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Cohomologies of Lie Algebras of Vector Fields with Coefficients in Adjoint Representations Foliated Case

By

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Introduction

Let $(M, \mathscr{F})$ be a foliated manifold. We have a natural Lie algebra $\mathscr{L}(M, \mathscr{F})$ of vector fields locally preserving the foliation $\mathscr{F}$, and its ideal $\mathscr{T}(M, \mathscr{F})$ of vector fields tangent to leaves of $\mathscr{F}$. Here we are interested in the first cohomologies of $\mathscr{L}(M, \mathscr{F})$ and $\mathscr{T}(M, \mathscr{F})$ with coefficients in their adjoint representations. This work is in a series of F. Takens' work [7] and the author's [3], [4]. In this paper, we use the latter for the general reference.

Our main result is

Main Theorem (i) $H^1(\mathscr{L}(M, \mathscr{F}); \mathscr{L}(M, \mathscr{F}))=0$.
(ii) $H^1(\mathscr{T}(M, \mathscr{F}); \mathscr{T}(M, \mathscr{F}))\cong\mathscr{L}(M, \mathscr{F})/\mathscr{T}(M, \mathscr{F})$.

If M is compact, $\mathscr{L}(M, \mathscr{F})$ is identical with the Lie algebra of vector fields preserving $\mathscr{F}$. There are compact foliated manifolds $(M, \mathscr{F})$ such that $H^1(\mathscr{T}(M, \mathscr{F}); \mathscr{T}(M, \mathscr{F}))$ are of dimension r for any r $(0\leqq r\leqq\infty)$.

The content of this paper is arranged as follows. In §1, we introduce Lie algebras $\mathscr{L}$ and $\mathscr{T}$ for a standard foliation on a eucildean space, and study their structures. In §2, we investigate properties of derivations of $\mathscr{L}$ and $\mathscr{T}$, and in §3, we prove Main Theorem for $\mathscr{L}$ and $\mathscr{T}$ (flat case). In §4, we give the proof of Main Theorem and

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some examples.

All manifolds, foliations, vector fields, etc. are assumed to be of C^∞-class, throughout this paper.

§1. Lie Algebras $\mathscr{L}$ and $\mathscr{T}$

1.1. Notations and Definitions. Fix a coordinate system $x_1, \ldots, x_p, y_1, \ldots, y_q$ in a $(p+q)$-dimensional euclidean space $V=\boldsymbol{R}^{p+q}$. Denote $\frac{\partial}{\partial x_i}$ by $\partial_i (i=1, \ldots, p)$, and $\frac{\partial}{\partial y_\alpha}$ by $\partial_\alpha (\alpha=1, \ldots, q)$. Use Latin indices $i, j, k, \ldots$ for $x_1, \ldots, x_p$, and Greek indices $\alpha, \beta, \ldots$ for $y_1, \ldots, y_q$, otherwise stated. Put

$$\mathscr{T} = \{\sum_{i=1}^{p} f_i(x, y)\partial_i\,; f_i(x, y) \text{ are } C^\infty\text{-functions of } x_1, \ldots, x_p, y_1, \ldots, y_q\},$$

$$\mathscr{L}' = \{\sum_{\alpha=1}^{q} g_\alpha(y)\partial_\alpha\,; g_\alpha(y) \text{ are } C^\infty\text{-functions of } y_1, \ldots, y_q\},$$

$$\mathscr{L} = \mathscr{T} + \mathscr{L}' \qquad \text{(as vector spaces)}.$$

Then they are subalgebras of the Lie algebra $\mathfrak{A}$ of all vector fields on V, and $\mathscr{T}$ is an ideal of $\mathscr{L}$.

Let $\mathscr{F}$ be a standard codimension-q foliation, defined by parallel p-planes: $y_1=\text{constant}, \ldots, y_q=\text{constant}$, in V. Any vector field X in $\mathscr{T}$ is tangent to leaves of $\mathscr{F}$, and X is called *leaf-tangent*. Let ϕ_t be the one-parameter group of diffeomorphisms generated by $Y \in \mathscr{L}$, then ϕ_t transforms every leaf to some leaf for each t, and Y is called *foliation preserving*.

Denote by $\mathscr{T}_x$ or $\mathscr{T}_y$, the subalgebra of $\mathscr{T}$ of all vector fields in $\mathscr{T}$ whose coefficient functions depend only on $x_1, \ldots, x_p$, or $y_1, \ldots, y_q$, respectively.

Here we summarize the facts which will be applied later.

Lemma 1.1. (*i*) *Let* $X \in \mathfrak{A}$. *If* $[\partial_i, X]=0$ *for all* $i=1, \ldots, p$, *then* X *is independent of the variables* $x_1, \ldots, x_p$.

(*ii*) $[\mathscr{T}_x, \mathscr{L}']=0$, *and* $[\mathscr{T}_x, \mathscr{L}] \subset \mathscr{T}$.

(*iii*) *Let* $X \in \mathscr{L}$. *If* $[\partial_i, X] \in \mathscr{L}'$ *for all* i, *then* X *is independent of the variables* $x_1, \ldots, x_p$.

(*iv*) *Let* $X \in \mathscr{T}_y$. *Then* $[X, I] = X$, *where* $I = \sum_{i=1}^{p} x_i \partial_i \in \mathscr{T}_x$.

(*v*) *Let* $X \in \mathscr{L}'$. *If* $[X, y_\alpha \partial_i] = 0$ *for all* i *and* α, *then* $X = 0$.

This can be proved by elementary calculations.

