$[1, 2)$ 上のベータ変換により生成される非再帰型擬似乱数の分布の研究

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序文

この論文では，($[0,1)$ 上の)β 変換を拡張した $[1,2)$ 上の β 変換により生成される乱数の分布についての解析を行っている．$[1,2)$ 上の β 変換は，($[0,1)$ 上の)linear mod 1 変換の特別な場合として捉えられるが，β 変換，linear mod 1 変換は Rényi や Parry など様々な数学者により研究されている．本論文ではこれらの研究結果を用いて乱数の分布を解析しようとするものである．本論文の構成は以下のとおりである．

第 1 章では，序論として本論文を読み進めるにあたって必要な用語の定義を述べている．
第 2 章では，f-展開の定義や f-展開可能な条件について述べている．
第 3 章では，f-展開の考え方を用いた β-展開とそれに伴う β 変換について述べ，β 変換の不変測度について新たに Perron-Frobenius 作用素の不動点の概念を用いた直截的な証明を行っている．
第 4 章では，第 3 章で述べた不変測度についての証明を linear mod 1 変換に拡張して証明をしている．
第 5 章では，乱数の分布を具体的な関数で表現するための解析を行っている．具体的には，各 β に対して無限級数で表現される不変測度について，有限和をとり，それを正規化した関数を考え，さらにその関数を β について平均する．そしてこの関数と実際の乱数分布との比較を行っている．

最後に，本論文の作成にあたり指導してくださった谷口礼偉特任教授，そして 6 年間共に研鑽し合った研究仲間の山口知也氏に深い感謝をし，序文とする．

1　序論

1.1　エルゴード性

X を集合とする．写像 T が $^{\forall}x \in X$ に対して，$T(x) \in X$ を満たすならば，T は**変換 (transformation)** と言われる．$T : X \longrightarrow X$ を変換とする．全単射である変換を**可逆変換 (invertible transformation)** という．集合 $\{T^n(x)\}_{n\geq 0}$ を x の**正軌道**と呼ぶ．また，T が可逆変換のとき x の全軌道 $\{T^n(x)\}_{n=-\infty}^{\infty}$ を考える．変換 $T : X \longrightarrow X$ に対して，$^{\exists}n, T^n(x) = x$ ならば，その点 x を変換 T の下での**周期点**と呼ぶ．そして，その整数 n を x の**周期**と呼ぶ．言い換えると，**最小周期**はそのような整数 n の中で最も小さい整数である．全ての点が T の周期点であれば変換 T を**周期変換**と呼び，全ての点が同じ周期を持てば**狭義周期的**であるという．

$(X, \mathcal{S}, \mu)$ を測度空間とする．変換 $T : X \longrightarrow X$ が，$^{\forall}A \in \mathcal{S}, T^{-1}(A) \in \mathcal{S}$ を満たすならば，T を**可測変換 (measurable transformation)** と言う．そして，可逆変換 T について T と T^{-1} が可測ならば T を**可逆可測変換 (inverse measurerable transformation)** と言う．また，T が可測で，

$$^{\forall}A \in \mathcal{S}; \mu(T^{-1}(A)) = \mu(A)$$

が成り立つばらば、変換 T は**保測 (measure-preserving)** と言われる．このような場合，μ を T に対する**不変測度 (invariant measure)** と言う．このように，保測変換は可測である必要があるが，可測性を強調するために変換が可測かつ保測と記述することもある．**可逆保測変換 (invertible measure-preserving)** は保測である可逆変換である．

$T : X \longrightarrow X$ を変換とする．$A \subset X$ が，

$$^{\forall}x \in A; T(x) \in A$$

を満たすならば，A を**正不変 (positively invariant 又は positively T-invariant)** という．

A が正不変のとき，T を A に制限すれば，A に関して，$T : A \longrightarrow A$ が定義される．変換 T を A に制限したものが A の変換となるならば，A は正不変集合である．特に $T^{-1}(A) = A$, つまり $x \in A \Leftrightarrow T(x) \in A$ ならば，集合 A は，**strictly T-invariant** である．エルゴード理論では，strictly invariant set を単に **invariant** 又は **T-invariant** と呼ぶ．

保測変換 T が strictly T-invariant である集合 A に対していつでも $\mu(A) = 0$ 又は $\mu(A^c) = 0$ を満たすならば，保測変換 T は**エルゴード的 (ergodic)** であるという．

定理 1.1 (Birkhoff)**.** T がエルゴード変換のとき，可積分関数 f に関して，

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f(x)\, d\mu \ a.e.\ x$$

が成り立つ．[3]

$A \subset X$ に対して，

$$\mathbb{I}_A(x) = \begin{cases} 1 & (x \in A) \\ 0 & (x \notin A) \end{cases}$$

を満たす関数 $\mathbb{I}_A$ を集合 X における部分集合 A の**特性関数**と呼ぶ．Birkhoff のエルゴード定理で $f(x) = \mathbb{I}_A(x)$ とおくと，

$$\lim_{n\to\infty} \frac{1}{N} \sharp\{n \in \mathbb{Z} \mid T^n(x) \in A,\ 0 \leq n \leq N-1\} = \mu(A)\ a.e.\ x$$

を得る．これは，ほとんど全ての x に対して $\{T^n(x)\}_{n=0}^{\infty}$ の分布が μ で記述されることを示している．したがって，T がエルゴード的なとき $\{T^n(x)\}_{n=0}^{\infty}$ の分布を知るためには，T の不変測度 μ を求めれば良いことが分かる．

2 実数表現とそれらのエルゴード的な性質

2.1 f-展開の導入

非負実数 x をある単調な非負関数 $y = f(x)$ を無限に反復させることによって，

$$x = \varepsilon_0 + f\left(\varepsilon_1 + f\left(\varepsilon_2 + \cdots\right)\right) \quad (\varepsilon_n \text{ は非負の整数 })$$

と表現することについて考える (f-展開)．この表現では，桁部分 $\varepsilon_n = \varepsilon_n(x)\,(n \in \mathbb{N} \cup \{0\})$ とその余り

$$r_n(x) = f\left(\varepsilon_{n+1} + f\left(\varepsilon_{n+2} + \cdots\right)\right)$$

は，次の帰納的な関係で定義される．

$$\begin{aligned} \varepsilon_0(x) &= [x] \quad , \qquad r_0(x) = \{x\} \\ \varepsilon_{n+1}(x) &= [\varphi\left(r_n(x)\right)] \quad , \qquad r_{n+1}(x) = \{\varphi\left(r_n(x)\right)\}. \end{aligned}$$

ただし，$[x]$ は x の整数部分，$\{x\}$ は x の小数部分，$x = \varphi(y)$ は $y = f(x)$ の逆関数とする．例えば，$f(x) = \frac{1}{x}$ とすると，

$$\begin{aligned} \sqrt{2} &= 1 + \left(f\left(2 + f\left(2 + f\left(2 + \cdots\right)\right)\right)\right) \\ &= 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \end{aligned}$$

と表現される (連分数展開)．本論文では f が単調増加である場合を扱う．

$f(x)$ を定義域が $\bigcup_{i=0}^{k}[a_i, b_i)\left(k \geq 1, [a_i, b_i) \subset [i, i+1), \bigcup_{i=0}^{k}[a_i - i, b_i - i) = [0,1)\right)$，値域が $[0,1)$ となるような狭義単調増加関数とする．$\mathfrak{T}$ を $f(x)$ の集まりとする．$X = [0,1)$, $\mathfrak{B}$ を X 上の Borel 集合族，λ を Lebesgue 測度，$f \in \mathfrak{T}$ に対して，

$$T(x) = f^{-1}(x) \bmod 1 \quad (= \{f^{-1}(x)\})$$

とすると，確率系 $(X, \mathfrak{B}, \lambda, T)$ を定義できる．系 $(X, \mathfrak{B}, \lambda, T)$ を f によって生成される**表現過程**と呼ぶ．$\overline{f}$ を f の定義域 $[0, \infty)$，値域 $[0,1]$ への一意的な単調拡張とする．すなわち，

$$\overline{f}(x) = \sup\{f(t) \mid t \leq x,\, t \in \bigcup_{i=0}^{k}[a_i, b_i)\} \tag{2.1}$$

とする．${}^{\forall}x \in [0,1)$

$$x_n = \left[f^{-1}\left(T^n(x)\right)\right] \quad (n = 0, 1, 2, \ldots)$$

ならば，

$$x = \overline{f}\left(x_0 + \overline{f}\left(x_1 + \ldots\right)\right)$$

が成り立つとき，表現過程は**有効**である，又は f-展開は**有効**であるという．

2.2 f-展開と諸定理

単調増加関数 f に関する f-展開は以下のような場合に有効になる (Rényi[4]).

定理 2.1 (表現定理). $f(x)$ について以下を仮定する.

$(\mathbf{A_1})$ $f(0) = 0$.

$(\mathbf{A_2})$ $1 < S \le +\infty$ とする. $0 \le t \le S$ で $f(t)$ は連続で狭義単調増加であり, $t \ge S$ で $f(t) = 1$ である. ($S = +\infty$ は $\lim_{t\to\infty} f(t) = 1$ を意味する.)S について 3 つの場合に分ける.

$(\mathbf{A_{2_1}})$ $S = +\infty$.

$(\mathbf{A_{2_2}})$ $1 < S < +\infty\,(S \in \mathbb{Z})$.

$(\mathbf{A_{2_3}})$ $1 < S < +\infty\,(S \notin \mathbb{Z})$.

いずれの場合でも, 次の条件を満たす.

$(\mathbf{A_3})$ $0 \le {}^\forall t_1 < {}^\forall t_2$ に対して, $\frac{f(t_1)-f(t_2)}{t_1-t_2} < 1$ が成り立つ.

$(\mathbf{A_1}) \sim (\mathbf{A_3})$ が満たされるとき, f-展開は任意の実数 x について有効である.

この定理において整数 ε_n のとり得る値は,
$(\mathbf{A_{2_1}})$ の場合は, $0, 1, 2, \cdots,$
$(\mathbf{A_{2_2}})$ の場合は, $0, 1, 2, \cdots, S-1,$
$(\mathbf{A_{2_3}})$ の場合は, $0, 1, 2, \cdots, [S],$
である. 有限列 $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ は定理 2.1 で $(\mathbf{A_1}) \sim (\mathbf{A_3})$ の条件を満たす f に関して, $\varepsilon_k(x) = \varepsilon_k\,(k = 1, 2, \cdots, n)$ となるような $x(0 \le x < 1)$ が存在するとき, f に関する **canonical 列**と言われる. S が整数あるいは $S = +\infty$ のときは canonical 列の値は独立に選ぶことができるが, S が有限で整数でないときには canonical 列の要素の間に従属性がある. この意味で $(\mathbf{A_{2_1}})$ あるいは $(\mathbf{A_{2_2}})$ を満たす展開を**独立桁を持つ** f-展開と呼ぶ. $(\mathbf{A_{2_3}})$ の条件を満たしているとき, その f-展開を**従属桁を持つ** f-展開と呼ぶ. 本論文で扱う β 変換, linear mod 1 変換は従属桁を持つ f-展開である.

定理 2.1 は, Parry[2] により次のように拡張される.

定理 2.2. $f \in \mathfrak{T}$ が, $0 < {}^\forall t_2 < {}^\forall t_1$ に対して,

$$\frac{\overline{f}(t_1) - \overline{f}(t_2)}{t_1 - t_2} < 1$$

を満たすならば, f-展開は有効である.

この定理は, 4 章で扱う linear mod 1 変換が有効であることを示すときに使われる. 次の定理も Parry[2] による.

定理 2.3. $f \in \mathfrak{T}$ とする. このとき, f-展開が有効であることと, $x \neq y$ ならば,

$$x_0, x_1, \ldots \neq y_0, y_1, \ldots$$

であることは同値である.

3　β-展開と不変測度

β を整数でない 1 より大きい数とし，$f(x)=\frac{1}{\beta}x$ とする．この $f(x)$ は定理 2.1 の $(\mathbf{A_1})$ を満たし，$x \geq \beta$ のときは，$f(x)=1$ と定めれば，$S=\beta$ で $(\mathbf{A_{2_3}})$ を満たす．また、$f'(x)=\frac{1}{\beta}$ であるから，$(\mathbf{A_3})$ も満たされる．したがって，${}^\forall x \geq 0$ に対して，非負の整数 $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ により，x を

$$x=\varepsilon_0+\frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_n}{\beta^n}+\cdots$$

と展開することができる (**β-展開**)．定理 2.3 により，このような展開は一意である．$\varepsilon_n=\varepsilon_n(x)$, $r_n(x)$ は次の帰納的な関係で定義される．

$$\begin{aligned}\varepsilon_0(x)=[x] \quad &, \qquad r_0(x)=\{x\},\\ \varepsilon_{n+1}(x)=[\beta(r_n(x))] \quad &, \qquad r_{n+1}(x)=\{\beta(r_n(x))\}.\end{aligned}$$

ここで，$[0,1)$ 上の変換 T を $T(x)=\{\beta x\}$ で定める．T は **β 変換**と言われる．$X=[0,1)$ 上の Borel 集合族を $\mathfrak{B}$, $(X,\mathfrak{B})$ 上の Lebesgue 測度を λ とする．$(X,\mathfrak{B},\lambda)$ は確率空間になる．Rényi[2] は，T に対して Lebesgue 測度 λ と equivalent で $T(x)=\{\beta x\}$ のもと不変で一意的な正規化された測度 ν が存在することを示した．ここで ν が λ と equivalent であるとは,．${}^\forall A \in \mathcal{S}$, $\nu(A)=0 \Longleftrightarrow \lambda(A)=0$ を満たすことをいう．一般に，$(X,\mathcal{S})$ を可測空間，μ,ν を $(X,\mathcal{S})$ 上の測度とするとき，ν が μ について**絶対連続**であるとは，任意の $A\in\mathcal{S}$ について，

$$\mu(A)=0 \Longrightarrow \nu(A)=0$$

が成り立つことをいう．(記号は $\nu \ll \mu$)．したがって，ν が μ に equivalent であるとは，$\nu \ll \mu$ かつ $\mu \ll \nu$ と同等である．$\nu \ll \mu$ のとき以下の定理が成り立つ．

定理 3.1 (Radon-Nikodym)．μ,ν を $\mathcal{S}$ 上の σ-有限な測度とするとき，$\nu \ll \mu \quad \Longrightarrow {}^\exists\phi$($X$ 上での μ 可積分関数),

$${}^\forall A\in\mathcal{S};\ \nu(A)=\int_A \phi(x)\,d\mu(x).$$

この定理から，$\nu \ll \lambda$ のとき，可積分関数 $h_\nu(x)$ が測度 0 を許して一意に定義され

$${}^\forall E\in\mathfrak{B};\ \nu(E)=\int_E h_\nu(x)dx \tag{3.1}$$

となる．

以下便宜上，

$$T^0(x)=x$$

と定義する．したがって任意の $n\in\mathbb{N}$ に対して，

$$T^n(1)=\{\beta T^{n-1}(1)\}$$

あるいは，

$$T^n(1)=T^{n-1}(\{\beta\}).$$

また，

$$\beta = \varepsilon_0(\beta) + \frac{\varepsilon_1(\beta)}{\beta} + \frac{\varepsilon_2(\beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(\beta)}{\beta^n} + \cdots, \tag{3.2}$$

$$\varepsilon_n(\beta) = [\beta T^{n-1}(\{\beta\})] = [\beta T^n(1)] \tag{3.3}$$

である．

$(X, \mathfrak{B}, \lambda)$ 上の可積分関数全体のなす空間を L^1 とおく．X 上の区分的に単調な C^1 級の写像 T に対応する **Perron-Frobenius** 作用素 $\mathfrak{L} : L^1 \longrightarrow L^1$ を以下で定義する．$f \in L^1$ に対し $\mathfrak{L}f \in L^1$ は,

$$(\mathfrak{L}f)(y) = \sum_{x \in T^{-1}(y)} \frac{f(x)}{|T'(x)|}$$

となる関数である．$\mathfrak{L}$ についてもう少し説明を付け加えると，T は各区間 $J_i = \langle b_i, b_{i+1} \rangle$ 上で単調，その内点で微分可能，かつ $X = \cup_i J_i$ とする．ただし，$J_i = \langle b_i, b_{i+1} \rangle$ は b_i, b_{i+1} を端点とする区間であり，各端点はその区間に属しても属さなくてもよい．また，端点における微係数は，その端点が属する区間の内点からの微分を意味する．このとき T の各 J_i の制限を T_i と記し，

$$h_i(y) = |T_i'(T_i^{-1}(x))|^{-1} \mathbb{I}_{T_i(J_i)}(x) \tag{3.4}$$

とおくとき，

$$(\mathfrak{L}f)(x) = \sum_i h_i(x) f(T_i^{-1}(x)) \tag{3.5}$$

となる．$\mathfrak{L}f$ については次の定理がある．

定理 3.2. **(1)** 有界可測関数 g に対して，

$$\int_X g(T(x)) f(x)\, dx = \int_X g(y)(\mathfrak{L}f)(y)\, dy \tag{3.6}$$

が成り立つ．

(2) 測度 $\mu (\mu \ll \lambda)$ は，T の不変測度である．

$\Longleftrightarrow$

$\mathfrak{L}f = f$ を満たす $f \in L^1$ が存在して，

$$\mu(A) = \int_A f(x)\, dx.$$

この定理の証明は，久保, 矢野 [7] などにあるが重要であるので証明を記しておく．

証明. **(1)** $y = T_i(x)$ と変数変換する．両辺を x で微分すると，

$$\begin{aligned} dy &= T_i'(x) dx, \\ dx &= \frac{1}{T_i'(x)} dy = \frac{1}{T_i'(T_i^{-1}(y))} \end{aligned} \tag{3.7}$$

となる．式 (3.6) の左辺を変形すると，

$$\int_X g(T(x)) f(x)\, dx = \sum_i \int_{J_i} g(T(x)) f(x)\, dx$$

$$
\begin{aligned}
&= \sum_i \int_{T_i(J_i)} g(y) f(T_i^{-1}(y)) \cdot \frac{1}{|T'(T_i^{-1}(y))|}\, dy \\
&= \int_X g(y) \sum_i \mathbb{I}_{T_i(J_i)}(y) \cdot \frac{1}{|T'(T_i^{-1}(y))|} f(T_i^{-1}(y))\, dy \\
&= \int_X g(y) \sum_i h_i(x) f(T_i^{-1}(y))\, dy \\
&= \int_X g(y) \mathfrak{L} f(y)\, dy.
\end{aligned}
$$

