N = \langle c, h, M|N, h, c\rangle \qquad (\|M\| = \|N\| = d). \tag{1-9}$$

Then the matrix $A_d(h,c)$ is a symmetric matrix and its determinant is given by V. G. Kac:

PROPOSITION 1-1. (V. G. Kac [8]). *For each $d \geqslant 0$,*

$$\det{}^2 A_d(h,c) = \text{const.} \prod_{k=1}^{d} \prod_{j|k} \Phi_{j,k/j}(h,c)^{p(d-k)}, \tag{1-10}$$

where const. means a positive constant and

(1-11)

$$\Phi_{k_1,k_2}(h,c) = \prod_{j=1}^{2} \left\{ h + \frac{c-13}{24}\left(k_j^2 - 1\right) + \frac{1}{2}(k_1 k_2 - 1)\right\} + \frac{\left(k_1^2 - k_2^2\right)^2}{2}.$$

Define the subspaces $I(h,c)$ and $I^\dagger(h,c)$ of $M(h,c)$ and $M^\dagger(h,c)$ as follows:

$$I(h,c) = \left\{ v \in M(h,c); \langle u|v\rangle = 0 \text{ for any } u \in M^\dagger(h,c)\right\}; \tag{1-12}$$

$$I^\dagger(h,c) = \left\{ v \in M^\dagger(h,c); \langle u|v\rangle = 0 \text{ for any } v \in M(h,c)\right\}.$$

Then we get

PROPOSITION 1-2. *Let $(h,c) \in \mathbf{C}^2$.*

(0) *$I(h,c)$ and $I^\dagger(h,c)$ are homogeneous subspaces of $M(h,c)$ and $M^\dagger(h,c)$ respectively.*

(i) *$I(h,c)$ and $I^\dagger(h,c)$ are maximal proper $\mathscr{L}$-submodules of $M(h,c)$ and $M^\dagger(h,c)$ respectively.*

(ii) *The quotient spaces $L(h,c) = M(h,c)/I(h,c)$ and $L^\dagger(h,c) = M^\dagger(h,c)/I^\dagger(h,c)$ are irreducible $\mathscr{L}$-modules.*

(iii) *The vacuum expectation factorizes through*

$$\langle | \rangle : L^\dagger(h,c) \times L(h,c) \to \mathbf{C} \tag{1-13}$$

and is homogeneous with respect to the grading $L(h,c) = \Sigma_{d \geqslant 0} L_d(h,c)$ and $L^\dagger(h,c) = \Sigma_{d \geqslant 0} L^\dagger_d(h,c)$.

(iv) *On the homogeneous components, the pairings*

$$\langle | \rangle_d : L_d^\dagger(h, c) \times L_d(h, c) \to \mathbb{C} \tag{1-14}$$

are nondegenerate for all $d \geqslant 0$.

(1.2) *Unitarizability.* Define the anti-$\mathbb{C}$-linear anti-automorphism of the universal enveloping algebra $U(\mathscr{L})$ of the Lie algebra $\mathscr{L}$:

$$\bar{\sigma} : U(\mathscr{L}) \to U(\mathscr{L}) \tag{1-15}$$

by the formulae:

$$\bar{\sigma}(e_m) = e_{-m} (m \in \mathbb{Z}) \quad \text{and} \quad \bar{\sigma}(e_0') = e_0'.$$

And for each $(h, c) \in \mathbb{R}^2$, define anti-$\mathbb{C}$-linear isomorphism of vector spaces

$$\bar{\sigma} : M(h, c) \to M^\dagger(h, c) \tag{1-16}$$

by setting $\sigma(|M, h, c\rangle) = \langle c, h, M|$ for all M with $\|M\| < \infty$. Then this map $\bar{\sigma}$ satisfies the following relations:

$$\begin{aligned} &\bar{\sigma}(|h, c\rangle) = \langle c, h|; \\ &\bar{\sigma}(eu) = \bar{\sigma}(u)\bar{\sigma}(e) \qquad (e \in U(\mathscr{L}), u \in M(h, c)). \end{aligned} \tag{1-17}$$

For each $(h, c) \in \mathbb{R}^2$, define the hermitian pairing

$$\{\, ,\} : M(h, c) \times M(h, c) \to \mathbb{C}, \tag{1-18}$$

by setting

$$\{u, v\} = \langle \bar{\sigma}(u) | v \rangle \qquad (u, v \in M(h, c)).$$

Then by Proposition 1-2, we get

PROPOSITION 1-3. *Let* $(h, c) \in \mathbb{R}^2$.

(i) *The hermitian pairing* (1-18) *factorizes through*

$$\{\, ,\} : L(h, c) \times L(h, c) \to \mathbb{C}, \tag{1-19}$$

and this pairing is homogeneous with respect to the grading $L(h, c) = \sum_{d \geqslant 0} L_d(h, c)$.

(ii) *On the homogeneous components, the pairing*

$$\{\, ,\}_d : L_d(h, c) \times L_d(h, c) \to \mathbb{C} \tag{1-20}$$

is nondegenerate for all $d \geqslant 0$.

(iii) *The hermitian pairing* (1-19) *satisfies the following relations*:

$$\{|h, c\rangle, |h, c\rangle\} = 1; \tag{1-21}$$

$$\{u, e_{-m}v\} = \{e_m u, v\}; \qquad \{u, e_0' v\} = \{e_0' u, v\}$$

$$(m \in \mathbb{Z},\ u, v \in L(h, c)).$$

Remark that the hermitian pairing $\{\,,\}$ on $L(h, c)$ are not necessarily positive-definite.

PROPOSITION 1-4. *Let* $(h, c) \in \mathbb{R}^2$. *If the hermitian form* $\{\,,\}$ *on* $L(h, c)$ *is positive-definite, then we get*

$$h \geqslant 0 \quad \text{and} \quad c \geqslant 0.$$

Proof. For each $n \geqslant 1$,

$$\{e_{-n}|h, c\rangle, e_{-n}|h, c\rangle\} = \{|h, c\rangle, e_n e_{-n}|h, c\rangle\} = 2nh + \frac{n^3 - n}{12}c.$$

Hence the positive-definiteness of $\{\,,\}$ implies $2nh + (n^3 - n)c/12 \geqslant 0$ for all $n \geqslant 1$. qed.

The following assertion, easily obtained from the Kac's determinant formula (1-10), is known to many people, for example, V. G. Kac [9].

PROPOSITION 1-5. *If* $h \geqslant 0$ *and* $c \geqslant 1$, *then the hermitian form* $\{\,,\}$ *on* $L(h, c)$ *is positive-definite.*

Moreover by deeply using the Kac's determinant formula, D. Friedan–Z. Qiu–S. Shenker [4] got the following assertion:

Assertion 1-6. (D. Friedan–Z. Qiu–S. Shenker [4]). Let $0 \leqslant c < 1$ and $h > 0$. If the hermitian form $\{\,,\}$ on $L(h, c)$ is positive-definite, then (h, c) has the following form

$$c = 1 - \frac{6}{(l+1)(l+2)} \quad \text{and} \quad h = \frac{[(l+2)p - (l+1)q]^2 - 1}{4(l+1)(l+2)} \tag{1-22}$$

for some $l = 1, 2, \ldots$ and $1 \leqslant q \leqslant p \leqslant l$.

Their proof of this assertion is very complicated, and it seems that the detailed proof does not appear yet. It is desirable that a simple proof of this beautiful formula should be given.

§2. Affine Lie algebras of type $C_l^{(1)}$. In this section, we summarize the basic known facts about affine Lie algebras of type $C_l^{(1)}$ and their irreducible representations after V. G. Kac [10].

(2.1) *Lie algebra of type C_l.* Let $\mathring{\mathfrak{g}}$ be the simple Lie algebra over $\mathbf{C}$ of type C_l ($l = 1, 2, \dots$). We take the following realization of $\mathring{\mathfrak{g}}$:

$$\mathring{\mathfrak{g}} = \mathfrak{sp}(l, \mathbf{C}) = \{A \in \mathfrak{gl}(2l, \mathbf{C}); {}^tAJ + JA = 0\}, \tag{2-1}$$

where

$$J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} \quad \text{and } I_l \text{ is the unit } l \times l\text{-matrix.}$$

Denote by E_{ij} the $2l \times 2l$-matrix whose (i, j)-component is 1, and all other components are zero. Put $h_j = E_{jj} - E_{j+l, j+l}$ $(1 \leqslant j \leqslant l)$ and $\mathring{\mathfrak{h}} = \sum_{j=1}^{l} \mathbf{C} h_j$. Then $\mathring{\mathfrak{h}}$ is a Cartan subalgebra of $\mathring{\mathfrak{g}}$. Denote by ε_i $(1 \leqslant i \leqslant l)$ the element of the dual space $\mathring{\mathfrak{h}}^*$ of $\mathring{\mathfrak{h}}$ defined by $\varepsilon_i(h_j) = \delta_{ij}$ $(1 \leqslant i,\ j \leqslant l)$. Then the set $\mathring{\Delta}$ of roots of $(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$ is given as

$$\mathring{\Delta} = \{\pm(\varepsilon_i \pm \varepsilon_j)(1 \leqslant i < j \leqslant l),\ \pm 2\varepsilon_i (1 \leqslant i \leqslant l)\}. \tag{2-2}$$

The set $\mathring{\Pi} = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l,\ \alpha_l = 2\varepsilon_l\}$ *gives a simple root system of* $\mathring{\Delta}$, *and the set* $\mathring{\Delta}_+$ *of corresponding positive roots is given as*

$$\mathring{\Delta}_+ = \{\varepsilon_i \pm \varepsilon_j\ (1 \leqslant i < j \leqslant l), 2\varepsilon_i\ (1 \leqslant i \leqslant l)\}. \tag{2-3}$$

For each $\alpha \in \mathring{\Delta}$, define the element $E_\alpha \in \mathring{\mathfrak{g}}$ by the following:

(i) $E_\alpha = E_{ij} - E_{j+l, i+l}$ for $\alpha = \varepsilon_i - \varepsilon_j$ $(1 \leqslant i \neq j \leqslant l)$;

(ii) $E_\alpha = E_{i, j+l} + E_{j, i+l}$, $E_{-\alpha} = E_{i+l, j} + E_{j+l, i}$ for $\alpha = \varepsilon_i + \varepsilon_j$ $(1 \leqslant i < j \leqslant l)$;

(iii) $E_\alpha = E_{i, i+l}$, $E_{-\alpha} = E_{i+l, i}$ for $\alpha = 2\varepsilon_i$ $(1 \leqslant i \leqslant l)$.

Put $\mathring{\mathfrak{g}}_\alpha = \mathbf{C} E_\alpha$ for each $\alpha \in \mathring{\Delta}$, then we get the root space decomposition of $\mathring{\mathfrak{g}}$ as

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{h}} \oplus \sum_{\alpha \in \mathring{\Delta}} \mathring{\mathfrak{g}}_\alpha. \tag{2-4}$$

The highest root θ of $\mathring{\Delta}$ and the half sum $\mathring{\rho} = \frac{1}{2}\sum_{\alpha \in \mathring{\Delta}_+} \alpha$ of positive roots are written respectively as

$$\theta = 2\varepsilon_1 \quad \text{and} \quad \mathring{\rho} = l\varepsilon_1 + (l-1)\varepsilon_2 + \cdots + \varepsilon_l. \tag{2-5}$$

Define the $\mathring{\mathfrak{g}}$-invariant symmetric bilinear form $(\,,)$ on $\mathring{\mathfrak{g}}$ by

$$(X, Y) = \operatorname{tr}(XY) \qquad (X, Y \in \mathring{\mathfrak{g}}), \tag{2-6}$$

where tr means the trace as an element of $\mathfrak{gl}(2l, \mathbf{C})$. Let $\nu : \mathring{\mathfrak{h}} \to \mathring{\mathfrak{h}}^*$ be the linear isomorphism defined by $\langle \nu(h_1), h_2 \rangle = (h_1, h_2)$ for $h_1, h_2 \in \mathring{\mathfrak{h}}$. For $\alpha, \beta \in \mathring{\mathfrak{h}}^*$ put $(\alpha, \beta) = (\nu^{-1}(\alpha), \nu^{-1}(\beta))$. Then we get

$$(2\text{-}7) \qquad (h_i, h_j) = 2\delta_{ij}, \quad (\varepsilon_i, \varepsilon_j) = \tfrac{1}{2}\delta_{ij} \qquad (1 \leqslant i, j \leqslant l);$$

$$\nu(h_j) = 2\varepsilon_j \qquad (1 \leqslant j \leqslant l).$$

This $\mathring{\mathfrak{g}}$-invariant form $(\,,)$ on $\mathring{\mathfrak{h}}^*$ is normalized in the sense that $(\theta, \theta) = 2$.