1. 2. Vector Fields with Polynomial Coefficients. The vector field $X = \sum_{i=1}^{p} f_i(x, y)\partial_i + \sum_{\alpha=1}^{q} g_\alpha(x, y)\partial_\alpha$ on V is said to be with polynomial coefficients, if all $f_i(x, y)$ and $g_\alpha(x, y)$ $(i = 1, \dots, p, \alpha = 1, \dots, q)$ are polynomials. Such vector fields form a Lie subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$. Put $\tilde{\mathscr{T}} = \mathscr{T} \cap \tilde{\mathfrak{A}}$, $\tilde{\mathscr{L}} = \mathscr{L} \cap \tilde{\mathfrak{A}}$, and $\tilde{\mathscr{L}}' = \mathscr{L}' \cap \tilde{\mathfrak{A}}$. Put

$\mathscr{T}_{n,m} = \{\sum_{i=1}^{p} f_i(x, y)\partial_i \in \tilde{\mathscr{T}}$; $f_i(x, y)$ are homogeneous polynomials of degree $n+1$ in $x_1, \dots, x_p$, and of degree $m+1$ in $y_1, \dots, y_q\}$.

Then

$$\tilde{\mathscr{T}} = \sum_{n, m \geqq -1} \mathscr{T}_{n,m} ;$$
$$\tilde{\mathscr{T}} \supset \tilde{\mathscr{T}}_x = \tilde{\mathscr{T}} \cap \mathscr{T}_x = \sum_{n \geqq -1} \mathscr{T}_{n,-1} ,$$
$$\tilde{\mathscr{T}}_y = \tilde{\mathscr{T}} \cap \mathscr{T}_y = \sum_{m \geqq -1} \mathscr{T}_{-1,m} .$$

Moreover, we have easily

Lemma 1. 2. (*cf.* [4]) *Let* I *be defined in Lemma* 1. 1 (*iv*), *then* $\mathscr{T}_{n,-1} = \{X \in \tilde{\mathscr{T}}_x ; [I, X] = nX\}$.

Put $\mathscr{L}'_m = \{\sum_{\alpha=1}^{q} g_\alpha(y)\partial_\alpha \in \tilde{\mathscr{L}}'$; $g_\alpha(y)$ are homogeneous of degree $m + 1\}$. Then $\tilde{\mathscr{L}}' = \sum_{m \geqq -1} \mathscr{L}'_m$, and we have

Lemma 1. 3. *Let* $J = \sum_{\alpha=1}^{q} y_\alpha \partial_\alpha \in \mathscr{L}'$, *then* $\mathscr{L}'_m = \{Y \in \tilde{\mathscr{L}} ; [J, Y] = mY\}$.

1. 3. Proposition 1. 4. *If a vector field* $X \in \mathscr{T}$ *satisfies* $j^3(X)(0) = 0$, *then there exist a finite number of vector fields* $X_1, \dots, X_{2r} \in \mathscr{T}$ *such that*

$$X = \sum_{i=1}^{r} [X_i, X_{i+r}] \quad \textit{and} \quad j^1(X_i)(0) = 0 \quad (i = 1, \dots, 2r).$$

Proof. Clearly it is enough to show the assertion for the case

$$X = x_1^{i_1} \cdots x_p^{i_p}\, y_1^{j_1} \cdots y_q^{j_q} f(x, y) \partial_i$$

for $\sum_{k=1}^{p} i_k + \sum_{\alpha=1}^{q} j_\alpha \geqq 4$. Put $h(x, y) = x_1^{i_1} \cdots x_p^{i_p}\, y_1^{j_1} \cdots y_q^{j_q}$.

Case 1. *The case where* $i_k \geqq 2$ *for some* k.

$$\begin{aligned}
&[x_k^2 \partial_k,\ x_k^{-1} X] - [x_k^3 \partial_k,\ x_k^{-2} X] \\
&= (i_k - 1 - 2\delta_{ik}) X - x_k h(x, y) (\partial_k f(x, y)) \partial_i \\
&\quad - (i_k - 2 - 3\delta_{ik}) X + x_k h(x, y) (\partial_k f(x, y)) \partial_i \\
&= (1 + \delta_{ik}) X.
\end{aligned}$$

Here δ_{ik} is Kronecker's delta, so $(1+\delta_{ik}) \geqq 1 > 0$. And $j^1(x_k^{-2} X)(0) = 0$ is obvious.

In the following, we can assume that $i_k \leqq 1$ for all k.

Case 2. *The case where* $\sum_k i_k \geqq 2$. We can assume $i_1 = i_2 = 1$. Let ϕ be a coordinate transformation

$$\phi : \begin{cases} \bar{x}_1 = x_1 + x_2, & \bar{x}_2 = x_1 - x_2, \\ \bar{x}_i = x_i \ (i \geqq 3), & \bar{y}_\alpha = y_\alpha \ (\text{all } \alpha), \end{cases}$$

then $\phi(\mathscr{T}) = \mathscr{T}$. So this case is reduced to Case 1.

In the following, we can assume that $i_k = 0$ for all k except at most one k_0.

Case 3. *The case where* $j_\alpha \geqq 2$ *for some* α. We get

$$[y_\alpha^2 \partial_{k_0},\ x_{k_0} y_\alpha^{-2} X] - [y_\alpha x_{k_0} \partial_{k_0},\ y_\alpha^{-1} X] = (1 + \delta_{ik_0}) X.$$

Obviously $j^1(Y)(0) = 0$ for all vector fields Y in the left hand.