以上より **(1)** は成り立つ．

(2) ($\Longrightarrow$) 測度 $\mu(\mu \ll \lambda)$ が T の不変測度と仮定する．定理 3.1 より，

$$
{}^{\exists} f \in L^1,\ {}^{\forall} A \in \mathfrak{B};\ \mu(A) = \int_X \mathbb{I}_A(x) f(x)\, dx \tag{3.8}
$$

となる．また，式 (3.6) で $g(x) = \mathbb{I}_A$ とすれば，

$$
\int_X \mathbb{I}_A(T(x)) f(x) dx = \int_X \mathbb{I}_A(y)(\mathfrak{L} f)(x)\, dy \tag{3.9}
$$

を得る．式 (3.9) の左辺は，

$$
\int_X \mathbb{I}_A(T(x)) f(x)\, dx = \int_X \mathbb{I}_{T^{-1}(A)}(x) f(x)\, dx = \mu(T^{-1}(A)) \tag{3.10}
$$

である．μ が不変測度であるので，${}^{\forall} A \in \mathfrak{B}$ に対して，$\mu(A) = \mu(T^{-1}(A))$ を満たすが，これは，式 (3.8), (3.9), (3.10) により，${}^{\forall} A$ に対して，

$$
\int_X \mathbb{I}_A(x) f(x)\, dx = \int_X \mathbb{I}_A(y)(\mathfrak{L} f)(y)\, dy,
$$

すなわち，

$$
\int_X \mathbb{I}_A(x)\{f(x) - (\mathfrak{L} f)(x)\}\, dx = 0
$$

を意味する．これから，$f(x) = (\mathfrak{L} f)(x)\, a.e.\ x$, を得る．
($\Longleftarrow$) 任意の $A \in \mathfrak{B}$ に対して，

$$
\begin{aligned}
\mu(A) &= \int_X \mathbb{I}_A(x) f(x)\, dx \\
&= \int_X \mathbb{I}_A(x)(\mathfrak{L} f)(x)\, dx \\
&= \int_X \mathbb{I}_A(T(x)) f(x)\, dx \quad (\text{ 式 (3.6) より }) \\
&= \int_X \mathbb{I}_{T^{-1}(A)}(x) f(x)\, dx \\
&= \mu(T^{-1}(A)).
\end{aligned}
$$

したがって，μ は不変測度である．

$\square$

Parry は [3] で

$$h_\beta(y)=\sum_{y<T^n(1)}\frac{1}{\beta^n}$$

が T の不変測度を与えることを証明しているが，その証明は長くて見通しが悪いので本論文では，定理 3.2 の (2) の $\mathfrak{L}f=f$ を直接示す新たな証明をつけておく．

定理 3.3. $T(x)=\{\beta x\}$ とする．$\mathfrak{L}$ を T に対する *Perron-Frobenius* 作用素とすれば，$h_\beta(y)=\sum_{y<T^n(1)}\frac{1}{\beta^n}$ は T の不動点関数である．すなわち，$h_\beta(y)=(\mathfrak{L}h_\beta)(y)$ *a.e.* y.

証明.

$$\begin{aligned}h_\beta(y)&=\sum_{y<T^n(1)}\frac{1}{\beta^n}\\&=1+\sum_{n=1}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}(y)\end{aligned}$$

と表されるので，

$$\begin{aligned}\beta(h_\beta)(y)&=\beta+\sum_{n=1}^{\infty}\frac{1}{\beta^{n-1}}\mathbb{I}_{[0,T^n(1))}(y)\\&=\beta+\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y)\end{aligned}\tag{3.11}$$

である．T に対する Perron-Frobenius 作用素は，

$$(\mathfrak{L}f)(y)=\frac{1}{\beta}f\left(\frac{y}{\beta}\right)+\frac{1}{\beta}f\left(\frac{y}{\beta}+\frac{1}{\beta}\right)+\cdots+\frac{1}{\beta}f\left(\frac{y}{\beta}+\frac{[\beta]-1}{\beta}\right)+\frac{1}{\beta}f\left(\frac{y}{\beta}+\frac{[\beta]}{\beta}\right)\mathbb{I}_{[0,T(1))}(y)$$

であるので，$f=h_\beta$ とおくと，

$$(\mathfrak{L}h_\beta)(y)=\frac{1}{\beta}h_\beta\left(\frac{y}{\beta}\right)+\frac{1}{\beta}h_\beta\left(\frac{y}{\beta}+\frac{1}{\beta}\right)+\cdots+\frac{1}{\beta}h_\beta\left(\frac{y}{\beta}+\frac{[\beta]-1}{\beta}\right)+\frac{1}{\beta}h_\beta\left(\frac{y}{\beta}+\frac{[\beta]}{\beta}\right)\mathbb{I}_{[0,T(1))}(y)$$

となる．$h_\beta(y)=(\mathfrak{L}h_\beta)(y)$ を証明する代わりに，$\beta h_\beta(y)=\beta(\mathfrak{L}h_\beta)(y)$ を証明する．

$$\begin{aligned}\beta(\mathfrak{L}h_\beta)(y)&=h_\beta\left(\frac{y}{\beta}\right)+h_\beta\left(\frac{y}{\beta}+\frac{1}{\beta}\right)+\cdots+h_\beta\left(\frac{y}{\beta}+\frac{[\beta]-1}{\beta}\right)+h_\beta\left(\frac{y}{\beta}+\frac{[\beta]}{\beta}\right)\mathbb{I}_{[0,T(1))}(y)\\&=1+\sum_{n=1}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}(y)\\&\quad+1+\sum_{n=1}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}+\frac{1}{\beta}\right)\\&\quad+1+\sum_{n=1}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}+\frac{[\beta]-1}{\beta}\right)\\&\quad\vdots\\&\quad+1+\sum_{n=1}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}+\frac{[\beta]}{\beta}\right)\\&=1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\frac{y}{\beta}\right)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}\right)\end{aligned}$$

$$
\begin{aligned}
&\qquad \vdots \\
&+1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\frac{y}{\beta}+\frac{1}{\beta}\right)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}+\frac{1}{\beta}\right)+\cdots \\
&+1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\frac{y}{\beta}+\frac{[\beta]-1}{\beta}\right)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}+\frac{[\beta]-1}{\beta}+\cdots\right) \\
&+\left\{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\frac{y}{\beta}+\frac{[\beta]}{\beta}\right)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\frac{y}{\beta}+\frac{[\beta]}{\beta}\right)+\cdots\right\}\mathbb{I}_{[0,T(1))}(y) \\
=\ & 1+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y)+\cdots \\
&+1+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y+1)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y+1)+\cdots \\
&\qquad \vdots \\
&+1+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y+[\beta]-1)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta]-1)+\cdots \\
&+\Big\{\mathbb{I}_{[0,T(1)}(y)+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y+[\beta])\,\mathbb{I}_{[0,T(1))}(y)+\cdots \\
&\qquad +\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])\,\mathbb{I}_{[0,T(1))}(y)+\cdots\Big\}.
\end{aligned}
$$

ここで，$\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])=1$ とする．$[\beta]\leq\beta T^n(1)<\beta$ より，$\frac{[\beta]}{\beta}\leq T^n(1)<\frac{\beta}{\beta}=1$ である．今 $\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])=1$ より，$0\leq[\beta]+y<\beta T^n(1)<\beta$，つまり，$[\beta]\leq[\beta]+y<\beta T^n(1)<\beta$ となる．両辺から $[\beta]$ を引いて，$0\leq y<\beta-[\beta]=T(1)$ となるから，$\mathbb{I}_{[0,T(1))}(y)=1)$ である．以上より，$\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])=1$ ならば，$\mathbb{I}_{[0,T(1))}=1$ が成り立つので，任意の $n(\in\mathbb{N})$ に対して，

$$
\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])\,\mathbb{I}_{[0,T(1))}(y)=\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])
$$

が成り立つ．これをふまえて書きなおすと，

$$
\begin{aligned}
\beta(\mathfrak{L}h_\beta)(y)=\ & 1+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y)+\cdots \\
&+1+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y+1)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y+1)+\cdots \\
&\qquad \vdots \\
&+1+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y+[\beta]-1)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta]-1)+\cdots \\
&+\left\{\mathbb{I}_{[0,T(1)}(y)+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}(y+[\beta])+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])+\cdots\right\} \\
=\ & 1+1+1+\cdots+\mathbb{I}_{[0,T(1))}(y) \\
&+\frac{1}{\beta}\left(\mathbb{I}_{[0,\beta T(1))}(y)+\mathbb{I}_{[0,\beta T(1))}(y+1)+\cdots+\mathbb{I}_{[0,\beta T(1))}(y+[\beta])\right) \\
&\qquad \vdots \\
&+\frac{1}{\beta^n}\left(\mathbb{I}_{[0,\beta T^n(1))}(y)+\mathbb{I}_{[0,\beta T^n(1))}(y+1)+\cdots+\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])\right) \\
&\qquad \vdots
\end{aligned}
$$

となる．ここで，任意の非負の整数 n に対して，

$$
\begin{aligned}
&\mathbb{I}_{[0,\beta T^n(1))}(y)+\mathbb{I}_{[0,\beta T^n(1))}(y+1)+\cdots+\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta]-1)+\mathbb{I}_{[0,\beta T^n(1))}(y+[\beta])\\
&=\sharp\{l \mid 0\le y+l<\beta T^n(1)\}\\
&=[\beta T^n(1)]+\mathbb{I}_{[0,\{\beta T^n(1)\})}(y)\\
&=[\beta T^n(1)]+\mathbb{I}_{[0,T^{n+1}(1))}(y).
\end{aligned}
$$

であるので，

$$
\begin{aligned}
\beta\left(\mathfrak{L}h_\beta\right)(y) &= [\beta T^0(1)]+\mathbb{I}_{[0,T(1))}(y)\\
&\qquad+\frac{1}{\beta}\left([\beta T(1)]+\mathbb{I}_{[0,T^2(1))}(y)\right)\\
&\qquad\qquad\vdots\\
&\qquad+\frac{1}{\beta^n}\left([\beta T^n(1)]+\mathbb{I}_{[0,T^{n+1}(1))}(y)\right)\\
&\qquad\qquad\vdots\\
&=[\beta T^0(1)]+\frac{1}{\beta}[\beta T(1)]+\cdots+\frac{1}{\beta^n}[\beta T^n(1)]+\cdots\\
&\qquad+\mathbb{I}_{[0,T(1)}(y)+\frac{1}{\beta}\mathbb{I}_{[0,\beta T(1))}\left(y+[\beta]\right)+\cdots+\frac{1}{\beta^n}\mathbb{I}_{[0,\beta T^n(1))}\left(y+[\beta]\right)+\cdots\\
&=\sum_{n=0}^{\infty}\frac{[\beta T^n(1)]}{\beta^n}+\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y)\\
&=\sum_{n=0}^{\infty}\frac{\beta T^n(1)-T^{n+1}(1)}{\beta^n}+\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y)\\
&=\beta+\sum_{n=1}^{\infty}\frac{\beta T^n(1)}{\beta^n}-\sum_{n=0}^{\infty}\frac{T^{n+1}(1)}{\beta^n}+\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y)\\
&=\beta+\sum_{n=1}^{\infty}\frac{T^n(1)}{\beta^{n-1}}-\sum_{n=0}^{\infty}\frac{T^{n+1}(1)}{\beta^n}+\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y)\\
&=\beta+\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y)
\end{aligned}
$$

となる．ここで，式 (3.11) より，$\beta\left(h_\beta\right)(y)$ と等しくなるから，$\beta\left(\mathfrak{L}h_\beta\right)(y)=\beta\left(h_\beta\right)(y)\,a.e.\ y$ が成り立つ．したがって，

$$
\left(\mathfrak{L}h_\beta\right)(y)=\left(h_\beta\right)(y)\,a.e.\ y
$$

が成り立つ． □

4 Linear mod 1 変換

β 変換の概念は linear mod 1 変換として拡張された．β を 1 より大きい整数でない数，$0 \leq \alpha < 1$ に対して，$f(x) = \frac{x-\alpha}{\beta}$ $(\alpha \leq x < \alpha + \beta)$ と定めれば，$f \in \mathfrak{T}$ であり ($\mathfrak{T}$ は 2.1 節参照)，$f'(x) = \frac{1}{\beta}$ であるので，定理 2.2 より f-展開は有効である．そこで，

$$T_{\beta,\alpha}(x) = \{f^{-1}(x)\} = \{\beta x + \alpha\}$$

とおく．以下簡略のため，$T_{\beta,\alpha}$ を T で表す．$T(x) = \beta x + \alpha - [\beta x + \alpha]$ であることに注意する．この $T : [0,1) \longrightarrow [0,1)$ は **linear mod 1 変換**と言われ，Parry は [2] で

$$h_{\beta,\alpha}(y) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \mathbb{I}_{[0,T^n(1))}(y) - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \mathbb{I}_{[0,T^n(0))}(y)$$

は T に対する不変測度を与えることを示した．本論文では，前章と同様に，$h_{\beta,\alpha}(y)$ が Perron-Frobenius 作用素の不動点関数であることを直接証明することにする．

定理 4.1. $T(x) = \{\beta x + \alpha\}$ とする．$\mathfrak{L}$ を T に対する *Perron-Frobenius* 作用素とすれば，$h_{\beta,\alpha}(y) = \sum_{n=0} \frac{1}{\beta^n} \mathbb{I}_{[0,T^n(1))}(y) - \sum_{n=1} \frac{1}{\beta^n} \mathbb{I}_{[0,T^n(0))}(y)$ は T の不動点関数である．すなわち，$h_{\beta,\alpha}(y) = (\mathfrak{L} h_{\beta,\alpha})(y)$ *a.e.* y.

定理 3.3 の考え方を拡張して，定理 4.1 を証明するが，丁寧な証明を心がけたので記述が少し長くなっている．定理 4.1 を証明するために，まず，次の補題 4.1 を証明する．

補題 4.1. β を 1 より大きな整数でない数，$0 \leq \alpha < 1$ に対して，以下の等式が成り立つ．

(1) $\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha]) \mathbb{I}_{[0,T(1))}(y) = \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha])$.

(2) $\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y + [\beta + \alpha]) \mathbb{I}_{[0,T(1))}(y) = \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y + [\beta + \alpha])$.

(3) $\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y) \mathbb{I}_{[T(0),1)}(y) = \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$.

(4) $\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y) \mathbb{I}_{[T(0),1)}(y) = \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)$.

証明. (1) $\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha]) \mathbb{I}_{[0,T(1))}(y) \leq \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha])$ であることは，自明だから，$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha]) \mathbb{I}_{[0,T(1))}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha])$ であることを示す．$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha]) = 1$ ならば，$\mathbb{I}_{[0,T(1))}(y) = 1$ であることを示す．

$$\begin{aligned} &\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y + [\beta + \alpha]) = 1 \\ &\Longrightarrow [\beta + \alpha] \leq y + [\beta + \alpha] < \beta T^n(1) + \alpha < \beta + \alpha. \end{aligned}$$

上式より，$[\beta + \alpha] \leq \beta T^n(1) + \alpha$ だから $[\beta + \alpha] \leq [\beta T^n(1) + \alpha]$ となる．また，$T^n(1) < 1$ だから，$\beta + \alpha \geq \beta T^n(1) + \alpha$ であるから，$[\beta + \alpha] \geq [\beta T^n(1) + \alpha]$ となる．したがって，$[\beta + \alpha] = [\beta T^n(1) + \alpha]$ となる．これをふまえ，$[\beta + \alpha] \leq y + [\beta + \alpha] < \beta T^n(1) + \alpha < \beta + \alpha$ の両辺から $[\beta + \alpha]$ を引くと，

$$\begin{aligned} &[\beta + \alpha] - [\beta + \alpha] \leq y + [\beta + \alpha] - [\beta + \alpha] < \beta T^n(1) + \alpha - [\beta + \alpha] < \beta + \alpha - [\beta + \alpha] \\ &\Longrightarrow [\beta + \alpha] - [\beta + \alpha] \leq y + [\beta + \alpha] - [\beta + \alpha] < \beta T^n(1) + \alpha - [\beta T^n(1) + \alpha] < \beta + \alpha - [\beta + \alpha] \\ &\Longrightarrow 0 \leq y < T^{n+1}(1) < T(1) \end{aligned}$$

$\Longrightarrow \mathbb{I}_{[0,T(1))}(y) = 1.$

よって，任意の $n \in \mathbb{N}$ に対して，

$$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])$$

が成り立つから，

$$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y) = \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])$$

が成り立つ．

(2) $\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y) \leq \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])$ であることは，自明だから，$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])$ であることを示す．$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]) = 1$ ならば，$\mathbb{I}_{[0,T(1))}(y) = 1$ であることを示す．

$$\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]) = 1 \\
&\Longrightarrow [\beta+\alpha] \leq y+[\beta+\alpha] < \beta T^n(1)+\alpha < \beta+\alpha.
\end{aligned}$$

上式より，$[\beta+\alpha] \leq \beta T^n(0)+\alpha$ だから $[\beta+\alpha] \leq [\beta T^n(0)+\alpha]$ となる．また，$T^n(0) < 1$ だから，$\beta+\alpha \geq \beta T^n(0)+\alpha$ であるから，$[\beta+\alpha] \geq [\beta T^n(0)+\alpha]$ となる．したがって，$[\beta+\alpha] = [\beta T^n(0)+\alpha]$ となる．これをふまえ，$[\beta+\alpha] \leq y+[\beta+\alpha] < \beta T^n(0)+\alpha < \beta+\alpha$ の両辺から $[\beta+\alpha]$ を引くと，

$$\begin{aligned}
&[\beta+\alpha]-[\beta+\alpha] \leq y+[\beta+\alpha]-[\beta+\alpha] < \beta T^n(0)+\alpha-[\beta+\alpha] < \beta+\alpha-[\beta+\alpha] \\
&\Longrightarrow [\beta+\alpha]-[\beta+\alpha] \leq y+[\beta+\alpha]-[\beta+\alpha] < \beta T^n(0)+\alpha-[\beta T^n(0)+\alpha] < \beta+\alpha-[\beta+\alpha] \\
&\Longrightarrow 0 \leq y < T^{n+1}(0) < T(1) \\
&\Longrightarrow \mathbb{I}_{[0,T(1))}(y) = 1.
\end{aligned}$$

よって，任意の $n \in \mathbb{N}$ に対して，

$$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])$$

が成り立つから，

$$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y) = \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])$$