The dual Coxeter number g of $\mathring{\mathfrak{g}}$ is defined by

$$(2\text{-}8) \qquad g = \tfrac{1}{2}(\theta, \theta) + (\theta, \mathring{\rho}) = 1 + (\theta, \mathring{\rho}).$$

Then $g = l + 1$ in the case $\mathring{\mathfrak{g}} = \mathfrak{sp}(l, \mathbf{C})$.

Finally define the anti-$\mathbf{C}$-linear anti-automorphism $\bar{\omega}_0$ of the Lie algebra $\mathring{\mathfrak{g}}$ by

$$(2\text{-}9) \qquad \bar{\omega}_0(E_\alpha) = E_{-\alpha} (\alpha \in \mathring{\Delta}) \quad \text{and} \quad \bar{\omega}_0(h_i) = h_i \ (1 \leqslant i \leqslant l).$$

Then the real subspace $\mathring{\mathfrak{g}}_0 = \{A \in \mathring{\mathfrak{g}};\ \bar{\omega}_0(A) = -A\}$ is a compact real form of $\mathring{\mathfrak{g}}$.

(2.2) *Affine Lie algebra of type $C_l^{(1)}$.* The nontwisted affine Lie algebra $\mathfrak{g}$ associated to $\mathring{\mathfrak{g}} = \mathfrak{sp}(l, \mathbf{C})$ is defined by the following formulae and is called of type $C_l^{(1)}$:

$$(2\text{-}10) \qquad \mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d,$$

$$[x(m), y(n)] = [x, y](m + n) + m\delta_{m+n,0}(x, y)c;$$

$$[d, x(m)] = mx(m),\ [c, x(m)] = [c, d] = 0 \qquad (m, n \in \mathbf{Z}),$$

where $x(m) = x \otimes t^m$ for $x \in \mathring{\mathfrak{g}}$.

We identify $\mathring{\mathfrak{g}}$ with the subalgebra $\mathring{\mathfrak{g}} \otimes 1$ of $\mathfrak{g}$. The abelian subalgebra $\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbf{C}c \oplus \mathbf{C}d$ is called Cartan subalgebra of $\mathfrak{g}$. The dual space $\mathring{\mathfrak{h}}^*$ is considered as a subspace of $\mathfrak{h}^*$ by setting $\langle \alpha, c \rangle = \langle \alpha, d \rangle = 0$ for $\alpha \in \mathring{\mathfrak{h}}^*$. Define the elements δ and $\alpha_0 \in \mathfrak{h}^*$ by

$$\langle \delta, \mathring{\mathfrak{h}} \rangle = \langle \delta, c \rangle = 0, \qquad \langle \delta, d \rangle = 1 \quad \text{and} \quad \alpha_0 = \delta - \theta.$$

The set Δ of roots of $(\mathfrak{g}, \mathfrak{h})$ is given as

$$(2\text{-}11) \qquad \Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}},$$

$$\Delta^{\mathrm{re}} = \{k\delta + \gamma;\ k \in \mathbf{Z}, \gamma \in \mathring{\Delta}\};$$

$$\Delta^{\mathrm{im}} = \{k\delta;\ k \in \mathbf{Z} \setminus \{0\}\}.$$

And the corresponding root spaces are given as

(2-12)
$$\mathfrak{g}_{k\delta+\gamma} = \mathring{\mathfrak{g}}_\gamma \otimes t^k \qquad (k \in \mathbb{Z}, \gamma \in \mathring{\Delta}),$$
$$\mathfrak{g}_{k\delta} = \mathring{\mathfrak{h}} \otimes t^k \qquad (k \in \mathbb{Z} \setminus \{0\}).$$

The set $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ gives a simple root system of Δ, and the set Δ_+ of corresponding positive roots is written as

(2-13)
$$\Delta_+ = \{k\delta + \gamma;\, k > 0, \gamma \in \mathring{\Delta} \cup \{0\}\} \cup \mathring{\Delta}_+.$$

Then the Lie algebra $\mathfrak{g}$ has the triangular decomposition:

(2-14)
$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+; \qquad \mathfrak{n}_\pm = \sum_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}.$$

Extend $(\,,)$ on $\mathring{\mathfrak{g}}$ to the symmetric $\mathfrak{g}$-invariant bilinear form $(\,,)$ on $\mathfrak{g}$ by the following:

(2-15)
$$(x(m), y(n)) = \delta_{m+n,0}(x, y) \qquad (m, n \in \mathbb{Z}, x, y \in \mathring{\mathfrak{g}});$$
$$(c, x(m)) = (d, x(m)) = 0;$$
$$(c, c) = (d, d) = 0, \qquad (c, d) = 1.$$

Then $(\,,)$ is nondegenerate on $\mathfrak{h}$ and on $\mathfrak{g}$, in particular we can define an isomorphism $\nu\colon \mathfrak{h} \to \mathfrak{h}^*$ by $\langle \nu^{-1}(h), h' \rangle = (h, h')$ for $h, h' \in \mathfrak{h}$.

The simple coroots $\alpha_i^\vee (0 \leqslant i \leqslant l)$ are defined by $\alpha_i^\vee = 2\nu^{-1}(\alpha_i)/(\alpha_i, \alpha_i)$. Define the *fundamental weight* $\Lambda_i \in \mathfrak{h}^*$ $(0 \leqslant i \leqslant l)$ by

$$\langle \Lambda_i, \alpha_j^v \rangle = \delta_{ij} \quad \text{and} \quad \langle \Lambda_i, d \rangle = 0 \qquad (0 \leqslant i, j \leqslant l).$$

Then we have

(2-16)
$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta.$$

Consider the element $\rho \in \mathfrak{h}^*$ defined by

$$\langle \rho, \alpha_i^\vee \rangle = 1 \; (0 \leqslant i \leqslant l) \quad \text{and} \quad \langle \rho, d \rangle = 0,$$

then we get the important relation:

(2-17)
$$\rho = g\Lambda_0 + \mathring{\rho},$$

where g is the dual Coxeter number of $\mathring{\mathfrak{g}}$.

Let $Q = \sum_{i=0}^{l} \mathbf{Z}\alpha_i$ be the *root lattice* of $\mathfrak{g}$ and put $Q_+ = \sum_{i=0}^{l} \mathbf{Z}_{\geqslant 0}\alpha_i$. And put

$$P = \{\lambda \in \mathfrak{h}^*; \langle\lambda, \alpha_i^\vee\rangle \in \mathbf{Z}\ (0 \leqslant i \leqslant l)\} \quad \text{and}$$

$$P_+ = \{\lambda \in P; \langle\lambda, \alpha_i^\vee\rangle \geqslant 0\ (0 \leqslant i \leqslant l)\}.$$

The elements from P (or P_+) are called *integral weights* (or *dominant integral weights* respectively). Then we get

$$\text{(2-18)} \qquad P = \sum_{i=0}^{l} \mathbf{Z}\Lambda_i \oplus \mathbf{C}\delta \supset Q \quad \text{and}$$

$$P_+ = \sum_{i=0}^{l} \mathbf{Z}_{\geqslant 0}\Lambda_i \oplus \mathbf{C}\delta.$$

The *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{r_i; 0 \leqslant i \leqslant l\}$, where the elements $r_i \in GL(\mathfrak{h}^*)$ are defined by

$$\text{(2-19)} \qquad r_i(\lambda) = \lambda - \langle\lambda, \alpha_i^\vee\rangle\alpha_i \qquad (\lambda \in \mathfrak{h}^*, 0 \leqslant i \leqslant l).$$

Let $\mathring{W}$ be the subgroup of W generated by $\{r_i; 1 \leqslant i \leqslant l\}$, then $\mathring{W}$ preserves $\mathring{\mathfrak{h}}^*$ and fixes each element of $\mathbf{C}\delta \oplus \mathbf{C}\Lambda_0$. The group $\mathring{W}$ as a subgroup of $GL(\mathring{\mathfrak{h}}^*)$ can be canonically identified with the Weyl group of $(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$, hence

$$\text{(2-20)} \qquad \mathring{W} = \mathfrak{S}_l \ltimes \{\pm 1\}^l,$$

where $\mathfrak{S}_l$ is the l-th symmetric group.

For each $\alpha \in \mathring{\mathfrak{h}}^*$, define the element $t_\alpha \in GL(\mathfrak{h}^*)$ by the formula

$$\text{(2-21)} \qquad t_\alpha(\lambda) = \lambda + \langle\lambda, c\rangle\alpha - \left((\lambda, \alpha) + \tfrac{1}{2}(\alpha, \alpha)\langle\lambda, c\rangle\right)\delta,$$

then we have

$$t_\alpha \cdot t_\beta = t_{\alpha+\beta} \quad \text{and} \quad t_{w(\alpha)} = w t_\alpha w^{-1} \qquad (\alpha, \beta \in \mathring{\mathfrak{h}}^*, w \in W).$$

Introduce the subgroup T of $GL(\mathfrak{h}^*)$ defined by

$$T = \{t_\alpha; \alpha \in M\}, \quad \text{where } M = \sum_{i=1}^{l} \mathbf{Z}2\varepsilon_i \subset \mathring{\mathfrak{h}}^*.$$

Then T is a normal subgroup of W, and the Weyl group W is the semi-direct product of $\mathring{W}$ and T:

$$\text{(2-22)} \qquad W = \mathring{W} \ltimes T.$$

Finally introduce the anti-$\mathbb{C}$-linear anti-automorphism $\bar{\omega}$ of the Lie algebra $\mathfrak{g}$ defined by

$$\bar{\omega}(x(m)) = \bar{\omega}_0(x)(-m) \qquad (x \in \mathring{\mathfrak{g}}, m \in \mathbb{Z}); \tag{2-23}$$

$$\bar{\omega}(c) = c \quad \text{and} \quad \bar{\omega}(d) = d.$$

(2.3) *Integrable highest weight modules.* In this paragraph, we discuss representations of the affine Lie algebra $\mathfrak{g}$ of type $C_l^{(1)}$. A left $\mathfrak{g}$-module V is called a *highest weight module with highest weight* $\Lambda \in \mathfrak{h}^*$, if there exists a nonero vector $v \in V$, called a *vacuum vector*, such that

$$\mathfrak{n}_+ v = 0 \quad \text{and} \quad hv = \langle \Lambda, h \rangle v \qquad (h \in \mathfrak{h}); \tag{2-24}$$

and

$$V = U(\mathfrak{g})v,$$

where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. Then V has the weight space decomposition:

$$V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where $V_\lambda = \{v \in V;\ hv = \langle \lambda, h \rangle v\ (h \in \mathfrak{h})\}$. Let $P(V)$ denote the set of weights of V, then we have

$$P(V) = \{\lambda \in \mathfrak{h}^*; \dim V_\lambda \neq 0\} \subset \Lambda - Q_+. \tag{2-25}$$

The *formal character* of a highest weight module V is defined by

$$ch_V = \sum_{\lambda \in P(V)} \dim V_\lambda e^\lambda. \tag{2-26}$$

A highest weight $\mathfrak{g}$-module V is called *integrable*, if any element of $\mathfrak{g}_\alpha$ acts locally nilpotently on V for each $\alpha \in \Delta^{re}$. This definition of the integrability seems a little sstronger than the one given in V. G. Kac's book [10], but these two definitions are indeed equivalent with each other since every real root is a W-conjugate of a simple root.