Case 4. *The case where* $j_\alpha \leqq 1$ *for all* α. Since we have $\sum_{\alpha=1}^{q} j_\alpha \geqq 4 - 1 = 3$, so this case is also reduced to Case 3, similarly as Case 2.

Q. E. D.

Proposition 1.5. *If a vector field* $Y \in \mathscr{L}'$ *safisfies* $j^3(Y)(0) = 0$, *then there exist a finite number of vector fields* $Y_1, \ldots, Y_{2r} \in \mathscr{L}'$ *such that*

$$Y=\sum_{i=1}^{r}[Y_i,\ Y_{i+r}] \quad \textit{and} \quad j^1(Y_i)(0)=0 \ (i=1,\dots,2r).$$

Proof. Similarly as in Cases 1 and 2 in the proof of the above proposition. Q. E. D.

§ 2. Derivations of $\mathscr{T}$ and $\mathscr{L}$ (I)

2.1. Let $\mathscr{D}=\mathscr{D}_{ex}(\mathscr{T};\mathscr{L})$ be the space of derivations of $\mathscr{T}$ with values in $\mathscr{L}$. And let $\mathscr{D}_{\mathscr{L}}$ or $\mathscr{D}_{\mathscr{T}}$ be the derivation algebra of $\mathscr{L}$ or $\mathscr{T}$ respectively. Remember that a derivation D satisfies the property $D[X,\ Y]=[D(X),\ Y]+[X,\ D(Y)]$.

Proposition 2.1. If a derivation D in $\mathscr{D}$ is zero on $\mathscr{T}_{n,m}$ for $n+m\leqq -1$, then D is zero on $\tilde{\mathscr{T}}$.

Proof. Step 1. To show that D is zero on $\tilde{\mathscr{T}}_x$. We prove this by the induction on n for the decomposition $\tilde{\mathscr{T}}_x=\sum_{n\geqq -1}\mathscr{T}_{n,-1}$. When n is non-positive, the assertion holds by the assumption. Assume that D is zero on $\mathscr{T}_{k,-1}(k\leqq n-1)$. Let $Z\in\mathscr{T}_{n,-1}(n\geqq 1)$, and define the vector fields $X\in\mathscr{T}$ and $Y\in\mathscr{L}'$ as $D(Z)=X+Y$.

Apply D to $[\partial_i,\ Z]\in\mathscr{T}_{n-1,-1}(1\leqq i\leqq p)$, then we get $X\in\mathscr{T}_y$, by Lemma 1.1(i) and the hypothesis of the induction.

We get $[I,\ Z]=nZ$, by Lemma 1.2. Apply D to the both sides of this equality, then by Lemma 1.1 (iv), we get

$$-X=nX+nY,$$

hence $X=Y=0$, so $D(Z)=0$.

Step 2. To show that D is zero on $\mathscr{T}_{0,0}$. Clearly it is enough to show the assertion for the case $X=x_iy_\alpha\partial_j\in\mathscr{T}_{0,0}$. Apply D to

$$X=x_iy_\alpha\partial_j=2^{-1}[y_\alpha\partial_i,\ x_i^2\partial_j],$$

then we have $D(X)=0$, because $y_\alpha\partial_i\in\mathscr{T}_{-1,0}$ and $x_i^2\partial_j\in\tilde{\mathscr{T}}_x$.

Step 3. To show that D is zero on $\tilde{\mathscr{T}}_y$. The proof is carried out by the induction on m for the decomposition $\tilde{\mathscr{T}}_y=\sum_{m\geqq -1}\mathscr{T}_{-1,m}$. When m is non-positive, the assertion holds by the assumption. Assume that D is zero on $\mathscr{T}_{-1,k}(k\leqq m-1)$. Clearly it is enough to show that $D(Y)=0$ for the case

$$Y=y_1^{j_1}\cdots\cdots y_q^{j_q}\partial_i$$

for $\sum_\alpha j_\alpha=m+1$. There is an index β such that $j_\beta\geqq 1$. Apply D to

$$Y=[y_\beta^{-1}Y,\ y_\beta x_i\partial_i],$$

then $D(Y)=0$, because $y_\beta^{-1}Y\in\mathscr{T}_{-1,m-1}$, and $y_\beta x_i\partial_i\in\mathscr{T}_{0,0}$.

Last Step. Decompose $\tilde{\mathscr{T}}$ as $\tilde{\mathscr{T}}=\sum_{n\geqq -1}(\sum_{m\geqq -1}\mathscr{T}_{n,m})$. We prove the assertion of the proposition by the induction on n. The assertion for $n=-1$ holds by Step 3. Assume that D is zero on $\sum_{m\geqq -1}\mathscr{T}_{n,m}(n\leqq n_0-1)$. It is enough to show that $D(X)=0$ for the case

$$X=x_1^{i_1}\cdots\cdots x_p^{i_p}f(y)\partial_k$$

for $\sum_j i_j=n_0+1$, and some polynomial $f(y)$ of $y_1,\dots,y_q$. Apply D to the equality

$$X=\begin{cases}(i_k+1)^{-1}[x_k^{-i_k}X,\ x_k^{i_k+1}\partial_k] & \text{if } i_k>0,\\ [x_{k_0}^{-1}X,\ x_{k_0}x_k\partial_k] & \text{if } i_k=0, \text{ and } i_{k_0}>0 \text{ for some } k_0,\end{cases}$$

we get $D(X)=0$, because $x_k^{-i_k}X,\ x_{k_0}^{-1}X\in\sum_{n\leqq n_0-1}(\sum_{m\geqq -1}\mathscr{T}_{n,m})$, and $x_k^{i_k+1}\partial_k$, $x_{k_0}x_k\partial_k\in\tilde{\mathscr{T}}_x$.
Q. E. D.