が成り立つ．

(3) $\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) \leq \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$ であることは自明だから，$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$ であること示す．$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y) = 1$ ならば，$\mathbb{I}_{[T(0),1)}(y) = 1$ であることを示す．

（イ） $\beta T^n(1)+\alpha < 1$ のとき，

$$\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y) = 1 \\
&\Longrightarrow \alpha \leq y < \beta T^n(1)+\alpha < 1 \\
&\Longrightarrow \mathbb{I}_{[T(0),1)}(y) = 1.
\end{aligned}$$

（ロ） $\beta T^n(1)+\alpha \geq 1$ のとき，

$$
\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)=1\\
&\Longrightarrow \alpha \leq y < 1 \leq \beta T^n(1)+\alpha\\
&\Longrightarrow \mathbb{I}_{[T(0),1)}(y)=1.
\end{aligned}
$$

よって，$^\forall n \in \mathbb{N}$ に対して，

$$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$$

が成り立つから，

$$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) = \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$$

が成り立つ．

(4) $\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) \leq \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)$ であることは自明だから，$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)$ であること示す．$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)=1$ ならば，$\mathbb{I}_{[T(0),1)}(y)=1$ であることを示す．

（イ） $\beta T^n(0)+\alpha < 1$ のとき，

$$
\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)=1\\
&\Longrightarrow \alpha \leq y < \beta T^n(0)+\alpha < 1\\
&\Longrightarrow \mathbb{I}_{[T(0),1)}(y)=1.
\end{aligned}
$$

（ロ） $\beta T^n(0)+\alpha \geq 1$ のとき，

$$
\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)=1\\
&\Longrightarrow \alpha \leq y < 1 \leq \beta T^n(0)+\alpha\\
&\Longrightarrow \mathbb{I}_{[T(0),1)}(y)=1.
\end{aligned}
$$

よって，$^\forall n \in \mathbb{N}$ に対して，

$$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) \geq \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)$$

が成り立つから，

$$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y) = \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)$$

が成り立つ．

□

定理 4.1 の証明．

$$h_{\beta,\alpha}(y)=\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}(y)-\sum_{n=1}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}(y)$$

と表されるので，

$$\begin{aligned}
\beta(h_{\beta,\alpha})(y) &= \beta\left(\sum_{n=0}^{\infty}\frac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}(y) - \sum_{n=1}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}(y)\right) \\
&= \beta\left(1 + \sum_{n=1}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}(y) - \tfrac{1}{\beta}\mathbb{I}_{[0,T(0))}(y) - \sum_{n=2}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}(y)\right) \\
&= \beta + \sum_{n=1}^{\infty}\tfrac{1}{\beta^{n-1}}\mathbb{I}_{[0,T^n(1))}(y) - \mathbb{I}_{[0,T(0))}(y) - \sum_{n=2}^{\infty}\tfrac{1}{\beta^{n-1}}\mathbb{I}_{[0,T^n(0))}(y) \\
&= \beta + \sum_{n=0}^{\infty}\tfrac{1}{\beta^{n}}\mathbb{I}_{[0,T^{n+1}(1))}(y) - \mathbb{I}_{[0,T(0))}(y) - \sum_{n=1}^{\infty}\tfrac{1}{\beta^{n}}\mathbb{I}_{[0,T^{n+1}(0))}(y) \qquad (4.1)
\end{aligned}$$

である．T に対する Perron-Frobenius 作用素は，

$$\begin{aligned}
(\mathfrak{L}f)(y) &= \tfrac{1}{\beta}f\left(\tfrac{y-\alpha}{\beta}\right)\mathbb{I}_{[T(0),1)}(y) + \tfrac{1}{\beta}f\left(\tfrac{y-\alpha}{\beta}+\tfrac{1}{\beta}\right) \\
&\qquad\vdots \\
&\quad + \tfrac{1}{\beta}f\left(\tfrac{y-\alpha}{\beta}+\tfrac{[\beta+\alpha]-1}{\beta}\right) + \tfrac{1}{\beta}f\left(\tfrac{y-\alpha}{\beta}+\tfrac{[\beta+\alpha]}{\beta}\right)\mathbb{I}_{[0,T(1))}(y)
\end{aligned}$$

であるので，$f = h_{\beta,\alpha}$ とおくと，

$$\begin{aligned}
(\mathfrak{L}h_{\beta,\alpha})(y) &= \tfrac{1}{\beta}h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}\right)\mathbb{I}_{[T(0),1)}(y) + \tfrac{1}{\beta}h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}+\tfrac{1}{\beta}\right) + \cdots \\
&\quad + \tfrac{1}{\beta}h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}+\tfrac{[\beta+\alpha]-1}{\beta}\right) + \tfrac{1}{\beta}h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}+\tfrac{[\beta+\alpha]}{\beta}\right)\mathbb{I}_{[0,T(1))}(y)
\end{aligned}$$

となる．$h_{\beta,\alpha}(y) = (\mathfrak{L}h_{\beta,\alpha})(y)$ を証明する代わりに，$\beta h_{\beta,\alpha}(y) = \beta(\mathfrak{L}h_{\beta,\alpha})(y)$ を証明する．

$$\begin{aligned}
\beta(\mathfrak{L}h_{\beta,\alpha})(y) &= h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}\right)\mathbb{I}_{[T(0),1)}(y) + h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}+\tfrac{1}{\beta}\right) + \cdots \\
&\quad + h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}+\tfrac{[\beta+\alpha]-1}{\beta}\right) + h_{\beta,\alpha}\left(\tfrac{y-\alpha}{\beta}+\tfrac{[\beta+\alpha]}{\beta}\right)\mathbb{I}_{[0,T(1))}(y) \\
&= \left(\sum_{n=0}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha}{\beta}\right) - \sum_{n=1}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha}{\beta}\right)\right)\mathbb{I}_{[T(0),1)}(y) \\
&\quad + \sum_{n=0}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha+1}{\beta}\right) - \sum_{n=1}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha+1}{\beta}\right) \\
&\qquad\vdots \\
&\quad + \sum_{n=0}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right) - \sum_{n=1}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right) \\
&\quad + \left(\sum_{n=0}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right) - \sum_{n=1}^{\infty}\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right)\right)\mathbb{I}_{[0,T(1))}(y) \\
&= \Bigg\{\left(\mathbb{I}_{[0,1)}\left(\tfrac{y-\alpha}{\beta}\right) + \tfrac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\tfrac{y-\alpha}{\beta}\right) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha}{\beta}\right) + \cdots\right) \\
&\quad - \left(\tfrac{1}{\beta}\mathbb{I}_{[0,T(0))}\left(\tfrac{y-\alpha}{\beta}\right) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha}{\beta}\right) + \cdots\right)\Bigg\}\mathbb{I}_{[T(0),1)}(y) \\
&\quad + \Bigg\{\mathbb{I}_{[0,1)}\left(\tfrac{y-\alpha+1}{\beta}\right) + \tfrac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\tfrac{y-\alpha+1}{\beta}\right) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha+1}{\beta}\right) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&\quad -\left(\tfrac{1}{\beta}\mathbb{I}_{[0,T(0))}\left(\tfrac{y-\alpha+1}{\beta}\right)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha+1}{\beta}\right)+\cdots\right)\Bigg\}\\
&\quad\vdots\\
&+\Bigg\{\Bigg(\mathbb{I}_{[0,1)}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right)+\tfrac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right)+\cdots\\
&+\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right)+\cdots\\
&\quad -\left(\tfrac{1}{\beta}\mathbb{I}_{[0,T(0))}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha+[\beta+\alpha]-1}{\beta}\right)+\cdots\right)\Bigg\}\\
&+\Bigg\{\Bigg(\mathbb{I}_{[0,1)}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right)+\tfrac{1}{\beta}\mathbb{I}_{[0,T(1))}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right)+\cdots\\
&+\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(1))}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right)+\cdots\Bigg)\\
&\quad -\left(\tfrac{1}{\beta}\mathbb{I}_{[0,T(0))}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[0,T^n(0))}\left(\tfrac{y-\alpha+[\beta+\alpha]}{\beta}\right)+\cdots\right)\Bigg\}\\
&\hfill\times\mathbb{I}_{[0,T(1))}(y)\\
=&\Bigg\{\left(\mathbb{I}_{[\alpha,\beta+\alpha)}(y)+\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)+\cdots\right)\\
&-\left(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)+\cdots\right)\Bigg\}\mathbb{I}_{[T(0),1)}(y)\\
&+\Bigg\{\mathbb{I}_{[\alpha,\beta+\alpha)}(y+1)+\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+1)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+1)+\cdots\\
&\quad -\left(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+1)+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+1)+\cdots\right)\Bigg\}\\
&\quad\vdots\\
&+\Bigg\{\mathbb{I}_{[\alpha,\beta+\alpha)}(y+[\beta+\alpha]-1)+\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+[\beta+\alpha]-1)+\cdots\\
&+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha]-1)+\cdots\\
&\quad -\Bigg(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+[\beta+\alpha]-1)+\cdots\\
&\quad +\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]-1)+\cdots\Bigg)\Bigg\}\\
&+\Bigg\{\mathbb{I}_{[\alpha,\beta+\alpha)}(y+[\beta+\alpha])+\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+[\beta+\alpha])+\cdots\\
&+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])+\cdots\\
&\quad -\left(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+[\beta+\alpha])+\cdots+\tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])+\cdots\right)\Bigg\}\\
&\hfill\times\mathbb{I}_{[0,T(1))}(y).
\end{aligned}
$$

補題 4.1 より，$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y)=\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])$，$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])\mathbb{I}_{[0,T(1))}(y)=\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha])$，$\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y)=\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$，$\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y)\mathbb{I}_{[T(0),1)}(y)=\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y)$ が成り立つから，これをふまえて，$\beta(\mathfrak{L}h_{\beta,\alpha})(y)$ を

書きなおすと，

$$
\begin{aligned}
\beta(\mathfrak{L}h_{\beta,\alpha})(y) = \Bigg\{ & \mathbb{I}_{[\alpha,\beta+\alpha)}(y) + \tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y) + \cdots \\
& - \left(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y) + \cdots\right)\Bigg\} \\
& + \Bigg\{\mathbb{I}_{[\alpha,\beta+\alpha)}(y+1) + \tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+1) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+1) + \cdots \\
& \quad - \left(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+1) + \cdots + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+1) + \cdots\right)\Bigg\} \\
& \quad \vdots \\
& + \Bigg\{\mathbb{I}_{[\alpha,\beta+\alpha)}(y+[\beta+\alpha]-1) + \tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+[\beta+\alpha]-1) + \cdots \\
& \quad + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha]-1) + \cdots \\
& \quad - \Bigg(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+[\beta+\alpha]-1) + \cdots \\
& \quad + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]-1) + \cdots\Bigg)\Bigg\} \\
& + \Bigg\{\mathbb{I}_{[\alpha,\beta+\alpha)}(y+[\beta+\alpha]) + \tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+[\beta+\alpha]) + \cdots \\
& \quad + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha]) + \cdots) \\
& \quad - \Bigg(\tfrac{1}{\beta}\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+[\beta+\alpha]) + \cdots \\
& \quad + \tfrac{1}{\beta^n}\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]) + \cdots\Bigg)\Bigg\} \\
= \Bigg\{ & \mathbb{I}_{[\alpha,\beta+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta+\alpha)}(y+1) + \cdots \\
& + \mathbb{I}_{[\alpha,\beta+\alpha)}(y+[\beta+\alpha]-1) + \mathbb{I}_{[\alpha,\beta+\alpha)}(y+[\beta+\alpha])\Bigg\} \\
& + \Bigg\{\tfrac{1}{\beta}\Big(\mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+1) + \cdots \\
& \quad + \mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+[\beta+\alpha]-1) + \mathbb{I}_{[\alpha,\beta T(1)+\alpha)}(y+[\beta+\alpha])\Big) \\
& \quad - \tfrac{1}{\beta}\Big(\mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+1) + \cdots \\
& \quad + \mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+[\beta+\alpha]-1) + \mathbb{I}_{[\alpha,\beta T(0)+\alpha)}(y+[\beta+\alpha])\Big)\Bigg\} \\
& \quad \vdots \\
& + \Bigg\{\tfrac{1}{\beta^n}\Big(\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+1) + \cdots \\
& \quad + \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha]-1) + \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha])\Big) \\
& \quad - \tfrac{1}{\beta^n}\Big(\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+1) + \cdots
\end{aligned}
$$

$$
\left. + \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}\left(y+[\beta+\alpha]-1\right) + \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}\left(y+[\beta+\alpha]\right) \Big) \right\}
$$

$$
\vdots
$$

となる．ここで，

$$
\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+1) + \cdots + \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha]-1) \\
&\qquad + \mathbb{I}_{[\alpha,\beta T^n(1)+\alpha)}(y+[\beta+\alpha]) \\
&= \sharp\{l \in \mathbb{Z} \mid \alpha \le y + l < \beta T^n(1) + \alpha\} \\
&= \sharp\{l \in \mathbb{Z} \mid \alpha - y \le l < \beta T^n(1) + \alpha - y\} \\
&= \sharp\{l \in \mathbb{Z} \mid \alpha - y \le l < [\beta T^n(1) + \alpha] + \{\beta T^n(1) + \alpha\} - y\}.
\end{aligned}
$$

l は 0 以上の整数で，$\alpha - y$ は $-1 < \alpha - y < 1$ を満たす実数だから，$\alpha - y \le l$ は $\alpha \le y$ のとき $0 \le l$，$\alpha > y$ のとき $1 \le l$ と同値でありこれらは $\mathbb{I}_{[0,\alpha)}(y)$ にまとめられる．また，$\{\beta T^n(1)+\alpha\}-y$ は $-1 < \{\beta T^n(1)+\alpha\} - y < 1$ を満たす実数だから，$l < [\beta T^n(1)+\alpha] + \{\beta T^n(1)+\alpha\} - y$ は $\{\beta T^n(1)+\alpha\} \le y$ のとき $l \le [\beta T^n(1)+\alpha] - 1$ となる．また，$\{\beta T^n(1)+\alpha\} \ge y$ のとき $l \le [\beta T^n(1)+\alpha]$ と同値であり，これらは $l \le [\beta T^n(1)+\alpha] - \mathbb{I}_{[\{\beta T^n(1)+\alpha\},1)}(y)$ とまとめられる．したがって，

$$
\begin{aligned}
&\sharp\{l \in \mathbb{Z} \mid \mathbb{I}_{[0,\alpha)}(y) \le l \le [\beta T^n(1)+\alpha] - \mathbb{I}_{[\{\beta T^n(1)+\alpha\},1)}(y)\} \\
&= \sharp\{l \in \mathbb{Z} \mid \mathbb{I}_{[0,\alpha)}(y) \le l \le [\beta T^n(1)+\alpha] - \mathbb{I}_{[T^{n+1}(1),1)}(y)\} \\
&= [\beta T^n(1)+\alpha] - \mathbb{I}_{[T^{n+1}(1),1)}(y) - \mathbb{I}_{[0,\alpha)}(y) + 1 \\
&= [\beta T^n(1)+\alpha] - (1 - \mathbb{I}_{[0,T^{n+1}(1))}(y)) - \mathbb{I}_{[0,\alpha)}(y) + 1 \\
&= [\beta T^n(1)+\alpha] + \mathbb{I}_{[0,T^{n+1}(1))}(y) - \mathbb{I}_{[0,\alpha)}(y).
\end{aligned}
$$

同様にして，

$$
\begin{aligned}
&\mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y) + \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+1) + \cdots + \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]-1) \\
&\qquad + \mathbb{I}_{[\alpha,\beta T^n(0)+\alpha)}(y+[\beta+\alpha]) \\
&= \sharp\{l \in \mathbb{Z} \mid \alpha \le y + l < \beta T^n(0) + \alpha\} \\
&= \sharp\{l \in \mathbb{Z} \mid \alpha - y \le l < \beta T^n(0) + \alpha - y\} \\
&= \sharp\{l \in \mathbb{Z} \mid \alpha - y \le l < [\beta T^n(0) + \alpha] + \{\beta T^n(0) + \alpha\} - y\}.
\end{aligned}
$$

$\{\beta T^n(0)+\alpha\} - y$ は $-1 < \{\beta T^n(0)+\alpha\} - y < 1$ を満たす実数だから，$l < [\beta T^n(0)+\alpha] + \{\beta T^n(0)+\alpha\} - y$ は $\{\beta T^n(0)+\alpha\} \le y$ のとき $l \le [\beta T^n(0)+\alpha] - 1$，$\{\beta T^n(0)+\alpha\} \ge y$ のとき $l \le [\beta T^n(0)+\alpha]$ と同値であり，これらは $l \le [\beta T^n(0)+\alpha] - \mathbb{I}_{[\{\beta T^n(0)+\alpha\},1)}(y)$ とまとめられる．したがって，

$$
\begin{aligned}
&\sharp\{l \in \mathbb{Z} \mid \alpha - y \le l < [\beta T^n(0)+\alpha] + \{\beta T^n(0)+\alpha\} - y\} \\
&= \sharp\{l \in \mathbb{Z} \mid \mathbb{I}_{[0,\alpha)}(y) \le l \le [\beta T^n(0)+\alpha] - \mathbb{I}_{[\{\beta T^n(0)+\alpha\},1)}(y)\} \\
&= \sharp\{l \in \mathbb{Z} \mid \mathbb{I}_{[0,\alpha)}(y) \le l \le [\beta T^n(0)+\alpha] - \mathbb{I}_{[T^{n+1}(0),1)}(y)\} \\
&= [\beta T^n(0)+\alpha] - \mathbb{I}_{[T^{n+1}(0),1)}(y) - \mathbb{I}_{[0,\alpha)}(y) + 1
\end{aligned}
$$

$$
\begin{aligned}
&= [\beta T^n(0)+\alpha] - (1 - \mathbb{I}_{[0,T^{n+1}(0))}(y)) - \mathbb{I}_{[0,\alpha)}(y) + 1 \\
&= [\beta T^n(0)+\alpha] + \mathbb{I}_{[0,T^{n+1}(0))}(y) - \mathbb{I}_{[0,\alpha)}(y).
\end{aligned}
$$