For each $\Lambda \in \mathfrak{h}^*$, there exists an irreducible highest weight $\mathfrak{g}$-modules $L(\Lambda)$ with highest weight Λ, unique up to isomorphisms. We fix a vacuum vector $v_\Lambda \in L(\Lambda)$. By a result of V. G. Kac [10], the highest weight module $L(\Lambda)$ is integrable, if and only if $\Lambda \in P_+$. Furthermore, any integrable highest weight module is isomorphic to $L(\Lambda)$ for some $\Lambda \in P_+$. Hence the module $L(\Lambda)$ with $\Lambda \in P_+$ may be called integrable highest weight module with highest weight Λ. For $\Lambda \in P_+$, $k = \langle \Lambda, c \rangle$ is a non-negative integer and is called the *level* of Λ.

The following assertion due to H. Garland [5] is very important for our work.

PROPOSITION 2-1. *For each $\Lambda \in P_+$ with $\langle \Lambda, d \rangle \in \mathbb{R}$, there exists uniquely a positive-definite hermitian form $\{\, , \}$ on $L(\Lambda)$ with the following properties*:

$$\{v_\Lambda, v_\Lambda\} = 1 \quad \text{and} \quad \{u, av\} = \{\bar{\omega}(a)u, v\} \qquad (u, v \in L(\Lambda), a \in \mathfrak{g}).$$

Remark that we take the hermitian form $\{\, , \}$ as anti-$\mathbb{C}$-linear in the first argument.

The character of the integrable highest weight module $L(\Lambda)$ was determined by V. G. Kac (see [10]) as follows:

PROPOSITION 2-2. *Let $\Lambda \in P_+$. Then*

$$ch_{L(\Lambda)} = \sum_{w \in W} \det(w) e^{w(\Lambda+\rho)} \Big/ \sum_{w \in W} \det(w) e^{w(\rho)}, \tag{2-27}$$

and

$$\sum_{w \in W} \det(w) e^{w(\rho)} = e^{\rho} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{mult(\alpha)}. \tag{2-28}$$

(2.4) *Casimir operator.* In this paragraph after V. G. Kac [10], we introduce the Casimir operator Ω for the affine Lie algebra $\mathfrak{g}$ of type $C_l^{(1)}$.

For each $\alpha \in \Delta_+$, choose a basis $\{u_\alpha^1, \dots, u_\alpha^{d_\alpha}\}$ of $\mathfrak{g}_\alpha$ and let $\{u_{-\alpha}^1, \dots, u_{-\alpha}^{d_\alpha}\}$ be its dual basis of $\mathfrak{g}_{-\alpha}$ with respect to $(\, ,)$. And choose dual bases $\{u_1, \dots, u_{l+2}\}$ and $\{u^1, \dots, u^{l+2}\}$ of $\mathfrak{h}$.

Define the *Casimir operator* Ω by

$$\Omega = 2\nu^{-1}(\rho) + \sum_{j=1}^{l} u_j u^j + 2 \sum_{\alpha \in \Delta_+} \sum_{j=1}^{d_\alpha} u_{-\alpha}^j u_\alpha^j. \tag{2-29}$$

Then the Casimir operator Ω does not depend on the choice of dual bases. And the operator Ω can act on any highest weight $\mathfrak{g}$-module.

PROPOSITION 2-4. (V. G. KAC [10]). *Let V be a highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$, then as an operator on V we have*

$$\Omega = ((\lambda, \lambda) + 2(\lambda, \rho)) id_V.$$

§3. Construction of unitary representations

(3.1) *Segal operators.* In this section we construct unitary representations of the Virasoro algebra $\mathscr{L}$ having the central charge $c = 1 - 6/(l+1)(l+2)$, $l = 2, 3, \dots,$ from unitary representations of the affine Lie algebra of type $C_l^{(1)}$. Let $\mathring{\mathfrak{g}}$ be the simple Lie algebra of type C_l, and let $\mathfrak{g}$ be the affine Lie algebra of type $C_l^{(1)}$.

For each $x \in \mathring{\mathfrak{g}}$, define the formal Laurent series $\hat{x}(z)$ by

$$\hat{x}(z) = \sum_{m \in \mathbf{Z}} x(m) z^{-m-1}. \tag{3-1}$$

Define the *normal order product* of elements from $\hat{\mathfrak{g}} = \mathring{\mathfrak{g}} \otimes \mathbf{C}[t, t^{-1}]$ by

$$\overset{\circ}{\underset{\circ}{}} x(m) y(n) \overset{\circ}{\underset{\circ}{}} = \begin{cases} x(m)y(n) & (m < n) \\ \frac{1}{2}\{x(m)y(n) + y(n)x(m)\} & (m = n) \\ y(n)x(m) & (m > n). \end{cases} \tag{3-2}$$

Choose elements $u_\alpha \in \mathring{\mathfrak{g}}_\alpha$ for all $\alpha \in \mathring{\Delta}$ satisfying the conditions $(u_\alpha, u_{-\alpha}) = 1$, and set $u^\alpha = u_{-\alpha}$. And choose a basis $\{u_1, \dots, u_l\}$ of $\mathring{\mathfrak{h}}$, and let $\{u^1, \dots, u^l\}$ be its dual basis with respect to $(,)$.

Consider the formal Laurent series of operators defined by

$$\begin{aligned} \hat{S}(z) &= \sum_{j=1}^{l} \overset{\circ}{\underset{\circ}{}} \hat{u}^j(z) \hat{u}_j(z) \overset{\circ}{\underset{\circ}{}} + \sum_{\alpha \in \mathring{\Delta}} \overset{\circ}{\underset{\circ}{}} \hat{u}^\alpha(z) \hat{u}_\alpha(z) \overset{\circ}{\underset{\circ}{}} \\ &= \sum_{m \in \mathbf{Z}} S(m) z^{-m-2}. \end{aligned} \tag{3-3}$$

Then its coefficients are written as

$$S(m) = \sum_{j=1}^{l} \sum_{r \in \mathbf{Z}} \overset{\circ}{\underset{\circ}{}} u^j(-r) u_j(r+m) \overset{\circ}{\underset{\circ}{}} + \sum_{\alpha \in \mathring{\Delta}} \sum_{r \in \mathbf{Z}} \overset{\circ}{\underset{\circ}{}} u^\alpha(-r) u_\alpha(r+m) \overset{\circ}{\underset{\circ}{}} \tag{3-4}$$

These operators $S(m)$'s are independent from the choice of elements u_α, u_j and u^j's. By the definition of the normal order products $\overset{\circ}{\underset{\circ}{}} \;\; \overset{\circ}{\underset{\circ}{}}$, the operators $S(m)$ can act any highest weight $\mathfrak{g}$-module.

These operators $S(m)$ were introduced firstly by G. Segal, and their following properties are proved in V. G. Kac and D. H. Peterson [11].

Proposition 3-1. *Let $x \in \mathring{\mathfrak{g}}$ and $m, n \in \mathbf{Z}$. Then*

$$[S(m), \hat{x}(z)] = 2(g + c) z^m \left\{ z \frac{d}{dz} + (m+1) \right\} \hat{x}(z);$$

$$[S(m), x(n)] = -2(g + c) n x(m+n).$$

Proposition 3-2. *Let $m, n \in \mathbf{Z}$. Then*

$$\begin{aligned} [S(m), S(n)] &= 2(m-n)(g+c) S(m+n) \\ &\quad + \frac{\dim \mathring{\mathfrak{g}}}{12} (m^3 - m) \delta_{m+n,0} 4(g+c)c. \end{aligned}$$

By Proposition 2-4 and (3-4), easily we get

PROPOSITION 3-3.

$$S(0) = \Omega - 2(g + c)d.$$

Now define the operators

$$\text{(3-5)} \qquad T(m) = \frac{1}{2(g+c)} S(m) \quad (m \in \mathbb{Z}); \qquad T'(0) = \frac{\dim \mathring{\mathfrak{g}}}{g+c} c$$

Then by Proposition 3-2, the operators $T(m)(m \in \mathbb{Z})$ and $T'(0)$ satisfy the following commutation relations:

$$\text{(3-6)} \qquad [T(m), T(n)] = (m-n)T(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} T'(0);$$

$$[T(m), T'(0)] = 0 \qquad (m, n \in \mathbb{Z}).$$

Consider an integrable highest weight $\mathfrak{g}$-module $L(\Lambda)$ with $\Lambda \in P_+$ and $\langle \Lambda, d \rangle \in \mathbb{R}$. Let $k = \langle \Lambda, c \rangle$ be the level of Λ.

PROPOSITION 3-4. *As operators on* $L(\Lambda)$,

$$\text{(i)} \qquad T'(0) = \frac{\dim \mathring{\mathfrak{g}}}{g+k} k \, id_{L(\Lambda)}.$$

$$\text{(ii)} \qquad \{u, T(-m)v\} = \{T(m)u, v\} \qquad (m \in \mathbb{Z}, u, v \in L(\Lambda)).$$

Proof. The statement (i) is clear, since $c = k \, id_{L(\Lambda)}$ on $L(\Lambda)$.

(ii) Put $u_\alpha = \dfrac{E_\alpha}{(E_\alpha, E_{-\alpha})}$ for each $\alpha \in \mathring{\Delta}$, then $u_\alpha \in \mathfrak{g}_\alpha$ and $(u_\alpha, u_{-\alpha}) = 1$ for all $\alpha \in \mathring{\Delta}$. On $\mathring{\mathfrak{h}}_{\mathbf{R}} = \sum_{j=1}^{l} \mathbb{R} h_j$, the invariant bilinear form $(,)$ is positive-definite. Choose an orthonormal basis $\{u_1, \ldots, u_l\}$ of $\mathring{\mathfrak{h}}_{\mathbf{R}}$ and set $u^i = u_i$ $(1 \leqslant i \leqslant l)$. Since $\bar{\omega}_0(u_\alpha) = u_{-\alpha}(\alpha \in \mathring{\Delta})$ and $\bar{\omega}_0(u_i) = u_i(1 \leqslant i \leqslant l)$, we get that $\bar{\omega}(u_\alpha(m)) = u_{-\alpha}(-m)(\alpha \in \mathring{\Delta})$ and $\bar{\omega}(u_i(m)) = u_i(-m)$ $(1 \leqslant i \leqslant l)$ for all $m \in \mathbb{Z}$. Then by the definition of $S(m)$, easily we get that $\bar{\omega}(S(m)) = S(-m)$, hence $\bar{\omega}(T(m)) = T(-m)$. Since $\{u, av\} = \{\bar{\omega}(a)u, v\}$ for all $a \in \mathfrak{g}$ and $u, v \in L(\Lambda)$, we get that $\{u, T(-m)v\} = \{\bar{\omega}(T(-m)u, v\} = \{T(m)u, v\}$. qed.