Corollary 2.2. *The derivation* $D\in\mathscr{D}$ *is zero on* $\mathscr{T}$, *under the same assumption as Proposition* 2.1.

Proof. It follows from Propositions 1.3 and 1.4 in [4], and Proposition 1.4.
Q. E. D.

2.2. **Proposition 2.3.** *If a derivation* $D\in\mathscr{D}_{\mathscr{L}}$ *is zero on* $\mathscr{T}$, *then* D *is zero on* $\tilde{\mathscr{L}}'$.

Proof. *Step* 1. To show that $D(\partial_\alpha)=0$ $(\alpha=1,\dots,q)$. Apply D to $[\partial_i, \partial_\alpha]=[I, \partial_\alpha]=0$, then we get $D(\partial_\alpha)\in\mathscr{L}'$, by Lemma 1.1(i), (iv).

Define the functions $g_\alpha^\beta(y)$ as $D(\partial_\alpha)=\sum_\beta g_\alpha^\beta(y)\partial_\beta\in\mathscr{L}'$. Apply D to $\delta_{\alpha\gamma}\partial_i=[\partial_\alpha, y_\gamma\partial_i]$, then we get

$$0=[\sum_\beta g_\alpha^\beta(y)\partial_\beta, y_\gamma\partial_i]=g_\alpha^\gamma(y)\partial_i,$$

hence $g_\alpha^\gamma(y)=0$, so $D(\partial_\alpha)=0$.

Step 2. To show that $D(J)=0$, where $J=\sum_{\alpha=1}^q y_\alpha\partial_\alpha$. Apply D to $[\partial_i, J]=[I, J]=0$, then we get $D(J)\in\mathscr{L}'$, by Lemma 1.1 (i), (iv).

Apply D to $[J, y_\alpha\partial_i]=y_\alpha\partial_i\in\mathscr{T}$, then we have $D(J)=0$, by Lemma 1.1 (v).

Last Step. Since $\tilde{\mathscr{L}}'$ is decomposed as $\tilde{\mathscr{L}}'=\sum_{m\geqq -1}\mathscr{L}'_m$ (cf. § 1.2), then by Lemma 1.3, this step is carried out similarly as Step 1 in the proof of Proposition 2.1. Q. E. D.

Corollary 2.4. *If a derivation D of $\mathscr{L}$ is zero on $\mathscr{T}_{n,m}$ for $n+m\leqq -1$, then D is zero on $\mathscr{L}$.*

Proof. Let D be a derivation of $\mathscr{L}$ such that D is zero on $\mathscr{T}_{n,m}$ for $n+m\leqq -1$. Let D' be the restriction of D to $\mathscr{T}$. Then by Coroallry 2.2, D' is zero on $\mathscr{T}$, hence by Proposition 2.3, D is zero on $\tilde{\mathscr{L}}'$. The assertion follows from Propositions 1.3 and 1.4 in [4] and Proposition 1.5. Q. E. D.

§ 3. Derivations of $\mathscr{T}$ and $\mathscr{L}$ (II)

3. 1. Determination of $\mathscr{D}$. Let Z be a vector field on V. We define ad$Z$ as $\mathrm{ad}Z(X)=[Z, X]$ for $X\in\mathfrak{A}$. Then we have

Lemma 3.1. *The map: $Z\longrightarrow \mathrm{ad}Z|_{\mathscr{T}}$, or $Z\longrightarrow\mathrm{ad}Z|_{\mathscr{L}}$ of $\mathscr{L}$ into $\mathscr{D}$ or $\mathscr{D}_{\mathscr{L}}$ respectively is an into isomorphism.*

Proof. It is sufficient to show the injectivity. Let $Z\in\mathscr{L}$. Assume that $\mathrm{ad}Z(\mathscr{T})=0$. By Lemma 1.1 (i), we get the vector fields $X\in\mathscr{T}_y$ and $Y\in\mathscr{L}'$ such that $Z=X+Y$. Then by Lemma 1.1 (ii), (iv), we have $X=[Z, \Gamma]=0$, whence $Y=0$, by Lemma 1.1 (v).

Q. E. D.

Theorem 3.2. *Let $D\in\mathscr{D}$. Then there exists a unique vector field W on V such that $D=\mathrm{ad}W|_{\mathscr{T}}$. Moreover, W is in $\mathscr{L}$.*

The proof of this theorem will be given in § 3.3.

Corollary 3.3. *Let $D\in\mathscr{D}_{\mathscr{T}}$ or $\mathscr{D}_{\mathscr{L}}$. Then there exists a unique vector field $W\in\mathfrak{A}$ such that $D=\mathrm{ad}W|_{\mathscr{T}}$ or $=\mathrm{ad}W|_{\mathscr{L}}$. Moreover, W is in $\mathscr{L}$.*

Proof. Obvious for the case $D\in\mathscr{D}_{\mathscr{T}}$. Let $D\in\mathscr{D}_{\mathscr{L}}$. The restriction of D to $\mathscr{T}$ belongs to $\mathscr{D}$. Then the assertion follows from Theorem 3.2 and Corollary 2.4. Q. E. D.