したがって，

$$
\begin{aligned}
\beta(\mathfrak{L}h_{\beta,\alpha})(y) &= [\beta T^0(1)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^1(1))}(y) \\
&\quad + \tfrac{1}{\beta}\Bigg([\beta T^1(1)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^2(1))}(y) \\
&\qquad - \big([\beta T^1(0)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^2(0))}(y)\big) \Bigg) \\
&\qquad \vdots \\
&\quad + \tfrac{1}{\beta^n}\Bigg([\beta T^n(1)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^{n+1}(1))}(y) \\
&\qquad - \big([\beta T^n(0)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^{n+1}(0))}(y)\big) \Bigg) \\
&\qquad \vdots \\
&= [\beta+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T(1))}(y) \\
&\quad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\big([\beta T^n(1)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^{n+1}(1))}(y)\big) \\
&\qquad - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\big([\beta T^n(0)+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T^{n+1}(0))}(y)\big) \\
&= [\beta+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T(1))}(y) \\
&\quad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\big([\beta T^n(1)+\alpha] - \mathbb{I}_{[0,\alpha)}(y)\big) + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y) \\
&\qquad - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\big([\beta T^n(0)+\alpha] - \mathbb{I}_{[0,\alpha)}(y)\big) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(0))}(y) \\
&= [\beta+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T(1))}(y) \\
&\quad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\big([\beta T^n(1)+\alpha] - \mathbb{I}_{[0,\alpha)}(y)\big) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\big([\beta T^n(0)+\alpha] - \mathbb{I}_{[0,\alpha)}(y)\big) \\
&\qquad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(0))}(y) \\
&= [\beta+\alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T(1))}(y) \\
&\quad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\left([\beta T^n(1)+\alpha] - ([\beta T^n(0)+\alpha])\right) \\
&\qquad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(1))}(y) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\mathbb{I}_{[0,T^{n+1}(0))}(y). \qquad (4.2)
\end{aligned}
$$

ここで，式 (4.2) の第 2 項 $\sum_{n=1}^{\infty} \frac{1}{\beta^n}\left([\beta T^n(1)+\alpha] - ([\beta T^n(0)+\alpha])\right)$ について計算する．

$$
\sum_{n=1}^{\infty} \tfrac{1}{\beta^n}\left([\beta T^n(1)+\alpha] - ([\beta T^n(0)+\alpha])\right)
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Bigg(\Big(\beta T^n(1) + \alpha - \{\beta T^n(1) + \alpha\}\Big) - \Big(\beta T^n(0) + \alpha - \{\beta T^n(0) + \alpha\}\Big) \Bigg) \\
&= \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Bigg(\Big(\beta T^n(1) + \alpha - T^{n+1}(1)\Big) - \Big(\beta T^n(0) + \alpha - T^{n+1}(0)\Big) \Bigg) \\
&= \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Bigg(\Big(\beta T^n(1) - T^{n+1}(1)\Big) - \Big(\beta T^n(0) - T^{n+1}(0)\Big) \Bigg) \\
&= \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Bigg(\beta\Big(T^n(1) - T^n(0)\Big) - \Big(T^{n+1}(1) - T^{n+1}(0)\Big) \Bigg) \\
&= \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \beta\Big(T^n(1) - T^n(0)\Big) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Big(T^{n+1}(1) - T^{n+1}(0)\Big) \\
&= \sum_{n=1}^{\infty} \tfrac{1}{\beta^{n-1}} \Big(T^n(1) - T^n(0)\Big) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Big(T^{n+1}(1) - T^{n+1}(0)\Big) \\
&= T(1) - T(0) + \sum_{n=2}^{\infty} \tfrac{1}{\beta^{n-1}} \Big(T^n(1) - T^n(0)\Big) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \Big(T^{n+1}(1) - T^{n+1}(0)\Big) \\
&= T(1) - T(0) \\
&= \{\beta + \alpha\} - \alpha
\end{aligned}
$$

となるので，これをふまえて式 (4.2) を書きなおすと，

$$
\begin{aligned}
\beta(\mathfrak{L}h_{\beta,\alpha})(y) &= [\beta + \alpha] - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T(1))}(y) \\
&\qquad + \{\beta + \alpha\} - \alpha \\
&\qquad\qquad + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(1))}(y) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(0))}(y) \\
&= \beta - \mathbb{I}_{[0,\alpha)}(y) + \mathbb{I}_{[0,T(1))}(y) + \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(1))}(y) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(0))}(y) \\
&= \beta - \mathbb{I}_{[0,\alpha)}(y) + \sum_{n=0}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(1))}(y) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(0))}(y)
\end{aligned}
$$

となる．式 (4.1) より，$\beta - \mathbb{I}_{[0,\alpha)}(y) + \sum_{n=0}^{\infty} \frac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(1))}(y) - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \mathbb{I}_{[0,T^{n+1}(0))}(y)$ は $\beta(h_{\beta,\alpha})(y)$ と等しくなるから，$\beta(\mathfrak{L}h_{\beta,\alpha})(y) = \beta(h_{\beta,\alpha})(y)\, a.e.\ y$ が成り立つ．したがって，

$$
(\mathfrak{L}h_{\beta,\alpha})(y) = (h_{\beta,\alpha})(y)\, a.e.\ y
$$

が成り立つ． $\square$

5 $h_{\beta,\{\beta\}}$ の積分

谷口，久保は [6] で，整数の乗算とシフトを用いた非再帰型 32 ビット擬似乱数 SSI32rand を構成した (SSI:Simplified Shift-Integer)．この SSI32rand は，実数の乗算と倍精度変数の仮数部のシフトから構成される非再帰型擬似乱数 SSR32(Simplified Shift-Real) の発展型として構成されたため，数学的背景は取り扱われてこなかった．その後，SSI32 アルゴリズムの本質は $[1,2)$ 上のベータ変換であることが示された [9]．$\beta > 1$ に対し，$[1,2)$ 上のベータ変換 M_β とは，

$$M_\beta ; [1,2) \longrightarrow [1,2), \quad M_\beta(t) = \beta t - [\beta t] + 1 \equiv \beta t \bmod [1,2)$$

で定義される関数である．SSI32 の乱数生成アルゴリズムは，$x \in [1,2)$ に対して $M^{23}_{2^2 x}(w_0)$ を計算し，上位 12 ビットを捨て，続く 32 ビットを取り出して乱数値とするものである．また x を $[1,2)$ 中で一様に変化させることにより，多くの乱数値を得ている．したがって，SSI32 乱数値の性質を調べるには，まず，**(1)** M_β のエルゴード性 (変換 M_β を繰り返し行ったときの性質)，**(2)** $M^k_\beta(w_0)$ の β をある区間で一様に変化させたときの性質を調べなくてはならない．**(1)** については，M_β は $[0,1)$ 上の linear mod 1 変換 $T_{\beta,\alpha}; [0,1) \longrightarrow [0,1)$，

$$T_{\beta,\alpha}(t) = \beta t + \alpha - [\beta t + \alpha]$$

と，

$$M_\beta(t) = T_{\beta,\{\beta\}}(t-1) + 1 \quad , \quad t \in [1,2)$$

の関係があるので ($\{\beta\}$ は β の小数部分)，

証明.

$$\begin{aligned}
T_{\beta,\{\beta\}}(t-1) &= \beta(t-1) + \{\beta\} - [\beta(t-1) + \{\beta\}] \\
&= \beta t - \beta + \{\beta\} - [\beta(t-1) + \{\beta\}] \\
&= \beta t - ([\beta] + \{\beta\}) + \{\beta\} - [\beta(t-1) + \{\beta\}] \\
&= \beta t - [\beta] - [\beta(t-1) + \{\beta\}] \\
&= \beta t - [\beta(t-1) + [\beta] + \{\beta\}] \\
&= \beta t - [\beta t] \qquad (\beta = [\beta] + \{\beta\} \text{ より }) \\
&= M_\beta(t) - 1.
\end{aligned}$$

□

$T_{\beta,\{\beta\}}$ の性質を調べればよいが，その不変測度 $h_{\beta,\{\beta\}}$

$$h_{\beta,\{\beta\}}(y) = \sum_{n=0}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^n(1))}(y) - \sum_{n=1}^{\infty} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^n(0))}(y) \tag{5.1}$$

は 3 章で示されている．ここで，$h_{\beta,\{\beta\}}$ を正規化して得られる確率密度関数を $\tilde{h}_{\beta,\{\beta\}}$ とおく; すなわち，

$$\tilde{h}_{\beta,\{\beta\}}(y) = \frac{h_{\beta,\{\beta\}}(y)}{\int_{[0,1)} h_{\beta,\{\beta\}}(y)\,dy}$$

とおく．**(2)** については，β がある区間を一様に変化する場合，$M_\beta^k(w_0)$ の分布 $H(y)$, $y \in [0,1)$, は，k を大きくしていくとき，$\tilde{h}_{\beta,\{\beta\}}(y)$ の β による平均に近づいていくことが数値計算，必要条件による推論などから分かっている．本論文の目的の 1 つは，この $H(y)$ を y の具体的な関数として求めることである．(もう 1 つの問題は linear mod 1 変換の不変測度について直截的な証明をつけることであった (4 章)).

$h_{\beta,\{\beta\}}(y)$ は T の不変測度を与えるが，必ずしも正規化されたものではないので，$h_{\beta,\{\beta\}}$ を正規化した確率密度関数 $\tilde{h}_{\beta,\{\beta\}}$ について，β が $[1,2)$ を一様に変化したとき，$M_\beta^k(w_0)$ の分布は k が大きいとき，

$$H(y) = \int_{[1,2)} \tilde{h}_{\beta,\{\beta\}}(y)\,d\beta$$

で与えられることを示したい．$H(y)$ を直接求めることは困難であるので，本論文では，その手始めとして，$\tilde{h}_{\beta,\{\beta\}}(y)$ の代わりに有限和 $S_k(y)$

$$S_k(y) = \sum_{n=0}^{k} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^n(1))}(y) - \sum_{n=1}^{k} \tfrac{1}{\beta^n} \mathbb{I}_{[0,T^n(0))}(y) \tag{5.2}$$

を用いて，それを正規化した $\tilde{S}_k(y)$ について，$k = 0, 1, 2$ の場合に，分布

$$H_k(y) = \int_{[1,2)} \tilde{S}_k(y)\,d\beta$$

と，β を $(1,2)$ で一様に動かしたときの実際の計算値 $M_\beta^n(w_0), n = 1, 2, \cdots, 1 < \beta < 2$，の分布との比較を行った．なお，以下の計算の記述では計算の正確さを期すために，計算過程を省略せずに記した．

5.1 $k = 0$ のとき

式 (5.2) に $k = 0$ を代入して，

$$\begin{aligned} S_0(y) &= \frac{1}{\beta^0} \mathbb{I}_{[0,1)}(y) = 1, \\ \tilde{S}_0(y) &= \frac{S_0(y)}{1}, \\ H_0(y) &= \int_{[1,2)} \tilde{S}_0(y)\,d\beta = 1. \end{aligned} \tag{5.3}$$

$H_0(y)$ は一番単純な $H(y)$ の近似であり，$H_0(y) =$ 定数，$\int_{[0,1)} H_0(y)\,dy = 1$ となるような確率密度関数である．

5.2　$k=1$ のとき

$k=1$ を式 (5.2) に代入すると，まず，

$$S_1(y) = 1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y).$$

$\int_{[0,1)} \left(1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)\right) dy$ について計算する．計算の中で $T(1)$ の値が出てくるが，$T(1)$ の値は β の値によって変わるため，この場でまとめておく．

$$T(1) = \{\beta + \{\beta\}\} = \{1 + \{\beta\} + \{\beta\}\} = \{2\{\beta\}\} = 2\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta) \tag{5.4}$$

$$\begin{aligned}
\int_{[0,1)} \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)\,dy &= \frac{1}{\beta}T(1) \\
&= \frac{1}{\beta}\{\beta + \{\beta\}\} \\
&= \frac{1}{\beta}\{1 + \{\beta\} + \{\beta\}\} \\
&= \frac{1}{\beta}\{2\{\beta\}\} \\
&= \frac{2\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}, \qquad (5.5) \\
\int_{[0,1)} \frac{1}{\beta}\mathbb{I}_{[0,T(0))}1(y)\,dy &= \frac{1}{\beta}T(0) \\
&= \frac{\{\beta\}}{\beta} \qquad (5.6)
\end{aligned}$$

となる．式 (5.5), (5.6) より，

$$\begin{aligned}
\int_{[0,1)} S_1(y)\,dy &= \int_0^1 \left(1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)\right) dy \\
&= 1 + \frac{2\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta} - \frac{\{\beta\}}{\beta} \\
&= 1 + \frac{\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta} \qquad (5.7)
\end{aligned}$$

となる。よって，$S_1(y)$ を正規化した $\tilde{S}_1(y)$ は，

$$\tilde{S}_1(y) = \frac{S_1}{1 + \frac{\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}} = \frac{1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)}{1 + \frac{\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}} \tag{5.8}$$

とおくことができる．今から，$H_1(y) = \int_{[1,2)} \tilde{S}_1(y)\,d\beta$ を計算する．

$$\begin{aligned}
&\int_{[1,2)} \tilde{S}_1(y)\,d\beta \\
&= \int_{[1,2)} \frac{1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)}{1 + \frac{\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}}\,d\beta \\
&= \int_{[1,2)} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{\beta + \{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}\,d\beta
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[1,2)} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{\beta + \beta - 1 - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}\, d\beta \\
&= \int_{[1,2)} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta - 1 - \mathbb{I}_{[\frac{3}{2},2)(\beta)}}\, d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta - 1}\, d\beta + \int_{[\frac{3}{2},2)} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta - 2}\, d\beta
\end{aligned} \tag{5.9}
$$

となるから，まず，$\int_{[1,\frac{3}{2})} \frac{\beta+\mathbb{I}_{[0,T(1))}(y)-\mathbb{I}_{[0,T(0))}(y)}{2\beta-1}\, d\beta$ について計算していく．

$$
\int_{[1,\frac{3}{2})} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta - 1}\, d\beta = \int_{[1,\frac{3}{2})} \frac{\beta}{2\beta - 1}\, d\beta + \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[0,T(1))}(y)}{2\beta - 1}\, d\beta - \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[0,T(0))}(y)}{2\beta - 1}\, d\beta \tag{5.10}
$$

となるから，まず，式 (5.10) の第 1 項目の積分を計算する．

$$
\begin{aligned}
\int_{[1,\frac{3}{2})} \frac{\beta}{2\beta - 1}\, d\beta &= \int_{[1,\frac{3}{2})} \left(\frac{1}{2} + \frac{\frac{1}{2}}{2\beta - 1} \right) d\beta \\
&= \frac{1}{2} \int_{[1,\frac{3}{2})} \left(1 + \frac{1}{2\beta - 1} \right) d\beta \\
&= \frac{1}{2} \left[\beta + \frac{1}{2} \log|2\beta - 1| \right]_{[1,\frac{3}{2})} \\
&= \frac{1}{2} \left[\left(\frac{3}{2} + \frac{1}{2}\log 2 \right) - \left(1 + \frac{1}{2}\log 1 \right) \right] \\
&= \frac{1}{2} \left[\frac{1}{2} + \frac{1}{2}\log 2 \right] \\
&= \frac{1}{4} + \frac{1}{4}\log 2.
\end{aligned} \tag{5.11}
$$

次に，式 (5.10) の第 2 項目の積分を計算する．

$$
\begin{aligned}
\int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[0,T(1))}(y)}{2\beta - 1}\, d\beta &= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(T(1))}{2\beta - 1}\, d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(2\{\beta\})}{2\beta - 1}\, d\beta && (\text{式 (5.4) より}) \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[\frac{y}{2},\frac{1}{2})}(\{\beta\})}{2\beta - 1}\, d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[\frac{y}{2},\frac{1}{2})}(\beta - 1)}{2\beta - 1}\, d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[\frac{y+2}{2},\frac{3}{2})}(\beta)}{2\beta - 1}\, d\beta \\
&= \int_{[1,\frac{3}{2}) \cap [\frac{y+2}{2},\frac{3}{2})} \frac{1}{2\beta - 1}\, d\beta \\
&= \int_{[\frac{y+2}{2},\frac{3}{2})} \frac{1}{2\beta - 1}\, d\beta \\
&= \left[\frac{1}{2} \log|2\beta - 1| \right]_{[\frac{y+2}{2},\frac{3}{2})}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\log 2 - \frac{1}{2}\log(y+1) \\
&= \frac{1}{2}\log\frac{2}{y+1}. \qquad (5.12)
\end{aligned}
$$

次に，式 (5.10) の第 3 項目の積分を計算する．

$$
\begin{aligned}
\int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[0,T(0))}(y)}{2\beta-1}\,d\beta &= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(T(0))}{2\beta-1}\,d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(\{\beta\})}{2\beta-1}\,d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(\beta-1)}{2\beta-1}\,d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[y+1,2)}(\beta)}{2\beta-1}\,d\beta \\
&= \int_{[1,\frac{3}{2})\cap[y+1,2)} \frac{1}{2\beta-1}\,d\beta \\
&= \int_{[1,\frac{3}{2})\cap[y+1,2)} \frac{1}{2\beta-1}\,d\beta \\
&= \left[\frac{1}{2}\log|2\beta-1|\right]_{[y+1,\frac{3}{2})} \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&= \left(\frac{1}{2}\log 2 - \frac{1}{2}\log(2y+1)\right) \\
&= \left(\frac{1}{2}\log\frac{2}{2y+1}\right)\mathbb{I}_{[0,\frac{1}{2})}(y). \qquad (5.13)
\end{aligned}
$$

式 (5.11), (5.12), (5.13) より，式 (5.9) の前半の積分の値は，

$$
\begin{aligned}
\int_{[1,\frac{3}{2})} \tilde{S}_1(y)\,d\beta &= \int_{[1,\frac{3}{2})} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta-1}\,d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\beta}{2\beta-1}\,d\beta + \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[0,T(1))}(y)}{2\beta-1}\,d\beta - \int_{[1,\frac{3}{2})} \frac{\mathbb{I}_{[0,T(0))}(y)}{2\beta-1}\,d\beta. \\
&= \frac{1}{4} + \frac{1}{4}\log 2 + \frac{1}{2}\log\frac{2}{y+1} - \left(\frac{1}{2}\log\frac{2}{2y+1}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) \qquad (5.14)
\end{aligned}
$$

となる．次に，$\int_{[\frac{3}{2},2)} \frac{\beta+\mathbb{I}_{[0,T(1))}(y)-\mathbb{I}_{[0,T(0))}(y)}{2\beta-2}\,d\beta$ について計算する．

$$
\int_{[\frac{3}{2},2)} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta-2}\,d\beta = \int_{[\frac{3}{2},2)} \frac{\beta}{2\beta-2}\,d\beta + \int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[0,T(1))}(y)}{2\beta-2}\,d\beta - \int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[0,T(0))}(y)}{2\beta-2}\,d\beta \qquad (5.15)
$$