(3.2) *Relative Segal operators or coset space representations.* Let $\mathring{\Delta}$ denote the root system of $(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$ (see the paragraph 2.1), and decompose the root system $\mathring{\Delta}$

as $\mathring{\Delta} = \mathring{\Delta}_1 \cup \mathring{\Delta}_2 \cup \mathring{\Delta}_\perp$, where

$$\begin{aligned} \text{(3-7)} \qquad \mathring{\Delta}_1 &= \{\pm 2\varepsilon_1\}; \\ \mathring{\Delta}_2 &= \left\{\pm 2\varepsilon_j (2 \leqslant j \leqslant l), \pm\left(\varepsilon_i \pm \varepsilon_j\right)(2 \leqslant i < j \leqslant l)\right\}; \\ \mathring{\Delta}_\perp &= \left\{\pm\left(\varepsilon_1 \pm \varepsilon_j\right)(2 \leqslant j \leqslant l)\right\}. \end{aligned}$$

Define the subspaces $\mathring{\mathfrak{h}}_1$ and $\mathring{\mathfrak{h}}_2$ of $\mathring{\mathfrak{h}}$ by

$$\text{(3-8)} \qquad \mathring{\mathfrak{h}}_1 = \mathbf{C} h_1 \quad \text{and} \quad \mathring{\mathfrak{h}}_2 = \sum_{j=2}^{l} \mathbf{C} h_j.$$

And define the subspaces $\mathring{\mathfrak{k}}_1$ and $\mathring{\mathfrak{k}}_2$ of $\mathring{\mathfrak{g}}$ by

$$\text{(3-9)} \qquad \mathring{\mathfrak{k}}_1 = \mathring{\mathfrak{h}}_1 \oplus \sum_{\alpha \in \mathring{\Delta}_1} \mathring{\mathfrak{g}}_\alpha \quad \text{and} \quad \mathring{\mathfrak{k}}_2 = \mathring{\mathfrak{h}}_2 \oplus \sum_{\alpha \in \mathring{\Delta}_2} \mathring{\mathfrak{g}}_\alpha.$$

Then $\mathring{\mathfrak{k}}_1$ and $\mathring{\mathfrak{k}}_2$ are Lie subalgebras of $\mathring{\mathfrak{g}}$ and are orthogonal with respect to $(\,,)$. The Lie subalgebra $\mathring{\mathfrak{k}}_1$ (or $\mathring{\mathfrak{k}}_2$) is isomorphic to the classical Lie algebra of type C_1 (or of type C_{l-1} respectively). The subspaces $\mathring{\mathfrak{h}}_1$ and $\mathring{\mathfrak{h}}_2$ are Cartan subalgebras of $\mathring{\mathfrak{k}}_1$ and $\mathring{\mathfrak{k}}_2$ respectively, and the decompositions (3-9) give the root space decomposition.

For $i = 1$ or 2, the intersection $\mathring{\Pi}_i = \mathring{\Pi} \cap \mathring{\Delta}_i$ gives a simple root system of $\mathring{\Delta}_i$, and the half sum $\mathring{\rho}_i$ of positive roots of $\mathring{\Delta}_i$ and the highest roots θ_i of $\mathring{\Delta}_i$ are given as

$$\begin{aligned} \text{(3-10)} \qquad &\mathring{\rho}_1 = \varepsilon_1 \quad \text{and} \quad \mathring{\rho}_2 = (l-1)\varepsilon_2 + \cdots + \varepsilon_l; \\ &\theta_1 = 2\varepsilon_1 \quad \text{and} \quad \theta_2 = 2\varepsilon_2. \end{aligned}$$

The restriction $(\,,)$ to $\mathring{\mathfrak{k}}_i \times \mathring{\mathfrak{k}}_i$ gives a $\mathring{\mathfrak{k}}_i$-invariant symmetric bilinear form on $\mathring{\mathfrak{k}}_i$ $(i = 1, 2)$, and is normalized in the sense that $(\theta_i, \theta_i) = 2$ $(i = 1, 2)$.

Put $\mathring{\mathfrak{k}} = \mathring{\mathfrak{k}}_1 + \mathring{\mathfrak{k}}_2$, and define the subspaces $\mathfrak{k}_1$, $\mathfrak{k}_2$ and $\mathfrak{k}$ of $\mathfrak{g}$ as follows:

$$\begin{aligned} \text{(3-11)} \qquad \mathfrak{k}_1 &= \mathring{\mathfrak{k}}_1 \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C} c \oplus \mathbf{C} d; \\ \mathfrak{k}_2 &= \mathring{\mathfrak{k}}_2 \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C} c \oplus \mathbf{C} d, \end{aligned}$$

and

$$\mathfrak{k} = \mathring{\mathfrak{k}} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C} c \oplus \mathbf{C} d.$$

Then these spaces $\mathfrak{k}_1$, $\mathfrak{k}_2$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, and are invariant under the anti-$\mathbf{C}$-linear involution $\bar{\omega}$. Since $\mathring{\mathfrak{k}}_1$ and $\mathring{\mathfrak{k}}_2$ are orthogonal with respect to $(\,,)$

and $[\mathring{\mathfrak{k}}_1, \mathring{\mathfrak{k}}_t] = 0$, we get

$$[\mathring{\mathfrak{k}}_1 \otimes \mathbf{C}[t, t^{-1}], \mathring{\mathfrak{k}}_2 \otimes \mathbf{C}[t, t^{-1}]] = 0. \tag{3-12}$$

From the affine Lie algebra $\mathfrak{k}_i$ ($i = 1$ or 2) of type $C_p^{(1)}$ ($p = 1$ or $l - 1$ respectively), we define the operators $T_i(m)$ ($m \in \mathbf{Z}$) and $T_i'(0)$ just analogously as $T(m)$ ($m \in \mathbf{Z}$) and $T'(0)$ from $\mathfrak{g}$. Then we have

PROPOSITION 3-5. (P. Goddard–A. Kent–D. Olive [6]). *For all $m, n \in \mathbf{Z}$,*

(i)
$$[T_1(m), T_2(n)] = 0;$$

(ii)
$$[T(m) - T_i(m), T_i(n)] = 0 \qquad (i = 1, 2).$$

Proof. The statement (i) is clear from the commutativity (3-12). Now we prove (ii). Fix i ($i = 1, 2$). From Proposition 3-1, we get

$$[T(m), x(k)] = -kx(m + k) \quad \text{and} \quad [T_i(m), x(k)] = -kx(m + k)$$

for $x \in \mathring{\mathfrak{k}}_i$ and $m, k \in \mathbf{Z}$, hence

$$[T(m) - T_i(m), x(k)] = 0.$$

This implies (ii), since $T_i(n)$'s are constructed from $x(k)(x \in \mathring{\mathfrak{k}}_i, k \in \mathbf{Z})$. q.e.d.

Now define the operators

$$T_\perp(m) = T(m) - T_1(m) - T_2(m) \qquad (m \in \mathbf{Z}); \tag{3-13}$$

$$T_\perp'(0) = \left(\frac{\dim \mathring{\mathfrak{g}}}{g + c} - \frac{\dim \mathring{\mathfrak{k}}_1}{g_1 + c} - \frac{\dim \mathring{\mathfrak{k}}_2}{g_2 + c} \right) c,$$

where g_i is the dual Coxeter number of $\mathring{\mathfrak{k}}_i$ ($i = 1, 2$), and in our case, $g_1 = 2$ and $g_2 = l$. Then by (3-6) and Proposition 3-5, easily we get

PROPOSITION 3-6. (P. Goddard–A. Kent–D. Olive [6]). *For all $m, n \in \mathbf{Z}$,*

$$[T_\perp(m), T_\perp(n)] = (m - n)T_\perp(m + n) + \frac{m^3 - m}{12} \delta_{m+n,0} T'(0);$$

$$[T_\perp(m), T_\perp'(0)] = 0.$$

Furthermore by the construction of $T_\perp(m)$, we have

PROPOSITION 3-7. *Let $x \in \mathring{\mathfrak{k}}$ and $m, n \in \mathbf{Z}$. Then*

$$[T_\perp(m), x(n)] = 0 \quad \textit{and} \quad [T_\perp'(0), x(n)] = 0.$$

And by the same way as in the proof of Proposition 3-4, we get

PROPOSITION 3-8. *For all $m \in \mathbb{Z}$ and $i = 1, 2$,*

$$\bar{\omega}(T_i(m)) = T_i(-m) \quad \text{and} \quad \bar{\omega}(T_\perp(m)) = T_\perp(-m).$$

(3.3) $\mathfrak{t}$-*singular vectors.* Now take a dominant integral weight $\Lambda \in P_+$ with $k = \langle \Lambda, c \rangle > 0$ and $\langle \Lambda, d \rangle \in \mathbb{R}$. Then the integrable highest weight $\mathfrak{g}$-module $L(\Lambda)$ has a left $\mathscr{L}$-module structure by defining

$$\pi(e_m) = T_\perp(m)(m \in \mathbb{Z}) \quad \text{and} \quad \pi(e_0') = T_\perp'(0). \tag{3-14}$$

By Proposition 3-8, the $\mathscr{L}$-module $(L(\Lambda), \pi)$ is unitary in the sense that

$$\{u, \pi(e_{-m})v\} = \{\pi(e_m)u, v\} \quad \text{and} \quad \{u, \pi(e_0')v\} = \{\pi(e_0')u, v\}$$

for all $m \in \mathbb{Z}$ and $u, v \in L(\Lambda)$.

Since $c = k id_{L(\Lambda)}$ on $L(\Lambda)$, we get

$$\pi(e_0') = T_\perp'(0) = c(l, k) id_{L(\Lambda)},$$

where

$$c(l, k) = \left(\frac{2l^2 + l}{l + 1 + k} - \frac{3}{2 + k} - \frac{2(l-1)^2 + (l-1)}{l + k} \right) k.$$

In particular, for $k = 1$ we have

$$c(l, 1) = 1 - \frac{6}{(l+1)(l+2)}. \tag{3-15}$$

Note that P. Goddard–A. Kent–D. Olive pointed out that these central charges appear in this coset representation.

Set $\mathfrak{n}_\pm(\mathfrak{t}) = \mathfrak{n}_\pm \cap \mathfrak{t}$, then we have $\mathfrak{t} = \mathfrak{n}_-(\mathfrak{t}) \oplus \mathfrak{h} \oplus \mathfrak{n}_+(\mathfrak{t})$.

Now introduce the subspace $\mathscr{S}(\Lambda; \mathfrak{t})$ of $\mathfrak{t}$-*singular vectors* in $L(\Lambda)$ defined by

$$\mathscr{S}(\Lambda; \mathfrak{t}) = \{u \in L(\Lambda); \mathfrak{n}_+(\mathfrak{t})u = 0\}. \tag{3-16}$$

By Proposition 3-7, this subspace $\mathscr{S}(\Lambda; \mathfrak{t})$ of $L(\Lambda)$ is invariant under the actions of $\mathscr{L}$. So the space $\mathscr{S}(\Lambda; \mathfrak{t})$ has a left $\mathscr{L}$-module structure which is unitary with respect to the positive-definite hermitian form $\{\,,\}$ restricted to $\mathscr{S}(\Lambda; \mathfrak{t})$.

Since $[\mathfrak{h}, \mathfrak{n}_+(\mathfrak{t})] \subset \mathfrak{n}_+(\mathfrak{t})$, the space $\mathscr{S}(\Lambda; \mathfrak{t})$ is also $\mathfrak{h}$-invariant and has the weight space decomposition with respect to $\mathfrak{h}$:

$$\mathscr{S}(\Lambda; \mathfrak{t}) = \sum_{\lambda \in \mathfrak{h}^*} \mathscr{S}_\lambda(\Lambda; \mathfrak{t}) = \sum_{\lambda \in P(\Lambda; \mathfrak{t})} \mathscr{S}_\lambda(\Lambda; \mathfrak{t}), \tag{3-17}$$

where $P(\Lambda; \mathfrak{t})$ denotes the set of weights of $\mathscr{S}(\Lambda; \mathfrak{t})$.