Theorem 3.4. (*i*) *All derivations of $\mathscr{L}$ are inner, that is, $\mathscr{D}_{\mathscr{L}}=\mathrm{ad}\,\mathscr{L}\cong\mathscr{L}$. Hence*

$$H^1(\mathscr{L}\,;\,\mathscr{L})=0.$$

(*ii*) *The derivation algebra of $\mathscr{T}$ is naturally isomorphic to $\mathscr{L}$, that is, $\mathscr{D}_{\mathscr{T}}=\{\mathrm{ad}W|_{\mathscr{T}};\ W\in\mathscr{L}\}\cong\mathscr{L}$. Hence*

$$H^1(\mathscr{T}\,;\,\mathscr{T})\cong\mathscr{L}/\mathscr{T}\cong\mathscr{L}'.$$

In particular, the space $H^1(\mathscr{T}\,;\,\mathscr{T})$ is of infinite dimension.

Proof. (ii) By Coroallry 3.3, we have $\mathscr{D}_{\mathscr{T}}\subset\{\mathrm{ad}W|_{\mathscr{T}};\ W\in\mathscr{L}\}$. The converse inclusion is obvious, because $\mathscr{T}$ is an ideal of $\mathscr{L}$. For the latter half, remember that $H^1(\mathscr{T}\,;\,\mathscr{T})\cong\mathscr{D}_{\mathscr{T}}/\mathrm{ad}\,\mathscr{T}$ (see § 1 in [3]). Q. E. D.

3.2. To prove Theorem 3.2, we prepare the following four lemmata.

Lemma 3.5. *Let $D \in \mathscr{D}$. Then there exists a vector field $W_1 \in \mathscr{T}$ such that $D(\partial_i) \equiv [W_1, \partial_i]$ (mod $\mathscr{L}'$) for $i = 1, \dots, p$.*

Proof. Define the functions $f_i^j(x, y)$, and the vector fields $Y_i \in \mathscr{L}'$, as

$$D(\partial_i) = \sum_{j=1}^{p} f_i^j(x, y)\partial_j + Y_i \qquad (1 \leqq i \leqq p).$$

Apply D to the both sides of $[\partial_i, \partial_k] = 0$, then we have, by Lemma 1.1 (ii),

$$\sum_{j=1}^{p} \{\partial_i(f_k^j(x, y)) - \partial_k(f_i^j(x, y))\}\partial_j = 0 \qquad (1 \leqq i,\ k \leqq p),$$

and so

$$\partial_i(f_k^j(x, y)) = \partial_k(f_i^j(x, y)) \qquad (1 \leqq i,\ j,\ k \leqq p).$$

Therefore, there are unique functions $h^j(x, y)$ $(1 \leqq j \leqq p)$ such that

$$\begin{cases} \partial_i(h^j(x, y)) = f_i^j(x, y) & (1 \leqq i,\ j \leqq p), \\ h^j(0, y) = 0 & (1 \leqq j \leqq p). \end{cases}$$

Put $W_1 = -\sum_{i=1}^{p} h^i(x, y)\partial_i$, then we have the assertion of the lemma.

Q. E. D.

Lemma 3.6. *Let $D \in \mathscr{D}$. Assume that $D(\partial_i) \in \mathscr{L}'$ $(1 \leqq i \leqq p)$. Then*

(i) $D(\partial_i) = 0$ $(1 \leqq i \leqq p)$,

(ii) there exists a vector field $W_2 \in \mathscr{T}$ such that $[\partial_i, W_2] = 0$ $(1 \leqq i \leqq p)$, and $D(I) \equiv [W_2, I]$ (mod $\mathscr{L}'$).

Proof. Define the vector fields $X \in \mathscr{T}$ and $Y \in \mathscr{L}'$ as $D(I) = X + Y$. Apply D to $[\partial_i, I] = \partial_i$, then by Lemma 1.1 (ii), (iii), we have that $D(\partial_i) = 0$ $(1 \leqq i \leqq p)$, and $X \in \mathscr{T}_y$. Hence, by Lemma 1.1 (iv), we get

$$[X, I] = X \equiv D(I) \quad (\text{mod } \mathscr{L}').$$

Therefore, we can put $W_2 = X$. Q. E. D.

Lemma 3.7. *Let $D\in\mathscr{D}$. Assume that $D(\partial_i)=0$ $(1\leqq i\leqq p)$, and $D(I)\in\mathscr{L}'$. Then, $D(x_i\partial_j)\in\mathscr{L}'$ $(1\leqq i, j\leqq p)$.*

Proof. Define the vector fields $X_{ij}\in\mathscr{T}$ and $Y_{ij}\in\mathscr{L}'$ as $D(x_i\partial_j) = X_{ij}+Y_{ij}(1\leqq i, j\leqq p)$.

Apply D to $[\partial_k, x_i\partial_j]=\delta_{ik}\partial_j$, then by Lemma 1.1 (i), we have $X_{ij} \in\mathscr{T}_y(1\leqq i, j\leqq p)$. Apply D to $[I, x_i\partial_j]=0$, then by Lemma 1.1 (ii), (iv), we get $X_{ij}=0(1\leqq i, j\leqq p)$. Q. E. D.

Lemma 3.8. *Let $D\in\mathscr{D}$. Assume that $D(\partial_i)=0$, and that $D(I) \in\mathscr{L}'$, $D(x_i\partial_j)\in\mathscr{L}'(1\leqq i, j\leqq p)$. Then,*

(i) $D(I)=0$, $D(x_i\partial_j)=0$ $(1\leqq i, j\leqq p)$,

(ii) there exists a unique vector field W_3 on V such that

$$[W_3, \partial_i]=[W_3, I]=[W_3, x_i\partial_j]=0,$$
$$[W_3, y_\alpha\partial_i]=D(y_\alpha\partial_i) \qquad (1\leqq i, j\leqq p, 1\leqq\alpha\leqq q).$$

Moreover, W_3 is in $\mathscr{L}'$.