となるから，まず，式 (5.15) の第 1 項目の積分を計算する．

$$
\begin{aligned}
\int_{[\frac{3}{2},2)} \frac{\beta}{2\beta-2}\,d\beta &= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\beta}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} 1 + \frac{1}{\beta-1}\,d\beta \\
&= \frac{1}{2}\left[\beta + \log|\beta-1|\right]_{[\frac{3}{2},2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left[(2+\log 1) - \left(\frac{3}{2} + \log\frac{1}{2}\right)\right] \\
&= \frac{1}{2}\left(2 - \frac{3}{2} - \log\frac{1}{2}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \log\frac{1}{2}\right) \\
&= \frac{1}{4} - \frac{1}{2}\log\frac{1}{2}. \qquad (5.16)
\end{aligned}
$$

次に，式 (5.15) の第 2 項目の積分を計算する．

$$
\begin{aligned}
\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[0,T(1))}(y)}{2\beta-2}\,d\beta &= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[0,T(1))}(y)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y,1)}(T(1))}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y,1)}(2\{\beta\}-1)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y,1)}(2(\beta-1)-1)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{\frac{3}{2}}^{2} \frac{\mathbb{I}_{[y,1)}(2\beta-3)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{\frac{3}{2}}^{2} \frac{\mathbb{I}_{[y+3,4)}(2\beta)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{\frac{3}{2}}^{2} \frac{\mathbb{I}_{[\frac{y+3}{2},2)}(\beta)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)\cap[\frac{y+3}{2},2)} \frac{1}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{y+3}{2},2)} \frac{1}{\beta-1}\,d\beta \\
&= \frac{1}{2}\left[\log|\beta-1|\right]_{[\frac{y+3}{2},2)} \\
&= \frac{1}{2}\left(\log 1 - \log\frac{y+1}{2}\right) \\
&= -\frac{1}{2}\log\frac{y+1}{2}. \qquad (5.17)
\end{aligned}
$$

まず，式 (5.15) の第 3 項目の積分を計算する．

$$
\begin{aligned}
\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[0,T(0))}(y)}{2\beta-2}\,d\beta &= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[0,T(0))}(y)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y,1)}(T(0))}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y,1)}(\{\beta\})}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y,1)}(\beta-1)}{\beta-1}\,d\beta \\
&= \frac{1}{2}\int_{[\frac{3}{2},2)} \frac{\mathbb{I}_{[y+1,2)}(\beta)}{\beta-1}\,d\beta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{[\frac{3}{2},2)\cap[y+1,2)} \frac{1}{\beta-1}\,d\beta \\
&= \left(\frac{1}{2}\int_{[\frac{3}{2},2)} \frac{1}{\beta-1}\,d\beta\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\int_{[y+1,2)} \frac{1}{\beta-1}\,d\beta\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2}\left[\log|\beta-1|\right]_{[\frac{3}{2},2)}\mathbb{I}_{[0,\frac{1}{2})}(y) + \frac{1}{2}\left[\log|\beta-1|\right]_{[y+1,2)}\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2}\left(\log 1 - \log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \frac{1}{2}\left(\log 1 - \log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= -\left(\frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) - \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y). \qquad (5.18)
\end{aligned}
$$

式 (5.16), (5.17), (5.18) より，式 (5.9) の後半の積分の値は，

$$
\begin{aligned}
&\int_{\frac{3}{2}}^{2} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta-2}\,d\beta \\
&= \frac{1}{4} - \frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{y+1}{2} - \left(-\left(\frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) - \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y)\right) \\
&= \frac{1}{4} - \frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{y+1}{2} + \left(\frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \qquad (5.19)
\end{aligned}
$$

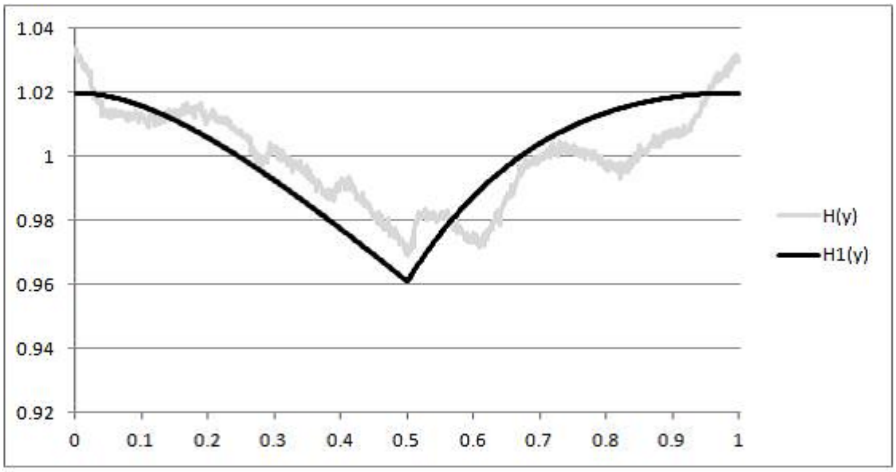
となる．

式 (5.14), (5.19) より，式 (5.9) の値は，

$$
\begin{aligned}
&\int_{[1,2)} \tilde{S}_1(y)\,d\beta \\
&= \int_{[1,\frac{3}{2})} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta-1}\,d\beta + \int_{[\frac{3}{2},2)} \frac{\beta + \mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y)}{2\beta-2}\,d\beta \\
&= \frac{1}{4} + \frac{1}{4}\log 2 + \frac{1}{2}\log\frac{2}{y+1} - \left(\frac{1}{2}\log\frac{2}{2y+1}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\qquad + \frac{1}{4} - \frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{y+1}{2} + \left(\frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\log 2 + \frac{1}{2}\log\frac{2}{y+1} - \frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{y+1}{2} \\
&\qquad + \left(-\frac{1}{2}\log\frac{2}{2y+1} + \frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{2}{y+1} - \frac{1}{2}\log\frac{y+1}{2} \\
&\qquad + \left(-\frac{1}{2}\log\frac{2}{2y+1} + \frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{4}{(y+1)^2} \\
&\qquad + \left(-\frac{1}{2}\log\frac{2}{2y+1} + \frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\left(\log 2 - 2\log\frac{1}{2}\right) + \frac{1}{2}\log\frac{4}{(y+1)^2} \\
&\qquad + \left(-\frac{1}{2}\log\frac{2}{2y+1} + \frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\left(\log 2 - \log\frac{1}{4}\right) + \frac{1}{2}\log\frac{4}{(y+1)^2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(-\frac{1}{2}\log\frac{2}{2y+1} + \frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\log 8 + \frac{1}{2}\log\frac{4}{(y+1)^2} \\
&\quad + \left(-\frac{1}{2}\log\frac{2}{2y+1} + \frac{1}{2}\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{1}{4}\log 2^3 + \frac{1}{2}\log\left(\frac{2}{y+1}\right)^2 \\
&\quad + \left(\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{2}{2y+1}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2} + \frac{3}{4}\log 2 + \log\frac{2}{y+1} \\
&\quad + \left(\frac{1}{2}\log\frac{2y+1}{4}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) + \left(\frac{1}{2}\log y\right)\mathbb{I}_{[\frac{1}{2},1)}(y). \qquad (5.20)
\end{aligned}
$$

求められた，$H_1(y) = \int_{[1,2)} \tilde{S}_1(y)\,d\beta$ の式 (5.20) のグラフを図示すると，



となる．図には β を $(1,2)$ で一様に分布させたときの $M_\beta^n(w_0), n = 1, 2, \cdots,$ の分布も重ねて図示してある．これによると式 (5.20) は，実際の分布と傾向を同じくするものの十分近似しているとは言い難い．さらに近似の精度を上げるために $k = 2$ の場合の $H_2(y) = \int_{[1,2)} \tilde{S}_2(y)\,d\beta$ を計算する．

5.3　$k = 2$ のとき

$k = 2$ を式 (5.2) に代入して，

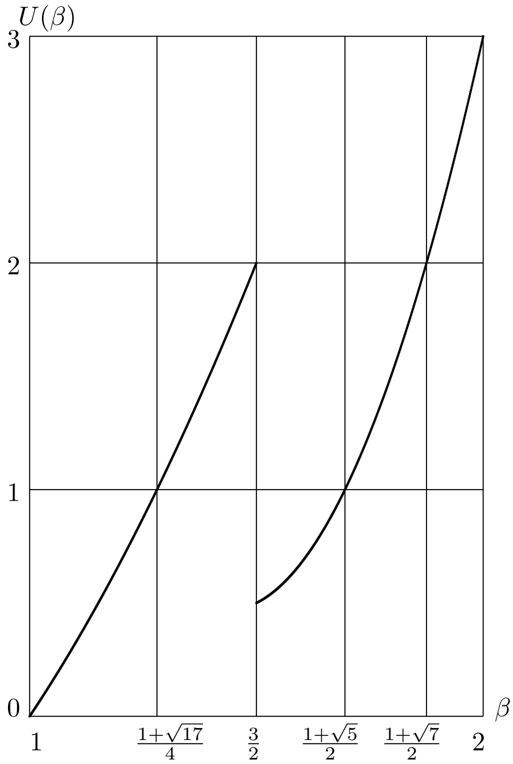
$$S_2(y) = 1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) + \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y) - \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)$$

となる．$S_2(y)$ を正規化した $\tilde{S}_2(y)$ を求めるために，
$\int_{[0,1)} \left(1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) + \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y) - \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)\right) dy$ を計算する．計算の中で $T^2(1), T^2(0)$ の値が出てくるが，$T^2(1), T^2(0)$ の値は β の値によって変わるため，この場でまとめておく．

$$T^2(1) = \{\beta T(1) + \{\beta\}\} = \{\beta\{\beta + \{\beta\}\} + \{\beta\}\}$$

$$
\begin{aligned}
&= \{\beta\{[\beta] + \{\beta\} + \{\beta\}\} + \{\beta\}\} = \{\beta\{1 + \{\beta\} + \{\beta\}\} + \{\beta\}\} \\
&= \{\beta\{2\{\beta\}\} + \{\beta\}\} = \left\{\beta\left(2\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)\right) + \{\beta\}\right\} \text{(式 (5.4))} \\
&= \left\{\beta\left(2(\beta - 1) - \mathbb{I}_{[\frac{3}{2},2)}(\beta)\right) + \beta - 1\right\} \\
&= \left\{2\beta^2 - \beta - 1 - \beta\mathbb{I}_{[\frac{3}{2},2)}(\beta)\right\}
\end{aligned}
$$

となる．$U(\beta) = 2\beta^2 - \beta - 1 - \beta\mathbb{I}_{[\frac{3}{2},2)}(\beta)$ とおき，$U(\beta)$ のグラフをかくと，$U(\beta)$ のグラフをかくと，



となる．

$2\beta^2 - \beta - 1 = 1$ を解くと，

$$
\begin{aligned}
2\beta^2 - \beta - 1 &= 1 \\
2\beta^2 - \beta - 2 &= 0 \\
\beta &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 2 \times (-2)}}{2 \times 2} \\
&= \frac{1 \pm \sqrt{17}}{4}
\end{aligned}
$$

となる．今 $\beta \geq 1$ だから，$\beta = \frac{1+\sqrt{17}}{4}$ である．

$2\beta^2 - 2\beta - 1 = 1$ を解くと，

$$
\begin{aligned}
2\beta^2 - 2\beta - 1 &= 1 \\
2\beta^2 - 2\beta - 2 &= 0
\end{aligned}
$$

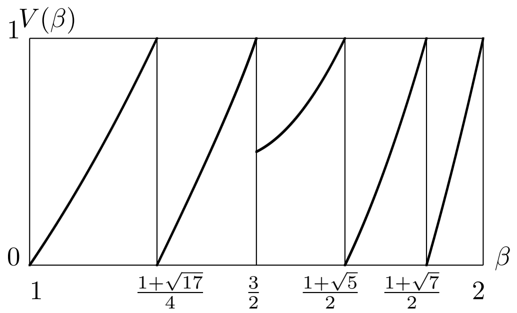
$$
\begin{aligned}
\beta^2 - \beta - 1 &= 0 \\
\beta &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\
&= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

となる．$\beta > 1$ だから，$\beta = \frac{1+\sqrt{5}}{2}$ である．

次に，$2\beta^2 - 2\beta - 1 = 2$ を解くと，

$$
\begin{aligned}
2\beta^2 - 2\beta - 1 &= 2 \\
2\beta^2 - 2\beta - 3 &= 0 \\
\beta &= \frac{-(-1) \pm \sqrt{(-1)^2 - 2 \times (-3)}}{2 \times 1} \\
&= \frac{1 \pm \sqrt{7}}{2}
\end{aligned}
$$

となる．$\beta > 1$ だから，$\beta = \frac{1+\sqrt{7}}{2}$ である．次に $V(\beta) = T^2(1)$ とおき，グラフをかくと，



となる．以上より，

$$
T^2(1) = \begin{cases}
2\beta^2 - \beta - 1 \ , \ \left(1 \leq \beta < \dfrac{1+\sqrt{17}}{4}\right) & (5.21) \\
2\beta^2 - \beta - 2 \ , \ \left(\dfrac{1+\sqrt{17}}{4} \leq \beta < \dfrac{3}{2}\right) & (5.22) \\
2\beta^2 - 2\beta - 1 \ , \ \left(\dfrac{3}{2} \leq \beta < \dfrac{1+\sqrt{5}}{2}\right) & (5.23) \\
2\beta^2 - 2\beta - 2 \ , \ \left(\dfrac{1+\sqrt{5}}{2} \leq \beta < \dfrac{1+\sqrt{7}}{2}\right) & (5.24) \\
2\beta^2 - 2\beta - 3 \ , \ \left(\dfrac{1+\sqrt{7}}{2} \leq \beta < 2\right) & (5.25)
\end{cases}
$$

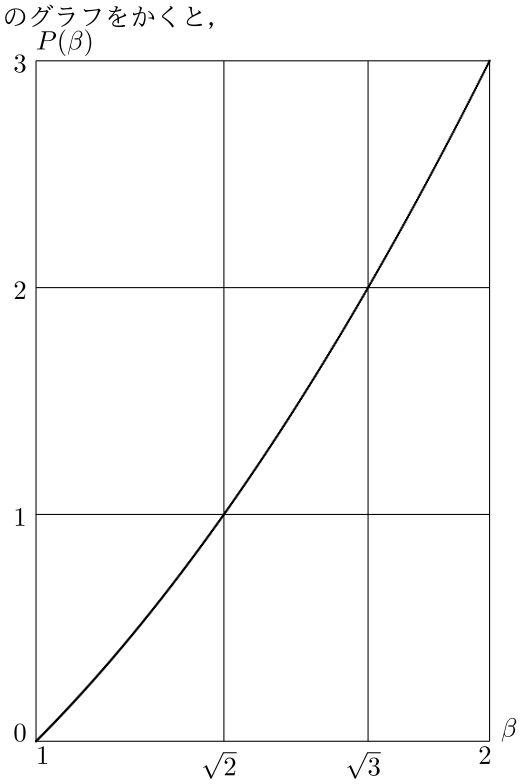
となる．

次に $T^2(\beta)$ について計算する．

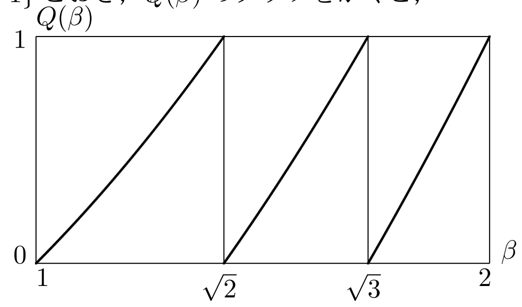
$$
\begin{aligned}
T^2(0) &= \{\beta T(0) + \{\beta\}\} = \{\beta\{\beta\} + \{\beta\}\} \\
&= \{\beta(\beta - [\beta]) + \beta - [\beta]\} = \{\beta(\beta - 1) + \beta - 1\}
\end{aligned}
$$

$$= \{\beta^2 - \beta + \beta - 1\} = \{\beta^2 - 1\}.$$

$P(\beta) = \beta^2 - 1$ とおき，$P(\beta)$ のグラフをかくと，



となり，$\beta^2 - 1 = 1$ を解くと，$\beta = \sqrt{2}\ (1 \leq \beta < 2)$ となり，$\beta^2 - 1 = 2$ を解くと，$\beta = \sqrt{3}\ (1 \leq \beta < 2)$ となる．次に，$Q(\beta) = \{\beta^2 - 1\}$ とおき，$Q(\beta)$ のグラフをかくと，



となる．以上より，

$$T^2(0) = \begin{cases} \beta^2 - 1\ , & \left(1 \leq \beta < \sqrt{2}\right) & (5.26) \\ \beta^2 - 2\ , & \left(\sqrt{2} \leq \beta < \sqrt{3}\right) & (5.27) \\ \beta^2 - 3\ , & \left(\sqrt{3} \leq \beta < 2\right) & (5.28) \end{cases}$$

となる．$T^2(1), T^2(0)$ の値から $T^2(1) - T^2(0)$ の値を求める．

(イ) $1 \leq \beta < \frac{1+\sqrt{17}}{4}$ のとき．((5.21) かつ (5.26))

$$T^2(1) - T^2(0) = 2\beta^2 - \beta - 1 - (\beta^2 - 1) = 2\beta^2 - \beta - 1 - \beta^2 + 1 = \beta^2 - \beta.$$

(ロ) $\frac{1+\sqrt{17}}{4} \le \beta < \sqrt{2}$ のとき．((5.22) かつ (5.26))

$$T^2(1) - T^2(0) = 2\beta^2 - \beta - 2 - (\beta^2 - 1) = 2\beta^2 - \beta - 2 - \beta^2 + 1 = \beta^2 - \beta - 1.$$

(ハ) $\sqrt{2} \le \beta < \frac{3}{2}$ のとき．((5.22) かつ (5.27))

$$T^2(1) - T^2(0) = 2\beta^2 - \beta - 2 - (\beta^2 - 2) = 2\beta^2 - \beta - 2 - \beta^2 + 2 = \beta^2 - \beta.$$