Let $\pi : \mathfrak{h}^* \to \mathring{\mathfrak{h}}^*$ denote the orthogonal projection with respect to $(\,,)$. And put $\overline{P}(\Lambda; \mathfrak{t}) = \{\pi(\lambda); \lambda \in P(\Lambda; \mathfrak{t})\} \subset \mathring{\mathfrak{h}}^*$. For each $\overline{\lambda} \in \overline{P}(\Lambda; \mathfrak{t})$, set

$$\mathscr{S}(\Lambda; \mathfrak{t}, \overline{\lambda}) = \Sigma \mathscr{S}_\lambda(\Lambda; \mathfrak{t}), \tag{3-18}$$

where the sum ranges over $\{\lambda \in P(\Lambda; \mathfrak{t});\ \pi(\lambda) = \overline{\lambda}\}$. Then

$$\mathscr{S}(\Lambda; \mathfrak{t}, \overline{\lambda}) = \{u \in \mathscr{S}(\Lambda; \mathfrak{t});\ hu = \langle \overline{\lambda}, h\rangle u \text{ for } h \in \mathring{\mathfrak{h}}\},$$

and the subspaces $\mathscr{S}(\Lambda; \mathfrak{t}, \overline{\lambda})$ is invariant under the action of $\mathscr{L}$, since

$$[\mathring{\mathfrak{h}}, T_\perp(m)] = [\mathring{\mathfrak{h}}, T'_\perp(0)] = 0 \qquad (m \in \mathbb{Z}).$$

Let $\{u_1, \ldots, u_l\}$ and $\{u^1, \ldots, u^l\}$ be dual bases of $\mathring{\mathfrak{h}}$ such that $u_1, u^1 \in \mathring{\mathfrak{h}}_1$ and $u_2, \ldots, u_l, u^2, \ldots, u^l \in \mathring{\mathfrak{h}}_2$. Then by Proposition 3-3 and the fact that $\mathfrak{n}_+(\mathfrak{t})\mathscr{S}(\Lambda; \mathfrak{t}) = 0$, we get

PROPOSITION 3-9. *For $\Lambda \in P_+$, put $k = \langle \Lambda, c\rangle$. Then*
(i) *as an operator on $\mathscr{S}(\Lambda; \mathfrak{t})$*

$$T_\perp(0) = \frac{\Omega}{2(l+1+k)} - d - \frac{u^1(0)u_1(0) + 2\nu^{-1}(\mathring{\rho}_1)}{2(2+k)} - \frac{\sum_{j=2}^{l} u^j(0)u_j(0) + 2\nu^{-1}(\mathring{\rho}_2)}{2(l+k)}.$$

(ii) *Decompose $\overline{\lambda} \in \overline{P}(\Lambda; \mathfrak{t})$ as $\overline{\lambda} = \overline{\lambda}_1 + \overline{\lambda}_2 \in \mathring{\mathfrak{h}}_1^* \oplus \mathring{\mathfrak{h}}_2^*$, then as an operator on $\mathscr{S}(\Lambda; \mathfrak{t}, \overline{\lambda})$,*

$$T_\perp(0) = \left(\frac{(\Lambda, \Lambda) + 2(\rho, \Lambda)}{2(l+1+k)} - \frac{(\overline{\lambda}_1, \overline{\lambda}_1) + 2(\mathring{\rho}_1, \overline{\lambda}_1)}{2(2+k)} - \frac{(\overline{\lambda}_2, \overline{\lambda}_2) + 2(\mathring{\rho}_2, \overline{\lambda}_2)}{2(l+k)}\right) - d.$$

This proposition shows that the operator $T_\perp(0)$ on $\mathscr{S}(\Lambda; \mathfrak{t}, \overline{\lambda})$ is equal to $-d$ up to a constant.

(3.4) *Branching law.* In this paragraph, we show that the subspace $\mathscr{S}(\Lambda; \mathfrak{t})$ of the $\mathfrak{g}$-module $L(\Lambda)$ describes the branching law of $L(\Lambda)$ as a $\mathfrak{t}$-module.

At first, define the subset $P_+(\mathfrak{t})$ of $P = \mathbb{Z}\Lambda_0 \oplus \sum_{j=1}^{l}\mathbb{Z}\varepsilon_j \oplus \mathbb{C}\delta$ by

$$P_+(\mathfrak{t}) = \left\{\lambda = k\Lambda_0 + \sum_{j=1}^{l} n_j\varepsilon_j + a\delta;\ k \geqslant n_1 \geqslant 0 \text{ and } k \geqslant n_2 \geqslant n_3 \geqslant \cdots \geqslant n_l \geqslant 0\right\}. \tag{3-19}$$

Elements from $P_+(\mathfrak{t})$ are called *dominant* $\mathfrak{t}$*-integral weights.*

By the same way as in the case of $\mathfrak{g}$-modules, we can define a highest weight $\mathfrak{t}$-module with highest weight $\lambda \in \mathfrak{h}^*$ due to the triangular decomposition $\mathfrak{t} = \mathfrak{n}_-(\mathfrak{t}) \oplus \mathfrak{h} \oplus \mathfrak{n}_+(\mathfrak{t})$.

The Lie algebra $\mathfrak{t}$ has the root space decomposition $\mathfrak{t} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta(\mathfrak{t})}\mathfrak{t}_\alpha$ with respect to the maximal abelian subalgebra $\mathfrak{h}$. Then the set $\Delta(\mathfrak{t})$ of roots of $(\mathfrak{t}, \mathfrak{h})$ is given by

$$\Delta(\mathfrak{t}) = \{\alpha \in \Delta;\ \mathfrak{g}_\alpha \subset \mathfrak{t}\}, \tag{3-20}$$

and $\mathfrak{t}_\alpha = \mathfrak{g}_\alpha$ for $\alpha \in \Delta(\mathfrak{t})$. Then define the set $\Delta^{\mathrm{re}}(\mathfrak{t})$ of real roots of $\Delta(\mathfrak{t})$ as $\Delta^{\mathrm{re}}(\mathfrak{t}) = \Delta^{\mathrm{re}} \cap \Delta(\mathfrak{t})$.

The highest weight $\mathfrak{t}$-module V is called *integrable*, if for each $\alpha \in \Delta^{\mathrm{re}}(\mathfrak{t})$ any element $a \in \mathfrak{t}_\alpha$ operates locally nilpotently on V. For each $\lambda \in \mathfrak{h}^*$, there exists an irreducible highest weight $\mathfrak{t}$-module $L(\mathfrak{t}, \lambda)$ with highest weight λ, unique up to isomorphisms. By the same method as for $\mathfrak{g}$-modules, it can be shown that the highest weight $\mathfrak{t}$-module $L(\mathfrak{t}, \lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ is integrable, if and only if $\lambda \in P_+(\mathfrak{t})$, and that any integrable highest weight $\mathfrak{t}$-module is isomorphic to $L(\mathfrak{k}, \lambda)$ for some $\lambda \in P_+(\mathfrak{t})$.

The integrable highest weight $\mathfrak{t}$-module $L(\mathfrak{t}, \lambda)$ can be constructed as follows. Write $\lambda \in P_+(\mathfrak{t})$ as $\lambda = k\Lambda_0 + \sum_{j=1}^{l} n_j\varepsilon_j + a\delta$, and put

$$\lambda_1 = k\Lambda_0 + n_1\varepsilon_1 + a\delta \quad \text{and} \quad \lambda_2 = k\Lambda_0 + n_2\varepsilon_2 + \cdots + n_l\varepsilon_l,$$

then λ_1 and λ_2 are dominant integral weights of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ respectively. Let $L(\mathfrak{t}_i, \lambda_i)$ be the integrable highest weight $\mathfrak{t}_i$-module with highest weight λ_i $(i = 1, 2)$. Define the $\mathfrak{t}$-module $L(\mathfrak{k}, \lambda)$ by

$$L(\mathfrak{t}, \lambda) = L(\mathfrak{t}_1, \lambda_1) \otimes_{\mathbb{C}} L(\mathfrak{t}_2, \lambda_2) \tag{3-21}$$

with the following $\mathfrak{t}$-action:

$$\begin{aligned} x_1(m)(v_1 \otimes v_2) &= x_1(m)v_1 \otimes v_2; \\ x_2(m)(v_1 \otimes v_2) &= v_1 \otimes x_2(m)v_2; \\ c(v_1 \otimes v_2) &= kv_1 \otimes v_2; \\ d(v_1 \otimes v_2) &= dv_1 \otimes v_2 + v_1 \otimes dv_2, \end{aligned} \tag{3-22}$$

for $x_i \in \mathring{\mathfrak{k}}_i$, $v_i \in L(\mathfrak{t}_i, \lambda_i)$, $i = 1, 2$ and $m \in \mathbf{Z}$. Then it can be shown that the left $\mathfrak{t}$-module $L(\mathfrak{t}, \lambda)$ is the integrable highest weight $\mathfrak{t}$-module with highest weight $\lambda \in P_+(\mathfrak{t})$.

Now we can apply the complete reducibility theorem of V. G. Kac and D. H. Peterson ([11] Proposition 2.9), since $\Delta^{\mathrm{re}}(\mathfrak{t}) \subset \Delta^{\mathrm{re}}$. Hence we get

PROPOSITION 3-10. *Take an element $\Lambda \in P_+$. Then the integrable highest weight $\mathfrak{g}$-module $L(\Lambda)$ is, as a $\mathfrak{t}$-module, uniquely decomposed into a direct sum of integrable highest weight $\mathfrak{t}$-modules.*

Note that for each $\lambda \in P_+(\mathfrak{t})$

$$\{u \in L(\mathfrak{t}, \lambda);\ \mathfrak{n}_+(\mathfrak{t})v = 0\} = \mathbf{C}v_\lambda,$$

where v_λ is a vacuum vector of $L(\mathfrak{t}, \lambda)$. Fix a vacuum vector v_λ of $L(\mathfrak{t}; \lambda)$ for each $\lambda \in P_+(\mathfrak{t})$ in the following. Then by Proposition 3-7, we get

PROPOSITION 3-11. *Let $\Lambda \in P_+$. Then*
(i) $P(\mathfrak{k}, \Lambda) \subset P_+(\mathfrak{t})$.
(ii) *Define the linear map*

$$\Phi\colon \sum_{\lambda \in P(\Lambda;\mathfrak{t})} L(t, \lambda) \otimes_{\mathbf{C}} \mathscr{S}_\lambda(\Lambda; \mathfrak{t}) \to L(\Lambda) \tag{3-23}$$

by the formula

$$\Phi((av_\lambda) \otimes u) = au$$

for $a \in U(\mathfrak{t})$, $u \in \mathscr{S}_\lambda(\Lambda; \mathfrak{t})$ and v_λ is the fixed vacuum vector of $L(\mathfrak{t}, \lambda)$. And introduce a $\mathfrak{t}$-module structure in $\sum_{\lambda \in P(\Lambda;\mathfrak{t})} L(\mathfrak{t}, \lambda) \otimes_{\mathbf{C}} \mathscr{S}_\lambda(\Lambda; \mathfrak{t})$ by

$$a(v \otimes u) = av \otimes u \qquad (a \in \mathfrak{k}, u \in \mathscr{S}_\lambda(\Lambda; \mathfrak{t}) \text{ and } v \in L(\mathfrak{k}, \lambda)).$$

Then this linear map Φ is an isomorphism as $\mathfrak{t}$-modules and commutes with Segal operators $T(m)$, $T_i(m)$, $T_\perp(m)$ for any $m \in \mathbf{Z}$.