Proof. Define the vector fields $X_{\alpha i}\in\mathscr{T}$ and $Y_{\alpha i}\in\mathscr{L}'$ as $D(y_\alpha\partial_i) = X_{\alpha i}+Y_{\alpha i}(1\leqq i\leqq p, 1\leqq\alpha\leqq q)$. Apply D to $[\partial_j, y_\alpha\partial_i]=0$, then by Lemma 1.1 (i), we have $X_{\alpha i}\in\mathscr{T}_y$ for all i and α. Apply D to $y_\alpha\partial_i=[y_\alpha\partial_i, I]$, then by Lemma 1.1 (ii), (iv), we get that $D(I)=0$ and $Y_{\alpha i}=0$ for all i and α.

Define the functions $f^j_{\alpha i}(y)$ $(1\leqq i, j\leqq p, 1\leqq\alpha\leqq q)$ as $X_{\alpha i}=\sum_j f^j_{\alpha i}(y)\partial_j$. Apply D to $y_\alpha\partial_i=[y_\alpha\partial_i, x_i\partial_i]$, then we get

$$\sum_j f^j_{\alpha i}(y)\partial_j=f^i_{\alpha i}(y)\partial_i+[y_\alpha\partial_i, D(x_i\partial_i)],$$

hence $D(x_i\partial_i)=0$ $(1\leqq i\leqq p)$, and $f^j_{\alpha i}(y)=0$ for all $i\neq j$ and α.

Apply D to $y_\alpha\partial_k=[y_\alpha\partial_i, x_i\partial_k]$ for $i\neq k$, then we get

$$f^k_{\alpha k}(y)\partial_k=f^i_{\alpha i}(y)\partial_k+[y_\alpha\partial_i, D(x_i\partial_k)],$$

hence $D(x_i\partial_k)=0$ $(1\leqq i, k\leqq p)$, and $f^i_{\alpha i}(y)=f^k_{\alpha k}(y)$ for all $i\neq k$ and α. Denote $f^i_{\alpha i}(y)$ by $f_\alpha(y)$ $(1\leqq\alpha\leqq q)$, then $D(y_\alpha\partial_i)=f_\alpha(y)\partial_i$.

Let W_3 be a vector field on V satisfying the equations in (ii). Since $[W_3, \partial_i]=[W_3, I]=0$ $(1\leqq i\leqq p)$, then we get $W_3\in\mathscr{L}'$, by Lemma

1. 1(i), (iv). Write W_3 as $W_3=\sum_\beta h_\beta(y)\partial_\beta$, then

$$[W_3,\ y_\alpha\partial_i]=h_\alpha(y)\partial_i \qquad (1\leqq i\leqq p,\ 1\leqq\alpha\leqq q).$$

Hence, $h_\alpha(y)$ must be equal to $f_\alpha(y)$ for all α.

Q. E. D.

3. 3. Proof of Theorem 3. 2. Let $D\in\mathscr{D}$. Then, by Lemmata 3. 5~3. 8, we have a unique vector field W on V such that $D=\mathrm{ad}W$ on $\mathscr{T}_{n,m}$ for $n+m\leqq-1$. We can determine W as $W=W_1+W_2+W_3$, where $W_i(i=1, 2, 3)$ are given in the above lemmata. Clearly $W\in\mathscr{L}$.

Hence, by Corollary 2. 2, we get that $D=\mathrm{ad}W$ on $\mathscr{T}$.

Q. E. D.

3. 4. *Remarks.* (i) Any derivation of $\mathscr{T}$ or $\mathscr{L}$ is continuous, because it is realized as $\mathrm{ad}W$ for some $W\in\mathscr{L}$.

(ii) Let V' be a subspace of V, spanned by $y_1, \dots, y_q$. Then Theorem 3. 4 (i) can be rewritten as in the following form in terms of $C^\infty(V')$, which is suggestive for calculations of cohomologies of $\mathscr{T}$ with various coefficients.

Theorem 3. 9. *Let $\mathscr{D}er\ (C^\infty(V'))$ be the derivation algebra of the associative algebra $C^\infty(V')$. Then*

$$H^1(\mathscr{T}\,;\,\mathscr{T})\cong\mathscr{D}er\ (C^\infty(V')).$$

This follows immediately from the following well-known fact.

Lemma 3. 10. *There is an natural Lie algebra isomorphism of $\mathscr{L}'$ onto $\mathscr{D}er(C^\infty(V'))$.*

We give here its elementary proof for completeness. Let $D\in\mathscr{D}er\ (C^\infty(V'))$. Define functions $g_\alpha(y)$ $(\alpha=1, \dots, q)$ as $D(y_\alpha)=g_\alpha(y)$. Let $Y=\sum_\alpha g_\alpha(y)\partial_\alpha\in\mathscr{L}'$. The vector field Y operates on $C^\infty(V')$ as a first-order partial differential operator, then it defines a derivation

D_Y of $C^\infty(V')$. Easily by induction, we can show that D coincides with D_Y on the polynomial algebra $\boldsymbol{R}[y_1, \dots, y_q]$. Hence we obtain Lemma 3. 10, because when $j^2(g)(0)=0$, g is expressed as $g(y) = \sum_{\alpha,\beta} y_\alpha y_\beta g_{\alpha\beta}(y)$ with $g_{\alpha\beta} \in C^\infty(V')$.