(ニ) $\frac{3}{2} \le \beta < \frac{1+\sqrt{5}}{2}$ のとき．((5.23) かつ (5.27))

$$T^2(1) - T^2(0) = 2\beta^2 - \beta - 1 - (\beta^2 - 2) = 2\beta^2 - 2\beta - 1 - \beta^2 + 2 = \beta^2 - 2\beta + 1.$$

(ホ) $\frac{1+\sqrt{5}}{2} \le \beta < \sqrt{3}$ のとき．((5.24) かつ (5.27))

$$T^2(1) - T^2(0) = 2\beta^2 - 2\beta - 2 - (\beta^2 - 2) = 2\beta^2 - 2\beta - 2 - \beta^2 + 2 = \beta^2 - 2\beta.$$

(ヘ) $\sqrt{3} \le \beta < \frac{1+\sqrt{7}}{2}$ のとき．((5.24) かつ (5.28))

$$T^2(1) - T^2(0) = 2\beta^2 - 2\beta - 2 - (\beta^2 - 3) = 2\beta^2 - 2\beta - 2 - \beta^2 + 3 = \beta^2 - 2\beta + 1.$$

(ト) $\frac{1+\sqrt{7}}{2} \le \beta < 2$ のとき．((5.25) かつ (5.28))

$$T^2(1) - T^2(0) = 2\beta^2 - 2\beta - 3 - (\beta^2 - 3) = 2\beta^2 - 2\beta - 3 - \beta^2 + 3 = \beta^2 - 2\beta.$$

(イ) ∼ (ト) をまとめると，

$$T^2(1) - T^2(0) = \begin{cases} \beta^2 - \beta\ , \ \left(1 < \beta \le \frac{1+\sqrt{17}}{4},\ \sqrt{2} \le \beta < \frac{3}{2}\right) \\ \beta^2 - \beta - 1\ , \ \left(\frac{1+\sqrt{17}}{4} \le \beta < \sqrt{2}\right) \\ \beta^2 - 2\beta + 1\ , \ \left(\frac{3}{2} \le \beta < \frac{1+\sqrt{5}}{2},\ \sqrt{3} \le \beta < \frac{1+\sqrt{7}}{2}\right) \\ \beta^2 - 2\beta\ , \ \left(\frac{1+\sqrt{5}}{2} \le \beta < \sqrt{3},\ \frac{1+\sqrt{7}}{2} \le \beta < 2\right) \end{cases} \tag{5.29}$$

式 (5.29) より，

$$\begin{aligned} &T^2(1) - T^2(0) \\ &= \beta^2 - \beta - \mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta) + (-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta) - \beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta) \end{aligned} \tag{5.30}$$

となるから，

$$\begin{aligned} &\int_{[0,1)} S_2(y)dy \\ &= \int_{[0,1)} \left(1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) + \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y) - \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)\right) dy \\ &= \int_{[0,1)} \left(1 + \frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y) - \frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)\right) dy \\ &\qquad + \int_{[0,1)} \left(\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y) - \frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)\right) dy \\ &= 1 + \frac{\{\beta\} - \mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\beta^2}\left(\beta^2-\beta-\mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta)+(-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta)-\beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta)\right) \\
&\hspace{20em}(\text{式 (5.7) より}) \\
&=1+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta} \\
&\quad+1-\frac{1}{\beta}+\frac{1}{\beta^2}\left(-\mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta)+(-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta)-\beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta)\right) \\
&=1+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta} \\
&\quad+1-\frac{1}{\beta}+\frac{1}{\beta^2}\left(-\mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta)+(-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta)-\beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta)\right) \\
&=2+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta} \\
&\quad-\frac{1}{\beta}+\frac{1}{\beta^2}\left(-\mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta)+(-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta)-\beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta)\right)
\end{aligned}
\tag{5.31}
$$

となる．よって，$S_2(y)$ を正規化した $\tilde{S}_2(y)$ は，

$$
\begin{aligned}
&\tilde{S}_2(y) \\
&=\frac{S_2(y)}{2+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}-\frac{1}{\beta}+\frac{1}{\beta^2}\left(-\mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta)+(-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta)-\beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta)\right)} \\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}-\frac{1}{\beta}+\frac{1}{\beta^2}\left(-\mathbb{I}_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}(\beta)+(-\beta+1)\mathbb{I}_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})}(\beta)-\beta\mathbb{I}_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}(\beta)\right)}
\end{aligned}
\tag{5.32}
$$

となる．今から，$\int_{[1,2)}\tilde{S}_2(y)\,d\beta$ を計算する．

(イ) $1<\beta<\frac{1+\sqrt{17}}{4}$, $\sqrt{2}\le\beta<\frac{3}{2}$ のとき．

$$
\begin{aligned}
\tilde{S}_2(y)&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}-\frac{1}{\beta}} \\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}}{\beta}-\frac{1}{\beta}} \\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\beta-1}{\beta}-\frac{1}{\beta}} \\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\beta-2}{\beta}} \\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{2\beta^2+\beta(\beta-2)} \\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-2\beta} \\
&=\frac{\beta^2+\beta\left(\mathbb{I}_{[0,T(1))}(y)-\mathbb{I}_{[0,T(0))}(y)\right)+\mathbb{I}_{[0,T^2(1))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-2\beta},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{[1,\frac{1+\sqrt{17}}{4})\cup[\sqrt{2},\frac{3}{2})} \tilde{S}_2(y)\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})\cup[\sqrt{2},\frac{3}{2})} \frac{\beta^2}{3\beta^2-2\beta}\,d\beta + \int_{[1,\frac{1+\sqrt{17}}{4})\cup[\sqrt{2},\frac{3}{2})} \frac{\beta\mathbb{I}_{[0,T(1))}(y)}{3\beta^2-2\beta}\,d\beta \\
&\quad- \int_{[1,\frac{1+\sqrt{17}}{4})\cup[\sqrt{2},\frac{3}{2})} \frac{\beta\mathbb{I}_{[0,T(0))}(y)}{3\beta^2-2\beta}\,d\beta + \int_{[1,\frac{1+\sqrt{17}}{4})\cup[\sqrt{2},\frac{3}{2})} \frac{\mathbb{I}_{[0,T^2(1))}(y)}{3\beta^2-2\beta}\,d\beta \\
&\quad- \int_{[1,\frac{1+\sqrt{17}}{4})\cup[\sqrt{2},\frac{3}{2})} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-2\beta}\,d\beta
\end{aligned} \tag{5.33}
$$

となる．$1 \le \beta < \frac{1+\sqrt{17}}{4}$, $\sqrt{2} \le \beta < \frac{3}{2}$ の場合で被積分関数 $\tilde{S}_2(y)$ は同じであるが，$T^2(1), T^2(0)$ の値が違うので 2 つの区間に分けて積分する．

(イ -1) $1 \le \beta < \frac{1+\sqrt{17}}{4}$ のとき．

まず，式 (5.33) の第 1 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\beta^2}{3\beta^2-2\beta}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\beta}{3\beta-2}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})} \frac{1}{3} + \frac{\frac{2}{3}}{3\beta-2}\,d\beta \\
&\quad= \frac{1}{3}\int_{[1,\frac{1+\sqrt{17}}{4})} 1 + \frac{2}{3\beta-2}\,d\beta \\
&\quad= \frac{1}{3}\left[\beta + \frac{2}{3}\log|3\beta-2|\right]_{[1,\frac{1+\sqrt{17}}{4})} \\
&\quad= \frac{1}{3}\left[\left(\frac{1+\sqrt{17}}{4} + \frac{2}{3}\log\left|3\cdot\frac{1+\sqrt{17}}{4}-2\right|\right) - \left(1+\frac{2}{3}\log 1\right)\right] \\
&\quad= \frac{1}{3}\left[\frac{1+\sqrt{17}}{4} + \frac{2}{3}\log\frac{3\sqrt{17}-5}{4} - 1\right] \\
&\quad= \frac{1}{3}\left(\frac{-3+\sqrt{17}}{4} + \frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right).
\end{aligned} \tag{5.34}
$$

次に，式 (5.33) の第 2 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\beta\mathbb{I}_{[0,T(1))}(y)}{3\beta^2-2\beta}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[y,1)}(T(1))}{3\beta-2}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[y,1)}(2\{\beta\})}{3\beta-2}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})\cap[\frac{y+2}{2},\frac{3}{2})} \frac{1}{3\beta-2}\,d\beta
\end{aligned}
$$

$$
\begin{cases}
y \le 2\{\beta\} < 1 \text{ をとくと，} \\
\begin{aligned}
y &\le 2\{\beta\} < 1, \\
y &\le 2(\beta - 1) < 1, \\
\frac{y}{2} &\le \beta - 1 < \frac{1}{2}, \\
\frac{y+2}{2} &\le \beta < \frac{3}{2}.
\end{aligned} \\
\tfrac{y+2}{2} \ge \tfrac{1+\sqrt{17}}{4} \text{ の場合は，} [1, \tfrac{1+\sqrt{17}}{4}) \cap [\tfrac{y+2}{2}, \tfrac{3}{2}) = \emptyset \text{ になり，} \\
\text{積分の値が } 0 \text{ になるため，今は } 0 \le y < \tfrac{-3+\sqrt{17}}{2} \text{ の場合だ} \\
\text{け計算する．積分範囲は } \tfrac{y+2}{2} \le \beta < \tfrac{1+\sqrt{17}}{4} \text{ である．}
\end{cases}
$$

$$
\begin{aligned}
&= \int_{[\frac{y+2}{2}, \frac{1+\sqrt{17}}{4})} \frac{1}{3\beta - 2} \, d\beta \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&= \left[\frac{1}{3} \log |3\beta - 2| \right]_{[\frac{y+2}{2}, \frac{1+\sqrt{17}}{4})} \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&= \left[\frac{1}{3} \log |3 \cdot \frac{1+\sqrt{17}}{4} - 2| - \left(\frac{1}{3} \log |3 \cdot \frac{y+2}{2} - 2| \right) \right] \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&= \left[\frac{1}{3} \log \frac{-5+3\sqrt{17}}{4} - \left(\frac{1}{3} \log \frac{3y+2}{2} \right) \right] \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&= \frac{1}{3} \left(\log \frac{-5+3\sqrt{17}}{4} - \log \frac{3y+2}{2} \right) \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y). \qquad (5.35)
\end{aligned}
$$

次に，式 (5.33) の第 3 項目を積分する．

$$
\begin{aligned}
&\int_{[1, \frac{1+\sqrt{17}}{4})} \frac{\beta \mathbb{I}_{[0, T(0))}(y)}{3\beta^2 - 2\beta} \, d\beta \\
&\qquad = \int_{[1, \frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[0, T(0))}(y)}{3\beta - 2} \, d\beta \\
&\qquad = \int_{[1, \frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[y, 1)}(\{\beta\})}{3\beta - 2} \, d\beta \\
&\qquad = \int_{[1, \frac{1+\sqrt{17}}{4}) \cap [y+1, 2)} \frac{1}{3\beta - 2} \, d\beta
\end{aligned}
$$

$$\left\{\begin{array}{l}y \leq \{\beta\} < 1 \text{ をとくと，} \\ \qquad\qquad \begin{aligned} y &\leq \{\beta\} < 1, \\ y &\leq \beta - 1 < 1, \\ y + 1 &\leq \beta < 2. \end{aligned} \\ 1 < \beta < \frac{1+\sqrt{17}}{4} \text{ だから，} y+1 \geq \frac{1+\sqrt{17}}{4} \text{ の場合は，} \\ [1, \frac{1+\sqrt{17}}{4}) \cap [y+1, 2) = \emptyset \text{ となり，積分の値は } 0 \text{ である．そ} \\ \text{のため今は } 0 \leq y < \frac{-3+\sqrt{17}}{4} \text{ の場合のみ計算する，積分範} \\ \text{囲は } y+1 \leq \beta < \frac{1+\sqrt{17}}{4} \text{ である．}\end{array}\right.$$

$$\begin{aligned}
&= \int_{[y+1, \frac{1+\sqrt{17}}{4})} \frac{1}{3\beta - 2}\, d\beta \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{4})}(y) \\
&= \left[\frac{1}{3}\log|3\beta - 2|\right]_{[y+1, \frac{1+\sqrt{17}}{4})} \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{4})}(y) \\
&= \left[\left(\frac{1}{3}\log\left|3 \cdot \frac{1+\sqrt{17}}{4} - 2\right|\right) - \left(\frac{1}{3}\log|3(y+1) - 2|\right)\right] \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{4})}(y) \\
&= \left[\left(\frac{1}{3}\log\frac{-5+3\sqrt{17}}{4}\right) - \left(\frac{1}{3}\log(3y+1)\right)\right] \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{4})}(y) \\
&= \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log(3y+1)\right) \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{4})}(y). \qquad (5.36)
\end{aligned}$$

次に，式 (5.33) の第 4 項目の積分を計算する．

$$\begin{aligned}
&\int_{[1, \frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[0, T^2(1))}(y)}{3\beta^2 - 2\beta}\, d\beta \\
&\quad = \int_{[1, \frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[y,1)}(T^2(1))}{3\beta^2 - 2\beta}\, d\beta \\
&\quad = \int_{[1, \frac{1+\sqrt{17}}{4}) \cap [\frac{1+\sqrt{8y+9}}{4}, \frac{1+\sqrt{17}}{4})} \frac{1}{3\beta^2 - 2\beta}\, d\beta
\end{aligned}$$

$$\left\{\begin{array}{l}y \leq T^2(1) < 1 \text{ をとくと，} \\ \qquad\qquad \begin{aligned} y &\leq 2\beta^2 - \beta - 1 < 1, \\ \frac{1+\sqrt{8y+9}}{4} &\leq \beta < \frac{1+\sqrt{17}}{4}. \end{aligned} \\ 1 < \beta < \frac{1+\sqrt{17}}{4} \text{ だから，積分範囲は } \frac{1+\sqrt{8y+9}}{4} \leq \beta < \frac{1+\sqrt{17}}{4} \\ \text{である．}\end{array}\right.$$

$$\begin{aligned}
&\quad = \int_{[\frac{1+\sqrt{8y+9}}{4}, \frac{1+\sqrt{17}}{4})} \frac{1}{3\beta^2 - 2\beta}\, d\beta \\
&\quad = \int_{[\frac{1+\sqrt{8y+9}}{4}, \frac{1+\sqrt{17}}{4})} \frac{\frac{-1}{2}}{\beta} + \frac{\frac{3}{2}}{3\beta - 2}\, d\beta \\
&\quad = -\frac{1}{2}\int_{[\frac{1+\sqrt{8y+9}}{4}, \frac{1+\sqrt{17}}{4})} \frac{1}{\beta} - \frac{3}{3\beta - 2}\, d\beta
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[\log\beta - \log|3\beta-2|\right]_{[\frac{1+\sqrt{8y+9}}{4},\frac{1+\sqrt{17}}{4})} \\
&= -\frac{1}{2}\Bigg[\Bigg(\log\frac{1+\sqrt{17}}{4} - \log\left|3\cdot\frac{1+\sqrt{17}}{4}-2\right|\Bigg) \\
&\qquad - \left(\log\frac{1+\sqrt{8y+9}}{4} - \log\left|3\cdot\frac{1+\sqrt{8y+9}}{4}-2\right|\right)\Bigg] \\
&= -\frac{1}{2}\Bigg[\Bigg(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4}\Bigg) - \left(\log\frac{1+\sqrt{8y+9}}{4} - \log\frac{-5+3\sqrt{8y+9}}{4}\right)\Bigg] \\
&= -\frac{1}{2}\Bigg(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\frac{1+\sqrt{8y+9}}{4} + \log\frac{-5+3\sqrt{8y+9}}{4}\Bigg). \qquad (5.37)
\end{aligned}
$$

次に，式 (5.33) の第 5 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-2\beta}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[y,1)}(T^2(0))}{3\beta^2-2\beta}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})} \frac{\mathbb{I}_{[y,1)}(\beta^2-1)}{3\beta^2-2\beta}\,d\beta \\
&\quad= \int_{[1,\frac{1+\sqrt{17}}{4})\cap[\sqrt{y+1},\sqrt{2})} \frac{1}{3\beta^2-2\beta}\,d\beta
\end{aligned}
$$

$y \le T^2(0) < 1$ をとくと，

$$
\begin{aligned}
y &\le \beta^2 - 1 < 1, \\
\sqrt{y+1} &\le \beta < \sqrt{2}.
\end{aligned}
$$

$1 < \beta < \frac{1+\sqrt{17}}{4}$ だから，$\sqrt{y+1} \ge \frac{1+\sqrt{17}}{4}$ の場合は，$[1,\frac{1+\sqrt{17}}{4})\cap[\sqrt{y+1},\sqrt{2}) = \emptyset$ となり積分の値が 0 になるから，$0 \le y < \frac{1+\sqrt{17}}{8}$ の場合だけ積分する．積分範囲は，$\sqrt{y+1} \le \beta < \frac{1+\sqrt{17}}{4}$ である．

$$
\begin{aligned}
&= \int_{[\sqrt{y+1},\frac{1+\sqrt{17}}{4})} \frac{1}{3\beta^2-2\beta}\,d\beta\,\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&= -\frac{1}{2}\left[\log\beta - \log|3\beta-2|\right]_{[\sqrt{y+1},\frac{1+\sqrt{17}}{4})}\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&= -\frac{1}{2}\Bigg[\Bigg(\log\frac{1+\sqrt{17}}{4} - \log\left|3\cdot\frac{1+\sqrt{17}}{4}-2\right|\Bigg) \\
&\qquad - \left(\log\sqrt{y+1} - \log\left|3\cdot\sqrt{y+1}-2\right|\right)\Bigg]\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&= -\frac{1}{2}\left[\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4}\right) - \left(\log\sqrt{y+1} - \log(3\sqrt{y+1}-2)\right)\right] \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&= -\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\sqrt{y+1} + \log(3\sqrt{y+1}-2)\right)
\end{aligned}
$$

$$\times \mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y). \tag{5.38}$$

よって，$1 \leq \beta < \frac{1+\sqrt{17}}{4}$ のとき，式 (5.34), (5.35), (5.36), (5.37), (5.38) より，

$$\begin{aligned}
&\int_{[1,\frac{1+\sqrt{17}}{4})} \tilde{S}_2(y)\, d\beta \\
&\quad = \frac{1}{3}\left(\frac{-3+\sqrt{17}}{4} + \frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right) \\
&\qquad\quad + \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log\frac{3y+2}{2}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad\quad - \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log(3y+1)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad\quad - \frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\frac{1+\sqrt{8y+9}}{4} + \log\frac{-5+3\sqrt{8y+9}}{4}\right) \\
&\qquad\quad - \left(-\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\sqrt{y+1} + \log(3\sqrt{y+1}-2)\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times \mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\right) \\
&\quad = \frac{1}{3}\left(\frac{-3+\sqrt{17}}{4} + \frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right) \\
&\qquad\quad + \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log\frac{3y+2}{2}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad\quad - \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log(3y+1)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad\quad - \frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\frac{1+\sqrt{8y+9}}{4} + \log\frac{-5+3\sqrt{8y+9}}{4}\right) \\
&\qquad\quad + \frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\sqrt{y+1} + \log(3\sqrt{y+1}-2)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y). \\
&\quad = \frac{1}{3}\left(\frac{-3+\sqrt{17}}{4} + \frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right) \\
&\qquad\quad - \frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\frac{1+\sqrt{8y+9}}{4} + \log\frac{-5+3\sqrt{8y+9}}{4}\right) \\
&\qquad\quad + \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log\frac{3y+2}{2}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad\quad - \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log(3y+1)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)
\end{aligned}$$

$$
+\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\sqrt{y+1}+\log(3\sqrt{y+1}-2)\right)
$$
$$
\times\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y). \tag{5.39}
$$