(3.5) *Unitarity of Virasoro modules.* Now we restrict ourselves to the level 1 case. Let $\Lambda \in P_+$ be a dominant integral weight with $\langle \Lambda, c \rangle = 1$ and $\langle \Lambda, d \rangle = 0$, then Λ is one of the following $l + 1$ elements:

$$\Lambda_0, \qquad \Lambda_1 = \Lambda_0 + \varepsilon_1, \ldots, \qquad \Lambda_l = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_l.$$

Now recall the facts on weights:

$$P(\Lambda_j; \mathfrak{t}) \subset P_+(\mathfrak{t}) \cap P(\Lambda_j), \qquad P(\Lambda_j) \subset \Lambda_j - Q_+, \qquad Q_+ = \sum_{j=0}^{l} \mathbf{Z}_{\geqslant 0}\alpha_j;$$

$$\alpha_0 = \delta - 2\varepsilon_1, \qquad \alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \qquad \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \qquad \alpha_l = 2\varepsilon_l.$$

Then easily we get

PROPOSITION 3-12. *For each j with $0 \leqslant j \leqslant l$,*

(i)
$$P(\Lambda_j; \mathfrak{t}) \subset \left\{\Lambda_0 + r\varepsilon_1 + \sum_{i=1}^{s} \varepsilon_{i+1} - a\delta;\ r, s, a \in \mathbb{Z}, 0 \leqslant r \leqslant 1,\right.$$
$$\left. 0 \leqslant s \leqslant l-1,\ a \geqslant 0,\ r+s \equiv j (\operatorname{mod} 2)\right\}.$$

(ii)
$$\bar{P}(\Lambda_j; \mathfrak{t}) \subset \left\{r\varepsilon_1 + \sum_{i=1}^{s} \varepsilon_{i+1};\ r, s \in \mathbb{Z}, 0 \leqslant r \leqslant 1, 0 \leqslant s \leqslant l-1,\right.$$
$$\left. r+s \equiv j (\operatorname{mod} 2)\right\}.$$

The main theorems of our paper are the following:

THEOREM 3-13. *For each j with $0 \leqslant j \leqslant l$,*
(i) $\bar{P}(\Lambda_j; \mathfrak{t}) = \{r\varepsilon_1 + \sum_{i=1}^{s}\varepsilon_{i+1};\ r, s \in \mathbb{Z}, 0 \leqslant r \leqslant 1, 0 \leqslant s \leqslant l-1, r+s \equiv j (\operatorname{mod} 2)\}$.
(ii) *For each* $\bar{\lambda}(r, s) = r\varepsilon_1 + \sum_{i=1}^{s}\varepsilon_{i+1} \in \bar{P}(\Lambda_j; \mathfrak{t})$, *the $\mathscr{L}$-module $\mathscr{S}(\Lambda_j, \mathfrak{t}, \bar{\lambda}(r, s))$ is isomorphic to the irreducible $\mathscr{L}$-module $L(h, c)$ with characteristic*

(3-24)
$$c = 1 - \frac{6}{(l+1)(l+2)} \quad \text{and} \quad h = \frac{\{(l+2)(s+1) - (l+1)(j+1)\}^2 - 1}{4(l+1)(l+2)}.$$

We will prove this theorem in the next section (4.4). From Theorem 3-13, easily we get

THEOREM 3-14. *For each $l, p, q \in \mathbb{Z}$ with $l \geqslant 1$ and $1 \leqslant q \leqslant p \leqslant l$, the irreducible $\mathscr{L}$-module $L(h(l; p, q), c(l))$ has a structure of a unitary $\mathscr{L}$-module, where*

(3-25)
$$c(l) = 1 - \frac{6}{(l+1)(l+2)}$$

and

$$h(l; p, q) = \frac{[(l+2)p - (l+1)q]^2 - 1}{4(l+1)(l+2)}.$$

Proof. In the case where $l \geqslant 2$, this is an immediate consequence of Theorem 3-13. Let $l = 1$, then $c(1) = 0$ and $h(1;1,1) = 0$. It is known that $L(0,0)$ is nothing but the trivial $\mathscr{L}$-module $\mathbf{C}$, hence $L(0,0)$ has a unitary $\mathscr{L}$-module structure. qed.

§4. Branching law of L(Λ) w.r.t. $(C_l^{(1)}, C_1^{(1)} + C_{l-1}^{(1)})$

(4.1) *Theta functions.* In [11], V. G. Kac and D. H. Peterson have shown that the character of $L(\Lambda)$ is expressed as a quotient of alternating sums of classical theta functions.

Introduce coordinates $(\tau, u_1, \ldots, u_l, t)$ in $\mathfrak{h}$ by

$$h = 2\pi\sqrt{-1}\left(-\tau d + \sum_{j=1}^{l} u_j h_j + tc\right) \in \mathfrak{h}, \tag{4-1}$$

and identify $\mathfrak{h}$ with $\mathbf{C}^{l+2} = \{(\tau, u, t)\}$. Consider the domain Y in $\mathfrak{h}$ defined by $Y = \{h \in \mathfrak{h}; \operatorname{Re}\langle \delta, h\rangle > 0\}$, then Y is written as

$$Y = \mathscr{H}_+ \times \mathbf{C}^l \times \mathbf{C}; \qquad \mathscr{H}_+ = \{\tau \in \mathbf{C}; \operatorname{Im}\tau > 0\}.$$

Write $\gamma \in \sum_{j=1}^{l} \mathbf{R}\,\varepsilon_j$ as $\gamma = \sum_{j=1}^{l} 2\gamma_j \varepsilon_j$, then we get

$$(\gamma, \gamma) = |\gamma|^2 = 2\sum_{j=1}^{l} \gamma_j^2 \quad\text{and}\quad \langle \gamma, u\rangle = 2\sum_{j=1}^{l} \gamma_j u_j,$$

since $(\varepsilon_i, \varepsilon_j) = \frac{1}{2}\delta_{ij}$ and $\langle \varepsilon_i, h_j\rangle = \delta_{ij}$.

For an integer $k > 0$ and each $\mu = \sum_{j=1}^{l} \mu_j \varepsilon_j \in \mathring{P} = \sum_{j=1}^{l} \mathbf{Z}\varepsilon_j \subset \mathring{\mathfrak{h}}^*$, we define the classical *theta function* $\Theta_{\mu,k}(\tau, u)$ of $(\tau, u) \in \mathscr{H}_+ \times \mathbf{C}^l$ by

$$\Theta_{\mu,k}(\tau, u) = \sum_{\gamma \in M + \mu/k} \mathrm{e}\left[\frac{\tau}{2} k(\gamma, \gamma) + k\langle \gamma, u\rangle\right], \tag{4-2}$$

where $\mathrm{e}(*) = \exp(2\pi\sqrt{-1}\,*)$. Note that $M = \sum_{j=1}^{l} \mathbf{Z}2\varepsilon_j$. Then the theta function $\Theta_{\mu,k}(\tau, u)$ is expressed as

$$\Theta_{\mu,k}(\tau, u) = \prod_{j=1}^{l} \vartheta_{\mu_j, 2k}(\tau, u_j), \tag{4-3}$$

where ϑ is the 1-dimensional theta function defined by

$$\vartheta_{\nu,k}(\tau, v) = \sum_{\gamma \in \mathbf{Z} + \nu/k} \mathrm{e}\left[\frac{k}{2}\tau\gamma^2 + k\gamma v\right], \tag{4-4}$$

for $\nu \in \mathbf{Z}$, $k \in 2\mathbf{Z}_{>0}$ and $(\tau, v) \in \mathscr{H}_+ \times \mathbf{C}$.

Also introduce the following alternating sum of theta functions:

$$A_{\mu,k}(\tau,u)=\sum_{\mathring{w}\in\mathring{W}}\det(\mathring{w})\Theta_{\mathring{w}(\mu),k}(\tau,u) \tag{4-5}$$

for $\mu\in\mathring{P}$ and $k\in Z_{>0}$. Let $\lambda=(k\Lambda_0+\bar{\lambda}+a\delta)\in P_+$ with $k\geqslant 1$, then by (2-20, 21) we get

$$\sum_{w\in W}\det(w)e^{\langle w(\lambda),h\rangle}=\mathrm{e}\left[-\left(\frac{|\bar{\lambda}|^2}{2k}+a\right)\tau+kt\right]A_{\bar{\lambda},k}(\tau,u).$$

Moreover by (2-20) and (4-3), we get

$$A_{\mu,k}(\tau,u)=\det\bigl(\vartheta_{\mu_i,2k}(\tau,u_j)-\vartheta_{-\mu_i,2k}(\tau,u_j)\bigr)_{1\leqslant i,j\leqslant l}. \tag{4-6}$$

For $\Lambda=(k\Lambda_0+\bar{\lambda}+a\delta)\in P_+$, we define the *normalized character* of $L(\Lambda)$ by

$$\chi_\Lambda(\tau,u)=A_{\bar{\lambda}+\mathring{\rho},k+g}(\tau,u)/A_{\mathring{\rho},g}(\tau,u), \tag{4-7}$$

where $g=g(c)$ is the dual Coxeter number of $\mathring{\mathfrak{g}}$, since $\rho=g\Lambda_0+\mathring{\rho}$.

Take $\Lambda=(k\Lambda_0+\bar{\lambda}+a\delta)\in P_+$, then by Proposition 2.2 we have

$$ch_{L(\Lambda)}(h)=\mathrm{e}[-s_\Lambda\tau+kt]\chi_\Lambda(\tau,u) \tag{4-8}$$

as a function of $h=(\tau,u,t)\in Y$, where

$$s_\Lambda=\frac{|\bar{\lambda}+\mathring{\rho}|^2}{2(k+g)}-\frac{|\mathring{\rho}|^2}{2g}+a. \tag{4-9}$$

Moreover the denominator identity (2-28) can be rewritten as (see M. Jimbo–T. Miwa [7] (2)-6°):

$$A_{\mathring{\rho},g}(\tau,u)=\eta(\tau)^{l(1-l)}\prod_{i=1}^{l}\theta(\tau,2u_i)\times\prod_{1\leqslant i<j\leqslant l}\theta(\tau,u_i+u_j)\theta(\tau,u_i-u_j), \tag{4-10}$$

where $\eta(\tau)$ is the Dedekind's eta function defined by

$$\eta(\tau)=q^{1/24}\prod_{n\geqslant 1}(1-q^n);\qquad q=\mathrm{e}[\tau], \tag{4-11}$$

and

$$\theta(\tau, v) = \sum_{n\in\mathbf{Z}} (-1)^n \mathrm{e}\left[\frac{1}{2}\left(n+\frac{1}{2}\right)^2 \tau + \left(n+\frac{1}{2}\right)v\right] \tag{4-12}$$

for $(\tau, v) \in \mathscr{H}_+ \times \mathbf{C}$.

(4.2) *Coefficients of branching law.* Now we restrict ourselves to the level 1 case. Take a dominant integral weight $\Lambda_j = \Lambda_0 + \varepsilon_1 + \cdots + \varepsilon_j$ $(0 \leqslant j \leqslant l)$, then

$$ch_{L(\Lambda_j)}(h) = \mathrm{e}\left[-s_{\Lambda_j}\tau + t\right]\chi_j^l(\tau, u) \tag{4-13}$$

as a function of $h = (\tau, u, t) \in Y$, where $\chi_j^l(\tau, u) = \chi_{L(\Lambda_j)}(\tau, u)$ and

$$s_{\Lambda_j} = \frac{4l+3}{24} - \frac{(l+1-j)^2}{4(l+2)}. \tag{4-14}$$

By Propositions 3-11 and 3-12, we get

PROPOSITION 4-2.

$$L(\Lambda_j) = \sum_{r=0}^{1}\sum_{s=0}^{l-1}\sum_{\pi(\lambda)=\bar{\lambda}(r,s)} L(\mathfrak{t}, \lambda) \otimes \mathscr{S}_\lambda(\Lambda_j; \mathfrak{t}), \tag{4-15}$$

and $\mathscr{S}_\lambda(\Lambda_j; \mathfrak{t}) = 0$ *unless* $\pi(\lambda) = \bar{\lambda}(r, s) = r\varepsilon_1 + \sum_{i=1}^{s}\varepsilon_{i+1}$ *and* $r + s \equiv j$ (mod 2).

Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are also the affine Lie algebras of same type $C_1^{(1)}$ and $C_{l-1}^{(1)}$ respectively, so we can define analogous objects in the paragraph above. The Cartan subalgebras $\mathfrak{h}_1$ of $\mathfrak{t}_1$ and $\mathfrak{h}_2$ of $\mathfrak{t}_2$ are imbedded in $\mathfrak{h}$ as

$$\mathfrak{h}_1 = \{(\tau, u, \mathrm{t}) \in \mathfrak{h};\, u' = 0\} \quad \text{and} \quad \mathfrak{h}_2 = \{(\tau, u, t) \in \mathfrak{h};\, u_1 = 0\},$$

where $u = (u_1, u')$ and $u' = (u_2, \ldots, u_l)$.
For $r = 0, 1$ and $s (0 \leqslant s \leqslant l-1)$, set

$$\lambda_r^1 = \Lambda_0 + r\varepsilon_1 \in P_+(\mathfrak{t}_1) \quad \text{and} \quad \lambda_s^2 = \Lambda_0 + \sum_{i=1}^{s} \varepsilon_{i+1} \in P_+(\mathfrak{t}_2). \tag{4-16}$$

Then the characters of $L(\mathfrak{t}_1, \lambda_r^1)$ and $L(\mathfrak{t}_2, \lambda_s^2)$ are given as

$$ch_{L(\mathfrak{t}_1, \lambda_r^1)}(\tau, u_1, t) = e\left[-s_{\lambda_r^1}\tau + t\right]\chi_r^1(\tau, u_1);$$

$$ch_{L(\mathfrak{t}_2, \lambda_s^2)}(\tau, u', t) = e\left[-s_{\lambda_s^2}\tau + t\right]\chi_s^{l-1}(\tau, u'),$$

where χ_r^1 and χ_s^{l-1} denote the normalized characters of $L(\mathfrak{t}_1, \lambda_r^1)$ and $L(\mathfrak{t}_2, \lambda_s^2)$ respectively, and

$$s_{\lambda_r^1} = \frac{r}{4} - \frac{1}{24} \quad \text{and} \quad s_{\lambda_s^2} = \frac{4l-1}{24} - \frac{(l-s)^2}{4(l+1)}.$$

Let $\lambda = \Lambda_0 + \bar{\lambda}(r,s) - a\delta \in P_+(\mathfrak{t})$, then

$$L(\mathfrak{t}, \lambda) = L(\mathfrak{t}_1, \lambda_r^1 - a\delta) \otimes_{\mathbf{C}} L(\mathfrak{t}_2, \lambda_s^2),$$

hence by the definition (3-22) of $\mathfrak{t}$-action, we get

$$ch_{L(\mathfrak{t},\lambda)}(h) = \mathrm{e}[-t]ch_{L(\mathfrak{t}_1, \lambda_r^1 - a\delta)}(\tau, u_1, t)ch_{L(\mathfrak{t}_2, \lambda_s^2)}(\tau, u', t).$$

For each integer r, s, j ($r = 0, 1$, $0 \leqslant s \leqslant l-1$, $0 \leqslant j \leqslant l$), set

$$E(j; r, s; \tau) = \sum_{a \geqslant 0} \mathrm{e}[a\tau] \dim \mathscr{S}_{\Lambda_0 + \bar{\lambda}(r,s) - a\delta}(\Lambda_j; \mathfrak{t}) \tag{4-17}$$

(note $E(j; r, s; \tau) = 0$ unless $r + s \equiv j \pmod 2$), and

$$e(j; r, s; \tau) = \mathrm{e}[s(j; r, s)\tau]E(j; r, s; \tau), \tag{4-18}$$

where

$$s(j; r, s) = s_{\Lambda_j} - s_{\lambda_r^1} - s_{\lambda_s^2}.$$

Then by Proposition 4-2, we get

PROPOSITION 4-3.

$$\chi_j^l(\tau, u) = \sum_{r=0}^{1} \sum_{s=0}^{l-1} e(j; r, s; \tau)\chi_r^1(\tau, u_1)\chi_s^{l-1}(\tau, u').$$

Finally we get

PROPOSITION 4-4. (i) $e(j; r, s; \tau) = 0$ *unless* $r + s \equiv j \pmod 2$.
(ii) *if* $r + s \equiv j \pmod 2$, *then*

$$\begin{aligned}\eta(\tau)\mathrm{e}&\left[-\frac{\{(l+2)(s+1)-(l+1)(j+1)\}^2}{4(l+1)(l+2)}\tau\right]e(j; r, s; \tau)\\ &= \sum_{n \in \mathbf{Z}} \mathrm{e}\left[\{(l+1)(l+2)n^2 + ((l+2)(s+1) - (l+1)(j+1))n\}\tau\right]\\ &\quad - \mathrm{e}[(j+1)(s+1)\tau] \sum_{n \in \mathbf{Z}} \mathrm{e}\big[\{(l+1)(l+2)n^2\\ &\qquad\qquad + ((l+2)(s+1) + (l+1)(j+1))n\}\tau\big].\end{aligned}$$

The proof of this proposition will be given in the paragraph (4.5).

Remark 4-5. This proposition allows us to represent $e(j; r, s; \tau)$ by theta constants as

$$\eta(\tau)e(j; r, s; \tau) = \vartheta_{(l+2)(s+1)-(l+1)(j+1),2(l+2)(l+1)}(\tau, 0)$$
$$-\vartheta_{(l+2)(s+1)+(l+1)(j+1),2(l+2)(l+1)}(\tau, 0).$$

(4.3) *Character formula for the Virasoro algebra.* F. L. Feigin–D. B. Fuks [3] determined the composition series for Verma modules of the Virasoro algebra $\mathscr{L}$. So we can write down the character formulae for the irreducible $\mathscr{L}$-modules $L(h, c)$ for all $(h, c) \in \mathbf{C}^2$. Here we give it in a suitable case for our work.

Now we are interested in the case

$$c = c(l) = 1 - \frac{6}{(l+1)(l+2)} \quad \text{and}$$

$$h = h(l; p, q) = \frac{\{p(l+1) - q(l+2)\}^2 - 1}{4(l+1)(l+2)},$$

for integers $l, p, q (l \geqslant 2,\ 0 < p < l+2,\ 0 < q < l+1)$. For an integer $n \geqslant 0$, put

$$h(l; p, q, n) = \frac{\{p(l+1) - q(l+2) + n(l+1)(l+2)\}^2 - 1}{4(l+1)(l+2)},$$

then $h(l; p, q) = h(l; p, q, 0)$.

Here we fix $l \geqslant 2$ and use abbreviation

$$L(p, q, n) = L(h(l; p, q, n), c(l));\ M(p, q, n) = M(h(l; p, q, n), c(l)),$$

for integers p, q, n $(n \geqslant 0,\ 0 < p < l+2,\ 0 < q < l+1)$. Set $\bar{p} = l + 2 - p$ and $\bar{q} = l + 1 - q$, then $0 < \bar{p} < l+2$, $0 < \bar{q} < l+1$).

By F. L. Feigin–D. B. Fuks [3], we get the resolution for the $\mathscr{L}$-module $L(h(l; p, q), c(l))$:

$$\begin{tikzcd}
0 & L(p,q,0) \arrow[l] & M(p,q,0) \arrow[l] & M(\bar{p},q,1) \arrow[l] & M(p,q,2) \arrow[l] \arrow[dl] & \cdots \arrow[l] \arrow[dl] \\
 & & & M(p,\bar{q},1) \arrow[ul] & M(\bar{p},\bar{q},2) \arrow[l] \arrow[ul] & \cdots \arrow[l] \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & M(p,q,2n) \arrow[l] \arrow[dl] & M(\bar{p},q,2n+1) \arrow[l] \arrow[dl] & \cdots \arrow[l] \arrow[dl] \\
\cdots & M(\bar{p},\bar{q},2n) \arrow[l] \arrow[ul] & M(p,\bar{q},2n+1) \arrow[l] \arrow[ul] & \cdots \arrow[l] \arrow[ul]
\end{tikzcd}$$

Let $\{\mu, \mu'\}$ be the dual basis of $\mathfrak{h}^*(\mathscr{L})$ to $\{e_0, e_0'\}$, i.e. $\mu(e_0) = \mu'(e_0') = 1$ and $\mu(e_0') = \mu'(e_0) = 0$. Then $L(h, c)$ corresponds to the weight $h\mu + c\mu'$. Then by the resolution above, we get

Proposition 4-6. *For integers l, p, q ($l \geqslant 1$, $0 \leqslant p \leqslant l+1$, $0 \leqslant q \leqslant l$), put $c = c(l)$ and $h = h(l; p, q)$. Then*

$$ch_{L(h,c)} = \frac{e^{h\mu + c\mu'}}{\prod_{n \geqslant 1}(1 - e^{n\mu})}\left[\sum_{n \in \mathbf{Z}} e^{[n\{p(l+1)-q(l+2)\}+n^2(l+1)(l+2)]\mu}\right.$$
$$\left. - e^{pq\mu}\sum_{n\in\mathbf{Z}} e^{[n\{p(l+1)+q(l+2)\}+n^2(l+1)(l+2)]\mu}\right].$$

(4.4) *Proof of Theorem 3-13.* By Proposition 3-9(ii), we get

(4-19)
$$T_\perp(0) = \left\{s(j; r, s) + \frac{l^2 + 3l - 4}{24(l+1)(l+2)}\right\} - d \qquad \text{on } \mathscr{S}(\Lambda_j; \mathrm{t}, \bar{\lambda}(r, s)).$$

Thus $T_\perp(0)$ acts on each summand of (4-15) as $-d$ up to a constant. Recall that $T'_\perp(0)$ acts on the whole $L(\Lambda_j)$ as $c(l, 1)id$ (see (3-15)). Hence we can identify $e^\mu = \mathrm{e}[\tau]$ and neglect the term $e^{c\mu'}$. Denote by $ch(j; r, s; \tau)$ the character of the $\mathscr{L}$-module $\mathscr{S}(\Lambda_j; \mathrm{t}, \bar{\lambda}(r, s))$ as a function of τ, then by (4-17, 19), we get

$$ch(j; r, s; \tau) = \mathrm{e}\left[\left\{s(j; r, s) + \frac{1}{24} - \frac{1}{4(l+1)(l+2)}\right\}\tau\right]E(j; r, s; \tau).$$

Then by Proposition 4.4, we get

$$\mathrm{e}\left[-\frac{\{(l+2)(s+1) - (l+1)(j+1)\}^2 - 1}{4(l+1)(l+2)}\tau - \frac{1}{24}\tau\right]\eta(\tau)ch(j; r, s; \tau)$$
$$= \sum_{n\in\mathbf{Z}} \mathrm{e}\left[\{(l+1)(l+2)n^2 + ((l+2)(s+1) - (l+1)(j+1))n\}\tau\right]$$
$$-\mathrm{e}[(j+1)(s+1)\tau]\sum_{n\in\mathbf{Z}}\mathrm{e}\left[\{(l+1)(l+2)n^2\right.$$
$$\left.+((l+2)(s+1) + (l+1)(j+1))n\}\tau\right].$$

Hence by Proposition 4.6, $ch(j; r, s; \tau)$ coincides with the character of the irreducible $\mathscr{L}$-module $L(h(l; j+1, s+1), c(l))$. Thus we get Theorem 3-13 by Proposition 4-2.

(4.5) *Proof of Proposition 4-4**. Here we use the following complementary decomposition formula for $(C_l^{(1)}, C_1^{(1)} + C_{l+1}^{(1)})$ given by M. Jimbo–T. Miwa [7,

*Calculations in this paragraph are due to Prof. M. Jimbo.