§ 4. Lie Algebras $\mathscr{L}(\mathbf{M}, \mathscr{F})$, $\mathscr{T}(\mathbf{M}, \mathscr{F})$, and Their Derivations

4. 1. Lie Algebras Associated with Foliations. Let M be a $(p+q)$-dimensional manifold and $\mathscr{F}$ a codimension-q foliation on M. Around any point of M, there is a distinguished coordinate neighborhood $(U; x_1, \dots, x_p, y_1, \dots, y_q)$, for which a plate represented as $y_1 = \text{constant}, \dots, y_q = \text{constant}$ in U is a connected component of $L \cap U$ for some leaf L of $\mathscr{F}$ (see e. g. [6] for definitions).

A vector field X on a foliated manifold $(M, \mathscr{F})$ is called *leaf-tangent*, if X is tangent to the leaf L through p for any point p of M, that is, the vector X_p belongs to the tangent space $\mathrm{T}_p L$ of L at p. A vector field X is called to be *locally foliation preserving* (or *l. f. p.*, in short), if ϕ_t maps every plate to some plate, where $\{\phi_t\}$ is a one-parameter group of local diffeomorphisms generated by X.

Locally for any distinguished coordinates $(x_1, \dots, x_p, y_1, \dots, y_q)$, a leaf-tangent vector field is represented as $\sum_{i=1}^{p} f_i(x, y)\partial_i$, and a *l. f. p.* vector field is represented as $\sum_{i=1}^{p} f_i(x, y)\partial_i + \sum_{\alpha=1}^{q} g_\alpha(y)\partial_\alpha$, where $f_i(x, y)$ $(i=1, \dots, p)$ are C^∞-functions of $x_1, \dots, x_p, y_1, \dots, y_q$, and $g_\alpha(y)$ $(\alpha=1, \dots, q)$ are C^∞-functions of $y_1, \dots, y_q$. Here we use the notations ∂_i or ∂_α instead of $\dfrac{\partial}{\partial x_i}$ or $\dfrac{\partial}{\partial y_\alpha}$ respectively, and the convention on indices (see § 1. 1).

All *l. f. p.* vector fields on $(M, \mathscr{F})$ form a Lie algebra $\mathscr{L}(M, \mathscr{F})$, and all leaf-tangent vector fields form its ideal $\mathscr{T}(M, \mathscr{F})$.

If a *l. f. p.* vector field X is complete, then X is foliation preserving, that is, the diffeomorphism ϕ_t maps every leaf of $\mathscr{F}$ to some leaf for each t. Simliarly, if a leaf-tangent vector field X is complete, ϕ_t leaves every leaf of $\mathscr{F}$ stable. Thus, when M is compact, *l. f. p.* vector fields are foliation preserving.

4.2. Derivations. Let $\mathscr{D}(M, \mathscr{F}) = \mathscr{D}_{ex}(\mathscr{T}(M, \mathscr{F})\,;\, \mathscr{L}(M, \mathscr{F}))$ be the space of derivations of $\mathscr{T}(M, \mathscr{F})$ with values in $\mathscr{L}(M, \mathscr{F})$. And let $\mathscr{D}_{\mathscr{L}}(M, \mathscr{F})$ or $\mathscr{D}_{\mathscr{T}}(M, \mathscr{F})$ be the derivation algebra of $\mathscr{L}(M, \mathscr{F})$ or $\mathscr{T}(M, \mathscr{F})$ respectively. Sometimes we omit $\mathscr{F}$ in the notations $\mathscr{T}(M, \mathscr{F})$, $\mathscr{D}(M, \mathscr{F})$, etc.

Lemma 4.1. *Let U be an open subset of M, and $X \in \mathscr{L}(M, \mathscr{F})$. Assume that $[X, Y]=0$ on U for any $Y \in \mathscr{T}(M, \mathscr{F})$ with support contained in U. Then, $X=0$ on U.*

Proof. Let $p \in U$. Take a sufficiently small neighborhood U' of p in U, and distinguished coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ in U'. Let a vector field Y' on U' be any one of ∂_i, $x_j\partial_i$, and $y_\alpha\partial_i$ $(1 \leqq i, j \leqq p,\ 1 \leqq \alpha \leqq q)$. Since $\mathscr{T}(M)$ is $C^\infty(M)$-module, there is a vector field $Y \in \mathscr{T}(M)$ such that $Y=Y'$ on U' and the support of Y is contained in U. Then we have $[X, Y]=0$ on U by the assumption. By the proof of Lemma 3.8, we have that $X=0$ on U', in particular, at p. Hence we get $X=0$ on U. Q. E. D.

From this lemma, we get the following two lemmata, similarly as Proposition 2.4 and Corollary 2.5 in [4].

Lemma 4.2. *Let $D \in \mathscr{D}(M, \mathscr{F})$ or $\mathscr{D}_{\mathscr{L}}(M, \mathscr{F})$. Then, D is local.*

Lemma 4.3. *Let $D \in \mathscr{D}(M, \mathscr{F})$. Then, D is localizable* (*see* § 1.2 *in* [4] *for definition*).