(イ -2) $\sqrt{2}\leq\beta<\frac{3}{2}$ のとき．

まず，式 (5.33) の第 1 項目を積分する．

$$
\begin{aligned}
&\int_{[\sqrt{2},\frac{3}{2})}\frac{\beta^2}{3\beta^2-2\beta}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})}\frac{\beta}{3\beta-2}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})}\frac{1}{3}+\frac{\frac{2}{3}}{3\beta-2}\,d\beta\\
&\quad=\frac{1}{3}\int_{[\sqrt{2},\frac{3}{2})}1+\frac{2}{3\beta-2}\,d\beta\\
&\quad=\frac{1}{3}\left[\beta+\frac{2}{3}\log|3\beta-2|\right]_{[\sqrt{2},\frac{3}{2})}\\
&\quad=\frac{1}{3}\left[\left(\frac{3}{2}+\frac{2}{3}\log|3\cdot\frac{3}{2}-2|\right)-\left(\sqrt{2}+\frac{2}{3}\log|3\sqrt{2}-2|\right)\right]\\
&\quad=\frac{1}{3}\left[\left(\frac{3}{2}+\frac{2}{3}\log\frac{5}{2}\right)-\left(\sqrt{2}+\frac{2}{3}\log(3\sqrt{2}-2)\right)\right]\\
&\quad=\frac{1}{3}\left(\frac{3}{2}+\frac{2}{3}\log\frac{5}{2}-\sqrt{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right).
\end{aligned}\tag{5.40}
$$

次に，式 (5.34) の第 2 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[[\sqrt{2},\frac{3}{2}))}\frac{\beta\mathbb{I}_{[0,T(1))}(y)}{3\beta^2-2\beta}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})}\frac{\mathbb{I}_{[y,1)}(T(1))}{3\beta-2}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})}\frac{\mathbb{I}_{[y,1)}(2\{\beta\})}{3\beta-2}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})\cap[\frac{y+2}{2},\frac{3}{2})}\frac{1}{3\beta-2}\,d\beta
\end{aligned}
$$

$$
\begin{cases}
y \leq 2\{\beta\} < 1 \text{ をとくと，} \\
\qquad\qquad\qquad\qquad y \leq 2\{\beta\} < 1, \\
\qquad\qquad\qquad\qquad y \leq 2(\beta - 1) < 1, \\
\qquad\qquad\qquad\qquad \dfrac{y}{2} \leq \beta - 1 < \dfrac{1}{2}, \\
\qquad\qquad\qquad\qquad \dfrac{y+2}{2} \leq \beta < \dfrac{3}{2}. \\
\text{積分範囲は } \max\{\sqrt{2}, \frac{y+2}{2}\} \leq \beta < \frac{3}{2} \text{ である．}
\end{cases}
$$

$$
\begin{aligned}
&= \int_{[\sqrt{2},\frac{3}{2})} \frac{1}{3\beta-2}\,d\beta \mathbb{I}_{[0,2\sqrt{2}-2)}(y) + \int_{[\frac{y+2}{2},\frac{3}{2})} \frac{1}{3\beta-2}\,d\beta \mathbb{I}_{[2\sqrt{2}-2,1)}(y) \\
&= \left[\frac{1}{3}\log|3\beta-2|\right]_{[\sqrt{2},\frac{3}{2})} \mathbb{I}_{[0,2\sqrt{2}-2)}(y) + \left[\frac{1}{3}\log|3\beta-2|\right]_{[\frac{y+2}{2},\frac{3}{2})} \mathbb{I}_{[2\sqrt{2}-2,1)}(y) \\
&= \left[\left(\frac{1}{3}\log|3\cdot\frac{3}{2}-2|\right) - \left(\frac{1}{3}\log|3\sqrt{2}-2|\right)\right] \mathbb{I}_{[0,2\sqrt{2}-2)}(y) \\
&\qquad + \left[\frac{1}{3}\log|3\cdot\frac{3}{2}-2| - \frac{1}{3}\log|3\cdot\frac{y+2}{2}-2|\right] \mathbb{I}_{[2\sqrt{2}-2,1)}(y) \\
&= \frac{1}{3}\left(\log\frac{5}{2} - \log(3\sqrt{2}-2)\right) \mathbb{I}_{[0,2\sqrt{2}-2)}(y) + \frac{1}{3}\left(\log\frac{5}{2} - \log\frac{3y+2}{2}\right) \mathbb{I}_{[2\sqrt{2}-2,1)}(y).
\end{aligned}
\tag{5.41}
$$

次に，式 (5.33) の第 3 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\sqrt{2},\frac{3}{2})} \frac{\beta\mathbb{I}_{[0,T(0))}(y)}{3\beta^2-2\beta}\,d\beta \\
&\qquad = \int_{[\sqrt{2},\frac{3}{2})} \frac{\mathbb{I}_{[0,T(0))}(y)}{3\beta-2}\,d\beta \\
&\qquad = \int_{[\sqrt{2},\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(\{\beta\})}{3\beta-2}\,d\beta \\
&\qquad = \int_{[\sqrt{2},\frac{3}{2})\cap[y+1,2)} \frac{1}{3\beta-2}\,d\beta
\end{aligned}
$$

$$
\begin{cases}
y \leq \{\beta\} < 1 \text{ をとくと，} \\
\qquad\qquad\qquad\qquad y \leq \{\beta\} < 1, \\
\qquad\qquad\qquad\qquad y \leq \beta - 1 < 1, \\
\qquad\qquad\qquad\qquad y + 1 \leq \beta < 2. \\
\beta < \frac{3}{2} \text{ だから，} y+1 \geq \frac{3}{2} \text{ の場合は } [\sqrt{2},\frac{3}{2}) \cap [y+1,2) = \emptyset \\
\text{であり積分の値が } 0 \text{ になるから，} 0 \leq y < \frac{1}{2} \text{ の場合だけ積} \\
\text{分の値を計算する．積分範囲は，} \max\{y+1,\sqrt{2}\} \leq \beta < \frac{3}{2} \\
\text{である．}
\end{cases}
$$

$$
= \int_{[\sqrt{2},\frac{3}{2})} \frac{1}{3\beta-2}\,d\beta \mathbb{I}_{[0,\sqrt{2}-1)}(y) + \int_{[y+1,\frac{3}{2})} \frac{1}{3\beta-2}\,d\beta \mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y)
$$

$$
\begin{aligned}
&= \left[\frac{1}{3}\log|3\beta-2|\right]_{[\sqrt{2},\frac{3}{2})} \mathbb{I}_{[0,\sqrt{2}-1)}(y) + \left[\frac{1}{3}\log|3\beta-2|\right]_{[y+1,\frac{3}{2})} \mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y) \\
&= \left[\frac{1}{3}\log|3\cdot\frac{3}{2}-2| - \frac{1}{3}\log|3\cdot\sqrt{2}-2|\right] \mathbb{I}_{[0,\sqrt{2}-1)}(y) \\
&\qquad + \left[\frac{1}{3}\log|3\cdot\frac{3}{2}-2| - \frac{1}{3}\log|3\cdot(y+1)-2|\right] \mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y) \\
&= \left(\frac{1}{3}\log\frac{5}{2} - \frac{1}{3}\log(3\sqrt{2}-2)\right) \mathbb{I}_{[0,\sqrt{2}-1)}(y) + \left(\frac{1}{3}\log\frac{5}{2} - \frac{1}{3}\log(3y+1)\right) \mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y) \\
&= \frac{1}{3}\left(\log\frac{5}{2} - \log(3\sqrt{2}-2)\right) \mathbb{I}_{[0,\sqrt{2}-1)}(y) + \frac{1}{3}\left(\log\frac{5}{2} - \log(3y+1)\right) \mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y).
\end{aligned}
\tag{5.42}
$$

次に，式 (5.33) の第 4 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\sqrt{2},\frac{3}{2})} \frac{\mathbb{I}_{[0,T^2(1))}(y)}{3\beta^2-2\beta}\,d\beta \\
&\qquad = \int_{[\sqrt{2},\frac{3}{2})} \frac{\mathbb{I}_{[y,1)}(T^2(1))}{3\beta^2-2\beta}\,d\beta \\
&\qquad = \int_{[\sqrt{2},\frac{3}{2})\cap[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})} \frac{1}{3\beta^2-2\beta}\,d\beta
\end{aligned}
$$

$$
\begin{cases}
y \le 2\beta^2-\beta-2 < 1 \text{ をとくと，} \\
\qquad\qquad y \le 2\beta^2-\beta-2 < 1, \\
\qquad\qquad \dfrac{1+\sqrt{8y+17}}{4} \le \beta < \dfrac{3}{2}. \\
\text{積分範囲は } \max\{\sqrt{2}, \frac{1+\sqrt{8y+17}}{4}\} \le \beta < \frac{3}{2}.
\end{cases}
$$

$$
\begin{aligned}
&= \int_{[\sqrt{2},\frac{3}{2})} \frac{1}{3\beta^2-2\beta}\,d\beta \mathbb{I}_{[0,2-\sqrt{2})}(y) + \int_{[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})} \frac{1}{3\beta^2-2\beta}\,d\beta \mathbb{I}_{[2-\sqrt{2},1)}(y) \\
&= \int_{[\sqrt{2},\frac{3}{2})} \frac{\frac{-1}{2}}{\beta} + \frac{\frac{3}{2}}{3\beta-2}\,d\beta \mathbb{I}_{[0,2-\sqrt{2})}(y) + \int_{[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})} \frac{\frac{-1}{2}}{\beta} + \frac{\frac{3}{2}}{3\beta-2}\,d\beta \mathbb{I}_{[2-\sqrt{2},1)}(y) \\
&= -\frac{1}{2}\int_{[\sqrt{2},\frac{3}{2})} \frac{1}{\beta} - \frac{3}{3\beta-2}\,d\beta \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\qquad - \frac{1}{2}\int_{[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})} \frac{1}{\beta} - \frac{3}{3\beta-2}\,d\beta \mathbb{I}_{[2-\sqrt{2},1)}(y) \\
&= -\frac{1}{2}\left[\log\beta - \log|3\beta-2|\right]_{[\sqrt{2},\frac{3}{2})} \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\qquad - \frac{1}{2}\left[\log\beta - \log|3\beta-2|\right]_{[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})} \mathbb{I}_{[2-\sqrt{2},1)}(y) \\
&= -\frac{1}{2}\left[\left(\log\frac{3}{2} - \log|3\cdot\frac{3}{2}-2|\right) - \left(\log\sqrt{2} - \log|3\sqrt{2}-2|\right)\right] \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\qquad - \frac{1}{2}\left[\left(\log\frac{3}{2} - \log|3\cdot\frac{3}{2}-2|\right)\right. \\
&\qquad \left. - \left(\log\frac{1+\sqrt{8y+17}}{4} - \log|3\cdot\frac{1+\sqrt{8y+17}}{4}-2|\right)\right] \mathbb{I}_{[2-\sqrt{2},1)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[\left(\log\frac{3}{2}-\log\frac{5}{2}\right)-\left(\log\sqrt{2}-\log(3\sqrt{2}-2)\right)\right]\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad -\frac{1}{2}\left[\left(\log\frac{3}{2}-\log\frac{5}{2}\right)-\left(\log\frac{1+\sqrt{8y+17}}{4}-\log\frac{-5+3\sqrt{8y+17}}{4}\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\mathbb{I}_{[2-\sqrt{2},1)}(y)\\
&= -\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad -\frac{1}{2}\left(\log\frac{3}{5}-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y). \qquad (5.43)
\end{aligned}
$$

次に，式 (5.33) の第 5 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\sqrt{2},\frac{3}{2})}\frac{\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-2\beta}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})}\frac{\mathbb{I}_{[y,1)}(T^2(0))}{3\beta^2-2\beta}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})}\frac{\mathbb{I}_{[y,1)}(\beta^2-2)}{3\beta^2-2\beta}\,d\beta\\
&\quad=\int_{[\sqrt{2},\frac{3}{2})\cap[\sqrt{y+2},\frac{3}{2})}\frac{1}{3\beta^2-2\beta}\,d\beta
\end{aligned}
$$

$y \le \beta^2-2<1$ をとくと，

$$
\begin{aligned}
y &\le \beta^2-2<1,\\
\sqrt{y+2} &\le \beta < \sqrt{3}.
\end{aligned}
$$

$\sqrt{2}\le\beta<\frac{3}{2}$ だから，$\sqrt{y+2}\ge\frac{3}{2}$ の場合は，$[\sqrt{2},\frac{3}{2})\cap[\sqrt{y+2},\frac{3}{2})=\emptyset$ で積分の値が 0 だから，$\sqrt{y+2}<\frac{3}{2}$，つまり，$0\le y<\frac{1}{4}$ の場合だけ積分する．積分範囲は $\sqrt{y+2}\le\beta<\frac{3}{2}$ である．

$$
\begin{aligned}
&\quad=\int_{[\sqrt{y+2},\frac{3}{2})}\frac{1}{3\beta^2-2\beta}\,d\beta\,\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&\quad=-\frac{1}{2}\left[\log\beta-\log|3\beta-2|\right]_{[\sqrt{y+2},\frac{3}{2})}\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&\quad=-\frac{1}{2}\Bigg[\left(\log\frac{3}{2}-\log\left|3\cdot\frac{3}{2}-2\right|\right)\\
&\qquad\quad-\left(\log\sqrt{y+2}-\log|3\sqrt{y+2}-2|\right)\Bigg]\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&\quad=-\frac{1}{2}\left(\log\frac{3}{2}-\log\frac{5}{2}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&\quad=-\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y). \qquad (5.44)
\end{aligned}
$$

よって，$\sqrt{2}\le\beta<\frac{3}{2}$ のとき．式 (5.40), (5.41), (5.42), (5.43), (5.44) より，

$$
\int_{[\sqrt{2},\frac{3}{2})}\tilde{S}_2(y)\,d\beta
$$

$$
\begin{aligned}
&= \frac{1}{3}\left(\frac{3}{2}+\frac{2}{3}\log\frac{5}{2}-\sqrt{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)\\
&\quad+\frac{1}{3}\left(\log\frac{5}{2}-\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y)+\frac{1}{3}\left(\log\frac{5}{2}-\log\frac{3y+2}{2}\right)\mathbb{I}_{[2\sqrt{2}-2,1)}(y)\\
&\quad-\left(\frac{1}{3}\left(\log\frac{5}{2}-\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y)+\frac{1}{3}\left(\log\frac{5}{2}-\log(3y+1)\right)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}\right)\\
&\quad+\left(-\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y)\right.\\
&\quad\left.-\frac{1}{2}\left(\log\frac{3}{5}-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y)\right)\\
&\quad-\left(-\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\right)\\
&= \frac{1}{3}\left(\frac{3}{2}+\frac{2}{3}\log\frac{5}{2}-\sqrt{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)\\
&\quad+\frac{1}{3}\left(\log\frac{5}{2}-\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y)+\frac{1}{3}\left(\log\frac{5}{2}-\log\frac{3y+2}{2}\right)\mathbb{I}_{[2\sqrt{2}-2,1)}(y)\\
&\quad-\frac{1}{3}\left(\log\frac{5}{2}-\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y)-\frac{1}{3}\left(\log\frac{5}{2}-\log(3y+1)\right)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y)\\
&\quad-\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad-\frac{1}{2}\left(\log\frac{3}{5}-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y)\\
&\quad+\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&= \frac{1}{3}\left(\frac{3}{2}+\frac{5}{3}\log\frac{5}{2}-\sqrt{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)-\frac{1}{2}\log\frac{3}{5}\\
&\quad-\frac{1}{3}\log\frac{5}{2}\mathbb{I}_{[0,\frac{1}{2})}(y)-\frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,2\sqrt{2}-2)}-\frac{1}{3}\log\frac{3y+2}{2}\mathbb{I}_{[2\sqrt{2}-2,1)}(y)\\
&\quad+\frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,\sqrt{2}-1)}(y)+\frac{1}{3}\log(3y+1)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y)\\
&\quad-\frac{1}{2}\left(-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad-\frac{1}{2}\left(-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y)\\
&\quad+\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y).
\end{aligned}
\tag{5.45}
$$

（ロ） $\frac{1+\sqrt{17}}{4}\le\beta<\sqrt{2}$ のとき．

$$
\begin{aligned}
\tilde{S}_2(y) &= \frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}}{\beta}-\frac{1}{\beta}+\frac{1}{\beta^2}(-1)}\\
&= \frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-1}{\beta}+\frac{1}{\beta^2}(-1)}\\
&= \frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{(\beta-1)-1}{\beta}+\frac{1}{\beta^2}(-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\beta^2 + \beta \mathbb{I}_{[0,T(1))}(y) + \mathbb{I}_{[0,T^2(1))}(y) - \beta \mathbb{I}_{[0,T(0))}(y) - \mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2 - 2\beta - 1} \\
&= \frac{\beta^2 + \beta \left(\mathbb{I}_{[0,T(1))}(y) - \mathbb{I}_{[0,T(0))}(y) \right) + \mathbb{I}_{[0,T^2(1))}(y) - \mathbb{I}_{[0,T^2(0))}(y)}{(3\beta+1)(\beta-1)}.
\end{aligned}
$$

$$
\begin{aligned}
\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \tilde{S}_2(y)\, d\beta = \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta^2}{(3\beta+1)(\beta-1)}\, d\beta + \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[0,T(1)}(y)}{(3\beta+1)(\beta-1)}\, d\beta \\
- \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[0,T(0)}(y)}{(3\beta+1)(\beta-1)}\, d\beta + \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[0,T^2(1)}(y)}{(3\beta+1)(\beta-1)}\, d\beta \\
- \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[0,T^2(1)}(y)}{(3\beta+1)(\beta-1)}\, d\beta. \qquad (5.46)
\end{aligned}
$$