(2)-7°]:

$$\eta(\tau)^{-2l}\prod_{i=1}^{l}\theta(\tau, u_i - u^\dagger)\theta(\tau, u_i - u^\dagger) = \sum_{\tilde\sigma\in\mathfrak{S}_{l+1}}' \det\tilde\sigma\chi^l_{\Lambda_{\tilde\sigma}}(\tau, u)\chi^1_{\Lambda^\dagger_{\tilde\sigma}}(\tau, u^\dagger), \tag{4-20}$$

where the sum Σ' extends over $\tilde\sigma \in \mathfrak{S}_{l+1}$ such that $\tilde\sigma(1) < \cdots < \tilde\sigma(l)$, and if $\tilde\sigma(l+1) = j+1$ $(0 \leqslant j \leqslant l)$, then

$$\Lambda_{\tilde\sigma} = \Lambda_j \text{ and } \Lambda^\dagger_{\tilde\sigma} = l\Lambda^\dagger_0 + (l-j)\varepsilon^\dagger_1 = j\Lambda^\dagger_0 + (l-j)\Lambda^\dagger_1.$$

Note that $u^\dagger$, $\Lambda^\dagger$, $\varepsilon^\dagger_1$ are the corresponding notions for $C_1^{(1)}$. Hence

$$\eta(\tau)^{-2l}\prod_{i=1}^{l}\theta(\tau, u_i + u^\dagger)\theta(\tau, u_i - u^\dagger) = \sum_{j=0}^{l}(-1)^{l-j}\chi^l_j(\tau, u)\chi^1_{j\Lambda^\dagger_0+(l-j)\Lambda^\dagger_1}(\tau, u^\dagger). \tag{4-21}$$

By substituting $l \mapsto l-1$ in (4-21), we get

$$\sum_{j=0}^{l}(-1)^{l-j}\chi^l_j(\tau, u)\chi^1_{j\Lambda^\dagger_0+(l-j)\Lambda^\dagger_1}(\tau, u^\dagger)$$

$$= \eta(\tau)^{-2}\theta(\tau, u_1 + u^\dagger)\theta(\tau, u_1 - u^\dagger)\sum_{s=0}^{l-1}(-1)^{l-1-s}\chi^{l-1}_s(\tau, u')$$

$$\chi^1_{s\Lambda^\dagger_0+(l-1-s)\Lambda^\dagger_1}(\tau, u^\dagger).$$

Then by Proposition 4-3 and the linear independence of normalized characters χ^{l-1}_s $(0 \leqslant s \leqslant l-1)$ of $C^{(1)}_{l-1}$, we get

(4-22)

$$\sum_{j=0}^{l}\sum_{r=0}^{1}(-1)^{l-j}e(j; r, s; \tau)\chi^1_r(\tau, u_1)\chi^1_{j\Lambda^\dagger_0+(l-j)\Lambda^\dagger_1}(\tau, u^\dagger)$$

$$= (-1)^{l-1-s}\eta(\tau)^{-2}\theta(\tau, u_1 + u^\dagger)\theta(\tau, u_1 - u^\dagger)\chi^1_{s\Lambda^\dagger_0+(l-1-s)\Lambda^\dagger_1}(\tau, u^\dagger).$$

In the following, we use the notation

$$\chi_{k,j}(\tau, u) = \chi^1_{j\Lambda_0+(k-j)\Lambda_1}(\tau, u) \qquad (0 \leqslant j \leqslant k)$$

for the normalized character of the level k irreducible representation $L(j\Lambda_0 + (k-j)\Lambda_1)$ of $C_1^{(1)}$, then by (4-6, 10), $\chi_{k,j}(\tau, u)$ is more explicitly given as

$$\chi_{k,j}(\tau,u) = \frac{\vartheta_{k-j+1,2(k+2)}(\tau,u) - \vartheta_{k-j+1,2(k+2)}(\tau,-u)}{\theta(\tau,2u)}, \tag{4-23}$$

in particular

$$\chi_{1,r}(\tau,u)\theta(\tau,2u) = \vartheta_{2-r,6}(\tau,u) - \vartheta_{2-r,6}(\tau,-u) \tag{4-24}$$

for $r = 0, 1$.

In these notations, (4-22) is rewritten as

$$\sum_{j=0}^{l}\sum_{r=0}^{1}(-1)^{j+s+1}e(j;r,s;\tau)\chi_{1,1-r}(\tau,u)\chi_{l,j}(\tau,v)$$
$$= \eta(\tau)^{-2}\theta(\tau,u+v)\theta(\tau,u-v)\chi_{l-1,s}(\tau,v),$$

hence by (4-24)

(4-25)

$$\sum_{j=0}^{l}\sum_{r=0}^{1}(-1)^{j+s+1}e(j;r,s;\tau)\chi_{l,j}(\tau,v)\{\vartheta_{1+r,6}(\tau,u) - \vartheta_{1+r,6}(\tau,-u)\}$$
$$= \eta(\tau)^{-2}\theta(\tau,2u)\theta(\tau,u+v)\theta(\tau,u-v)\chi_{l-1,s}(\tau,v).$$

Now we compare the coefficients of e$[u]$ and e$[2u]$ in the above equality (4-25). At first note the expansion:

$$\theta(\tau,2u)\theta(\tau,u+v)\theta(\tau,u-v)$$
$$= \sum_{k,m,n}(-1)^{k+m+n}\mathrm{e}\left[\frac{\tau}{2}\left\{\left(k+\frac{1}{2}\right)^2 + \left(m+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right)^2\right\}\right.$$
$$\left. + (2k+m+n+2)u + (m-n)v\right].$$

and the Euler's identity

$$\prod_{n\geqslant 1}(1-q^n) = \sum_{k\in\mathbf{Z}}(-1)^k q^{(3k^2+k)/2}. \tag{4-26}$$

Then the equality for the coefficients of e$[u]$ in (4-25) is

$$\sum_{j=0}^{l}(-1)^{j+s+1}e(j;0,s;\tau)\chi_{l,j}(\tau,v)\mathrm{e}\left[\frac{\tau}{12}\right]$$

$$=\eta(\tau)^{-2}\chi_{l-1,s}(\tau,v)\sum_{k,m}(-1)^{k+1}\mathrm{e}\left[\frac{\tau}{2}\left\{2\left(k+m+\tfrac{1}{2}\right)^2+3k^2+k+\frac{1}{4}\right\}\right.$$
$$\left.+2\left(k+m+\frac{1}{2}\right)v\right]$$

$$=\eta(\tau)^{-2}\chi_{l-1,s}(\tau,v)\mathrm{e}\left[\frac{\tau}{8}\right]\sum_{k\in\mathbf{Z}}(-1)^{k+1}\mathrm{e}\left[\frac{3k^2+k}{2}\tau\right]$$
$$\times\sum_{m\in\mathbf{Z}}\mathrm{e}\left[\left(m+\frac{1}{2}\right)^2\tau+2\left(m+\frac{1}{2}\right)v\right]$$

$$=-\mathrm{e}\left[\frac{\tau}{12}\right]\eta(\tau)^{-1}\chi_{l-1,s}(\tau,v)\vartheta_{1,2}(\tau,v).$$

Then by (4-23)

$$\sum_{j=0}^{l}(-1)^{j+s}e(j;0,s;\tau)\left\{\vartheta_{l-j+1,2(l+2)}(\tau,v)-\vartheta_{l-j+1,2(l+2)}(\tau,-v)\right\}$$
$$=\eta(\tau)^{-1}\vartheta_{1,2}(\tau,v)\left\{\vartheta_{l-s,2(l+1)}(\tau,v)-\vartheta_{l-s,2(l+1)}(\tau,-v)\right\}.$$

Compare the coefficients of e$[(l-j+1)v]$ in this equality, then we get that $j-s$ is even and

$$e(j;0,s;\tau)\mathrm{e}\left[\frac{(l-j+1)^2}{4(l+2)}\tau\right]\eta(\tau)$$

$$=\sum_{m\in\mathbf{Z}}\mathrm{e}\left[\tau\left\{\frac{s-j+1}{4}-m(l+1)\right\}^2+\tau(l+1)\left\{m+\frac{l-s}{2(l+1)}\right\}^2\right]$$

$$-\sum_{m\in\mathbf{Z}}\mathrm{e}\left[\tau\left\{\frac{1-j-s}{4}+l+m(l+1)\right\}^2+\tau(l+1)\left\{m+\frac{l-s}{2(l+1)}\right\}^2\right],$$

hence

$$e(j;0,s;\tau)\eta(\tau)\mathrm{e}\left[-\frac{[(j+1)(l+1)-(s+1)(l+2)]^2}{4(l+1)(l+2)}\tau\right]$$

$$= \sum_{m\in\mathbf{Z}} \mathrm{e}\big[\tau\{m^2(l+1)(l+2)+m[(j+1)(l+1)-(s+1)(l+2)]\}\big]$$

$$-\mathrm{e}[(j+1)(s+1)] \sum_{m\in\mathbf{Z}} \mathrm{e}\big[\tau\{m^2(l+1)(l+2)$$

$$+m[(j+1)(l+1)+(s+1)(l+2)]\}\big].$$

Similarly by comparing the coefficients of $\mathrm{e}[2u]$ in (4-25), we get

$$\sum_{j=0}^{l}(-1)^{j+s+1}e(j;1,s;\tau)\chi_{l,j}(\tau,v)=\eta(\tau)^{-1}\chi_{l-1,s}(\tau,v)\vartheta_{0,2}(\tau,v),$$

then by using (4-23) and comparing the coefficients of $\mathrm{e}[(l-j+1)\tau]$, we get that $j-s$ is odd and the formula for $e(j;1,s;\tau)$ is of the exactly same form as for $e(j;0,s;\tau)$. q.e.d.

(4.6) *Tensor products of integrable highest modules of $A_1^{(1)}$.* We take $\mathfrak{g}=C_1^{(1)}=A_1^{(1)}$ in this paragraph. Consider the tensor product $L(\Lambda_{1,r})\otimes L(\Lambda_{l-1,s})$ of the level 1 and $l-1$ integrable highest weight $\mathfrak{g}$-modules, where

$$\Lambda_{l,j}=j\Lambda_0+(l-j)\Lambda_1 \qquad (0\leqslant j\leqslant l).$$

Decompose it into a direct sum of level l integrable highest weight $\mathfrak{g}$-modules, and define the branching coefficients $\tilde{e}(j;r,s;\tau)$ by

$$\chi_{1,r}(\tau,u)\chi_{l-1,s}(\tau,u)=\sum_{j=0}^{l}\tilde{e}(j;r,s;\tau)\chi_{l,j}(\tau,u). \tag{4-27}$$

Then we get the following remarkable identity:

PROPOSITION 4-7. *For integers l,r,s,j ($2\leqslant l$, $0\leqslant r\leqslant 1$, $0\leqslant s\leqslant l-1$, $0\leqslant j\leqslant l$),*
$\tilde{e}(j;r,s;\tau)=e(j;r,s;\tau)$.

Proof. In §4.5, we get

$$\sum_{j=0}^{l}e(j;0,s;\tau)\chi_{l,j}(\tau,u)\eta(\tau)=\chi_{l-1,s}(\tau,u)\vartheta_{1,2}(\tau,u); \tag{4-28}$$

$$\sum_{j=0}^{l}e(j;1,s;\tau)\chi_{l,j}(\tau,u)\eta(\tau)=\chi_{l-1,s}(\tau,u)\vartheta_{0,2}(\tau,u).$$

Hence by the definition (4-27) and the linear independence of $\{\chi_{l,j}(\tau, u); 0 \leqslant j \leqslant l)\}$, it is enough to show

$$\chi_{1,r}(\tau, u)\eta(\tau) = \vartheta_{1-r,2}(\tau, u). \tag{4-29}$$

By the formula (4-24) and $\theta(\tau, 2u) = \vartheta_{1,4}(\tau, u) - \vartheta_{-1,4}(\tau, u)$, the formula (4-29) turns to be

$$\eta(\tau)\{\vartheta_{2-r,\sigma}(\tau, u) - \vartheta_{2-r,\sigma}(\tau, -u)\} = \vartheta_{1-r,2}(\tau, u)\{\vartheta_{1,4}(\tau, u) - \vartheta_{-1,4}(\tau, u)\}$$

Now the Euler's identity (4-26) implies

$$\eta(\tau) = \vartheta_{1,12}(\tau, 0) - \vartheta_{7,12}(\tau, 0) = -\vartheta_{5,12}(\tau, 0) + \vartheta_{11,12}(\tau, 0).$$

Then we can show (4-29) by the product formula of theta functions:

$$\vartheta_{a,2}(\tau, u)\vartheta_{b,4}(\tau, u) = \sum_{i=0}^{2} \vartheta_{4i+2a-b,12}(\tau, 0)\vartheta_{2i+a+b,6}(\tau, u).$$

q.e.d.

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Added. After the submission of this paper, the authors were informed of the following two papers. P. Goddard, A. Kent, D. Olive, *Unitary representations of the Virasoro and super-Virasoro algebra* (preprint, 1985 now published in Comm. Math. Phys. **103** (1986), 105–119.) and V. G. Kac, M. Wakimoto, *Unitarizable highest weight representations of the Virasoro, Neveu-Schwarz and Ramond algebras* (preprint, 1986). They also constructed unitary representations of the Virasoro algebra corresponding to Friedan-Qiu-Shenker parameters by using representations of the affine Lie algebra $A_1^{(1)}$.

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