4.3. Proposition 4.4. *Let $D \in \mathscr{D}(M, \mathscr{F})$. Then, there exists a vector field W on M such that $D = \mathrm{ad}\, W|_{\mathscr{T}(M, \mathscr{F})}$. Moreover, W is in $\mathscr{L}(M, \mathscr{F})$.*

Proof. Take a distinguished coordinate neighborhood system $\{U_\lambda\,;\,(x_1^\lambda, \ldots, x_p^\lambda, y_1^\lambda, \ldots, y_q^\lambda)\}_{\lambda \in \Lambda}$ on $(M, \mathscr{F})$. Since D is localizable, the derivation $D_{U_\lambda} \in D(U_\lambda, \mathscr{F}|_{U_\lambda})$ can be defined for all $\lambda \in \Lambda$ in such a way that $D(X)|_{U_\lambda} = D_{U_\lambda}(X|_{U_\lambda})$ for all $X \in \mathscr{T}(M)$. Then by Theorem

3. 2, there exists a unique vector field W_λ on U_λ such that $D_{U_\lambda} = \mathrm{ad}\, W_\lambda|_{\mathscr{T}(U_\lambda)}$ for any $\lambda \in \Lambda$. On the other hand, we have $D_{U_\lambda}|_{U_\lambda \cap U_\mu} = D_{U_\mu}|_{U_\lambda \cap U_\mu}$, so $W_\lambda = W_\mu$ on $U_\lambda \cap U_\mu$. Hence there is a vector field W on M such that $W = W_\lambda$ on U_λ for all $\lambda \in \Lambda$ and that $D = \mathrm{ad}\, W|_{\mathscr{T}(M)}$. Moreover, we have $W \in \mathscr{L}(M)$, because $W_\lambda \in \mathscr{L}(U_\lambda)$ for all $\lambda \in \Lambda$.

Q. E. D.

Corollary 4. 5. *Let $D \in \mathscr{D}_{\mathscr{T}}(M, \mathscr{F})$ or $\mathscr{D}_{\mathscr{L}}(M, \mathscr{F})$. Then there exists a vector field W on M such that $D = \mathrm{ad}\, W|_{\mathscr{T}(M, \mathscr{F})}$ or $= \mathrm{ad}\, W|_{\mathscr{L}(M, \mathscr{F})}$ respectively. Moreover, W is in $\mathscr{L}(M, \mathscr{F})$.*

Proof. Obvious for the case $D \in \mathscr{D}_{\mathscr{T}}(M)$. Let $D \in \mathscr{D}_{\mathscr{L}}(M)$. The restriction of D to $\mathscr{T}(M)$ belongs to $\mathscr{D}(M)$. Then the assertion follows from Proposition 4. 4 and Lemma 4. 1. Q. E. D.

Then we get Main Theorem similarly as Theorem 3. 4.

Theorem 4. 6. (*i*) *All derivations of $\mathscr{L}(M, \mathscr{F})$ are inner, that is, $\mathscr{D}_{\mathscr{L}}(M, \mathscr{F}) = \mathrm{ad}\mathscr{L}(M, \mathscr{F}) \cong \mathscr{L}(M, \mathscr{F})$. Hence*

$$H^1(\mathscr{L}(M, \mathscr{F});\ \mathscr{L}(M, \mathscr{F})) = 0.$$

(*ii*) *The derivation algebra of $\mathscr{T}(M, \mathscr{F})$ is naturally isomorphic to $\mathscr{L}(M, \mathscr{F})$, that is, $\mathscr{D}_{\mathscr{T}}(M, \mathscr{F}) = \{\mathrm{ad}\, W|_{\mathscr{T}(M, \mathscr{F})};\ W \in \mathscr{L}(M, \mathscr{F})\} \cong \mathscr{L}(M, \mathscr{F})$. Hence*

$$H^1(\mathscr{T}(M, \mathscr{F});\ \mathscr{T}(M, \mathscr{F})) \cong \mathscr{L}(M, \mathscr{F}) / \mathscr{T}(M, \mathscr{F}).$$

4. 4. Examples. Let $H^1 = H^1(\mathscr{T}(M, \mathscr{F});\ \mathscr{T}(M, \mathscr{F})) \cong \mathscr{L}(M, \mathscr{F}) / \mathscr{T}(M, \mathscr{F})$ for a foliated manifold $(M, \mathscr{F})$. In many cases, H^1 are of infinite dimension.

Proposition 4. 7. *Assume that there is a compact leaf L of $\mathscr{F}$ such that there is a saturated neighborhood U of L, which is a product foliation $D^q \times L$, where D^q is a q-dimensional disk. Then, H^1 is of infinite dimension.*

Proof. Every leaf in U is represented by a point of D^q. Let f be a function supported in D^q. Then $f\cdot\mathscr{L}(M, \mathscr{F})\subset\mathscr{L}(M, \mathscr{F})$. Hence the assertion follows from Theorem 3. 4. Q. E. D.

However, H^1 may be of finite dimension. Assume that M is compact. J. Leslie [5] gives examples of dim $H^1=0$, or 1: (i) an Anosov flow with an integral invariant for dim $H^1=0$, and (ii) irrational flows on a two dimensional torus T^2 for dim $H^1=1$. We can modify the latter to get a foliated manifold with dim $H^1=n$ (for arbitrary $n<+\infty$), that is, irrational flows on an $(n+1)$-dimensional torus T^{n+1}.

We have also other examples. Fukui and Ushiki [2] shows that dim $H^1=2$ for the Reeb foliation on a 3-shpere S^3. Further, Fukui [1] shows that the following: let $(M, \mathscr{F})$ be a Reeb foliated 3-manifold, then dim H^1 is finite, and equals to the number of generalized Reeb components.

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