まず，式 (5.46) の第 1 項目について積分する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta^2}{(3\beta+1)(\beta-1)}\, d\beta \\
&\qquad = \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{1}{3} + \frac{\frac{2}{3}\beta + \frac{1}{3}}{3\beta^2 - 2\beta - 1}\, d\beta \\
&\qquad = \frac{1}{3} \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} 1 + \frac{2\beta + 1}{3\beta^2 - 2\beta - 1}\, d\beta \\
&\qquad = \frac{1}{3} \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} 1 + \frac{\frac{3}{4}}{\beta - 1} + \frac{-\frac{1}{4}}{3\beta + 1}\, d\beta \\
&\qquad = \frac{1}{3} \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} 1 + \frac{1}{4}\left(\frac{3}{\beta - 1} - \frac{1}{3\beta + 1} \right) d\beta \\
&\qquad = \frac{1}{12} \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} 4 + \frac{3}{\beta - 1} - \frac{1}{3\beta + 1}\, d\beta \\
&\qquad = \frac{1}{12} \left[4\beta + 3\log|\beta - 1| - \frac{1}{3}\log|3\beta + 1| \right]_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \\
&\qquad = \frac{1}{12} \Bigg[4\sqrt{2} + 3\log(\sqrt{2} - 1) - \frac{1}{3}\log(3\sqrt{2} + 1) \\
&\qquad\qquad - \left(4 \cdot \frac{1+\sqrt{17}}{4} + 3\log\frac{-3+\sqrt{17}}{4} - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} \right) \Bigg] \\
&\qquad = \frac{1}{12} \Bigg(4\sqrt{2} + 3\log(\sqrt{2} - 1) - \frac{1}{3}\log(3\sqrt{2} + 1) \\
&\qquad\qquad - 1 - \sqrt{17} - 3\log\frac{-3+\sqrt{17}}{4} + \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} \Bigg). \qquad (5.47)
\end{aligned}
$$

次に，式 (5.46) の第 2 項目について積分する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[0,T(1))}(y)}{3\beta^2 - 2\beta - 1}\, d\beta \\
&\qquad = \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[y,1)}(T(1))}{3\beta^2 - 2\beta - 1}\, d\beta \\
&\qquad = \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2}) \cap [\frac{y+2}{2},\frac{3}{2})} \frac{\beta}{3\beta^2 - 2\beta - 1}\, d\beta
\end{aligned}
$$

$$
\begin{cases}
y \le 2\{\beta\} < 1 \text{ をとくと，} \\
\qquad\qquad \begin{aligned} & y \le 2\{\beta\} < 1, \\ & \frac{y}{2} \le \{\beta\} < \frac{1}{2}, \\ & \frac{y}{2} \le \beta - 1 < \frac{1}{2}, \\ & \frac{y+2}{2} \le \beta < \frac{3}{2}. \end{aligned} \\
\frac{1+\sqrt{17}}{4} \le \beta < \sqrt{2} \text{ だから，} \frac{y+2}{2} \ge \sqrt{2} \text{ の場合は、} \\
[\frac{1+\sqrt{17}}{4}, \sqrt{2}) \cap [\frac{y+2}{2}, \frac{3}{2}) = \emptyset \text{ になり、積分の値が } 0 \text{ になるた} \\
\text{め，} \frac{y+2}{2} < \sqrt{2}\text{，つまり，} 0 \le y < 2\sqrt{2} - 2 \text{ の場合だけ積分} \\
\text{の値を計算する．積分範囲は } \max\{\frac{1+\sqrt{17}}{4}, \frac{y+2}{2}\} \le \beta < \sqrt{2} \\
\text{である．}
\end{cases}
$$

$$
\begin{aligned}
&= \int_{[\frac{1+\sqrt{17}}{4}, \sqrt{2})} \frac{\frac{1}{4}}{3\beta+1} + \frac{\frac{1}{4}}{\beta-1}\, d\beta \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \int_{[\frac{y+2}{2}, \sqrt{2})} \frac{\frac{1}{4}}{3\beta+1} + \frac{\frac{1}{4}}{\beta-1}\, d\beta \mathbb{I}_{[\frac{-3+\sqrt{17}}{2}, 2\sqrt{2}-2)}(y) \\
&= \frac{1}{4}\left[\frac{1}{3}\log|3\beta+1| + \log|\beta-1|\right]_{[\frac{1+\sqrt{17}}{4}, \sqrt{2})} \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \frac{1}{4}\left[\frac{1}{3}\log|3\beta+1| + \log|\beta-1|\right]_{[\frac{y+2}{2}, \sqrt{2})} \mathbb{I}_{[\frac{-3+\sqrt{17}}{2}, 2\sqrt{2}-2)}(y) \\
&= \frac{1}{4}\left[\frac{1}{3}\log|3\cdot\sqrt{2}+1| + \log|\sqrt{2}-1| - \left(\frac{1}{3}\log|3\cdot\frac{1+\sqrt{17}}{4}+1| + \log|\frac{1+\sqrt{17}}{4}-1|\right)\right] \\
&\qquad\qquad \times \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \frac{1}{4}\left[\frac{1}{3}\log|3\cdot\sqrt{2}+1| + \log|\sqrt{2}-1| - \left(\frac{1}{3}\log|3\cdot\frac{y+2}{2}+1| + \log|\frac{y+2}{2}-1|\right)\right] \\
&\qquad\qquad \times \mathbb{I}_{[\frac{-3+\sqrt{17}}{2}, 2\sqrt{2}-2)}(y) \\
&= \frac{1}{4}\left[\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \left(\frac{1}{3}\log\frac{7+3\sqrt{17}}{4} + \log\frac{-3+\sqrt{17}}{4}\right)\right] \\
&\qquad\qquad \times \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \frac{1}{4}\left[\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \left(\frac{1}{3}\log\frac{3y+8}{2} + \log\frac{y}{2}\right)\right] \mathbb{I}_{[\frac{-3+\sqrt{17}}{2}, 2\sqrt{2}-2)}(y) \\
&= \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0, \frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{3y+8}{2} - \log\frac{y}{2}\right) \mathbb{I}_{[\frac{-3+\sqrt{17}}{2}, 2\sqrt{2}-2)}(y).
\end{aligned}
\tag{5.48}
$$

次に，式 (5.46) の第 3 項目について積分する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[0,T(0))}(y)}{3\beta^2-2\beta-1}\,d\beta \\
&\qquad= \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta \mathbb{I}_{[y,1)}(T(0))}{3\beta^2-2\beta-1}\,d\beta \\
&\qquad= \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})\cap[y+1,2)} \frac{\beta}{3\beta^2-2\beta-1}\,d\beta
\end{aligned}
$$

$y \le \{\beta\} < 1$ をとくと，

$$
\begin{aligned}
y &\le \{\beta\} < 1, \\
y &\le \beta - 1 < 1, \\
y + 1 &\le \beta < 2.
\end{aligned}
$$

$\frac{1+\sqrt{17}}{4} \le \beta < \sqrt{2}$ だから，$y+1 \ge \sqrt{2}$ の場合は、$[\frac{1+\sqrt{17}}{4},\sqrt{2})\cap[y+1,2) = \emptyset$ になり、積分の値が 0 になるため，$y+1<\sqrt{2}$，つまり，$0 \le y < \sqrt{2}-1$ の場合だけ積分の値を計算する．積分範囲は $\max\{\frac{1+\sqrt{17}}{4}, y+1\} \le \beta < \sqrt{2}$ である．

$$
\begin{aligned}
&\qquad= \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\beta}{3\beta^2-2\beta-1}\,d\beta \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad\qquad+ \int_{[y+1,\sqrt{2})} \frac{\beta}{3\beta^2-2\beta-1}\,d\beta \mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\qquad= \frac{1}{4}\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{1}{3\beta+1} + \frac{1}{\beta-1}\,d\beta \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad\qquad+ \frac{1}{4}\int_{[y+1,\sqrt{2})} \frac{1}{3\beta+1} + \frac{1}{\beta-1}\,d\beta \mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\qquad= \frac{1}{4}\left[\frac{1}{3}\log|3\beta+1| + \log|\beta-1|\right]_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad\qquad+ \frac{1}{4}\left[\frac{1}{3}\log|3\beta+1| + \log|\beta-1|\right]_{[y+1,\sqrt{2})} \mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&= \frac{1}{4}\left[\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \left(\frac{1}{3}\log\left(3\cdot\frac{1+\sqrt{17}}{4}+1\right) + \log\left(\frac{1+\sqrt{17}}{4}-1\right)\right)\right] \\
&\qquad\qquad\times \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad+ \frac{1}{4}\left[\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \left(\frac{1}{3}\log(3(y+1)+1) + \log(y+1-1)\right)\right] \\
&\qquad\qquad\times \mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\qquad= \frac{1}{4}\left[\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \left(\frac{1}{3}\log\frac{7+3\sqrt{17}}{4} + \log\frac{-3+\sqrt{17}}{4}\right)\right]\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad\qquad+ \frac{1}{4}\left[\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \left(\frac{1}{3}\log(3y+4) + \log y\right)\right]\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\qquad= \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)
\end{aligned}
$$

$$
+\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)-\frac{1}{3}\log(3y+4)-\log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y).
\tag{5.49}
$$

次に，式 (5.46) の第 4 項目について積分する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}\frac{\mathbb{I}_{[0,T^2(1))}(y)}{3\beta^2-2\beta-1}\,d\beta\\
&\quad=\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}\frac{\mathbb{I}_{[y,1)}(T^2(1))}{3\beta^2-2\beta-1}\,d\beta\\
&\quad=\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})\cap[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})}\frac{1}{3\beta^2-2\beta-1}\,d\beta
\end{aligned}
$$

$y\le T^2(1)<1$ をとくと，

$$
\begin{aligned}
&y\le 2\beta^2-\beta-2<1,\\
&\frac{1+\sqrt{8y+17}}{4}\le\beta<\frac{3}{2}.
\end{aligned}
$$

$\frac{1+\sqrt{17}}{4}\le\beta<\sqrt{2}$ だから，$\frac{1+\sqrt{8y+17}}{4}\ge\sqrt{2}$ の場合は、$[\frac{1+\sqrt{17}}{4},\sqrt{2})\cap[\frac{1+\sqrt{8y+17}}{4},\frac{3}{2})=\emptyset$ になり、積分の値が 0 になるため，$\frac{1+\sqrt{8y+17}}{4}<\sqrt{2}$，つまり，$0\le y<2-\sqrt{2}$ の場合だけ積分の値を計算する．積分範囲は $\frac{1+\sqrt{8y+17}}{4}\le\beta<\sqrt{2}$ である．

$$
\begin{aligned}
&\quad=\int_{[\frac{1+\sqrt{8y+17}}{4},\sqrt{2})}\frac{1}{3\beta^2-2\beta-1}\,d\beta\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad=\int_{[\frac{1+\sqrt{8y+17}}{4},\sqrt{2})}\frac{-\frac{3}{4}}{3\beta+1}+\frac{\frac{1}{4}}{\beta-1}\,d\beta\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad=\frac{1}{4}\int_{[\frac{1+\sqrt{8y+17}}{4},\sqrt{2})}-\frac{3}{3\beta+1}+\frac{1}{\beta-1}\,d\beta\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad=\frac{1}{4}\left[-\log|3\beta+1|+\log|\beta-1|\right]_{[\frac{1+\sqrt{8y+17}}{4},\sqrt{2})}\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad=\frac{1}{4}\Bigg[-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\\
&\qquad\qquad-\left(-\log\left(3\cdot\frac{1+\sqrt{8y+17}}{4}+1\right)+\log\left(\frac{1+\sqrt{8y+17}}{4}-1\right)\right)\Bigg]\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad=\frac{1}{4}\left[-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)-\left(-\log\frac{7+3\sqrt{8y+17}}{4}+\log\frac{-3+\sqrt{8y+17}}{4}\right)\right]\\
&\qquad\qquad\times\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad=\frac{1}{4}\left(-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)+\log\frac{7+3\sqrt{8y+17}}{4}-\log\frac{-3+\sqrt{8y+17}}{4}\right)\\
&\qquad\qquad\times\mathbb{I}_{[0,2-\sqrt{2})}(y).
\end{aligned}
\tag{5.50}
$$

次に，式 (5.46) の第 5 項目について積分する．

$$
\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-2\beta-1}\,d\beta
$$

$$
= \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})\cap[\sqrt{y+1},\sqrt{2})} \frac{1}{3\beta^2-2\beta-1}\,d\beta
$$

$$
\left(\begin{array}{l}
y \le T^2(0) < 1 \text{ をとくと，} \\
\qquad\qquad y \le \beta^2 - 1 < 1, \\
\qquad\qquad \sqrt{y+1} \le \beta < \sqrt{2}. \\
\frac{1+\sqrt{17}}{4} \le \beta < \sqrt{2} \text{ だから，積分範囲は} \\
\max\{\frac{1+\sqrt{17}}{4}, \sqrt{y+1}\} \le \beta < \sqrt{2} \text{ である.}
\end{array}\right.
$$

$$
\begin{aligned}
&= \int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \frac{1}{3\beta^2-2\beta-1}\,d\beta\,\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \int_{[\sqrt{y+1},\sqrt{2})} \frac{1}{3\beta^2-2\beta-1}\,d\beta\,\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&= \frac{1}{4}\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} -\frac{3}{3\beta+1} + \frac{1}{\beta-1}\,d\beta\,\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \frac{1}{4}\int_{[\sqrt{y+1},\sqrt{2})} -\frac{3}{3\beta+1} + \frac{1}{\beta-1}\,d\beta\,\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&= \frac{1}{4}\left[-\log|3\beta+1| + \log|\beta-1|\right]_{[\frac{1+\sqrt{17}}{4},\sqrt{2})}\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \frac{1}{4}\left[-\log|3\beta+1| + \log|\beta-1|\right]_{[\sqrt{y+1},\sqrt{2})}\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&= \frac{1}{4}\Bigg[-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) \\
&\qquad - \left(-\log\left(3\cdot\frac{1+\sqrt{17}}{4}+1\right) + \log\left(\frac{1+\sqrt{17}}{4}-1\right)\right)\Bigg]\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \frac{1}{4}\Bigg[-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) \\
&\qquad - \left(-\log\left(3\sqrt{y+1}+1\right) + \log\left(\sqrt{y+1}-1\right)\right)\Bigg]\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&= \frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) + \log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) + \log\left(3\sqrt{y+1}+1\right) - \log\left(\sqrt{y+1}-1\right)\right) \\
&\qquad\qquad \times \mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&= \frac{1}{4}\Bigg(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) \\
&\qquad + \left(\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \left(\log\left(3\sqrt{y+1}+1\right) - \log\left(\sqrt{y+1}-1\right)\right)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y)\Bigg).
\end{aligned}
\tag{5.51}
$$

よって，$\frac{1+\sqrt{17}}{4} \le \beta < \sqrt{2}$ のとき，式 (5.47), (5.48), (5.49), (5.50), (5.51) より，

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{17}}{4},\sqrt{2})} \tilde{S}_2\, d\beta \\
&\quad = \frac{1}{12}\left(4\sqrt{2} + 3\log(\sqrt{2}-1) - \frac{1}{3}\log(3\sqrt{2}+1) - 1 - \sqrt{17} - 3\log\frac{-3+\sqrt{17}}{4} + \frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right) \\
&\qquad + \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{3y+8}{2} - \log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y) \\
&\qquad - \left(+\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\right. \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad + \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log(3y+4) - \log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y)\Bigg) \\
&\qquad + \frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) + \log\frac{7+3\sqrt{8y+17}}{4} - \log\frac{-3+\sqrt{8y+17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\qquad - \left(\frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1)\right.\right. \\
&\qquad + \left(\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\qquad + \left.\left.\left(\log(3\sqrt{y+1}+1) - \log(\sqrt{y+1}-1)\right)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y)\right)\right) \\
&\quad = \frac{1}{12}\left(4\sqrt{2} + 3\log(\sqrt{2}-1) - \frac{1}{3}\log(3\sqrt{2}+1) - 1 - \sqrt{17} - 3\log\frac{-3+\sqrt{17}}{4} + \frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right) \\
&\qquad + \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\qquad + \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{3y+8}{2} - \log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y) \\
&\qquad - \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\qquad - \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) - \frac{1}{3}\log(3y+4) - \log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\qquad + \frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) + \log\frac{7+3\sqrt{8y+17}}{4} - \log\frac{-3+\sqrt{8y+17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,2-\sqrt{2})}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{4}\Bigg(-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\\
&\quad +\left(\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\\
&\quad +\Big(\log(3\sqrt{y+1}+1)-\log(\sqrt{y+1}-1)\Big)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y)\Bigg)\\
&=\frac{1}{12}\left(4\sqrt{2}+\frac{8}{3}\log(3\sqrt{2}+1)-1-\sqrt{17}-3\log\frac{-3+\sqrt{17}}{4}+\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right)\\
&\quad +\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y)\\
&\quad +\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y)\\
&\quad +\frac{1}{4}\left(-\frac{1}{3}\log\frac{3y+8}{2}-\log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y)\\
&\quad -\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y)\\
&\quad -\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)\\
&\quad -\frac{1}{4}\left(-\frac{1}{3}\log(3y+4)-\log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y)\\
&\quad +\frac{1}{4}\left(-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)+\log\frac{7+3\sqrt{8y+17}}{4}-\log\frac{-3+\sqrt{8y+17}}{4}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad -\frac{1}{4}\left(\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\\
&\quad -\frac{1}{4}\Big(\log(3\sqrt{y+1}+1)-\log(\sqrt{y+1}-1)\Big)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y)
\end{aligned}
\tag{5.52}
$$

（ハ） $\frac{3}{2}\leq\beta<\frac{1+\sqrt{5}}{2}$, $\sqrt{3}\leq\beta<\frac{1+\sqrt{7}}{2}$ のとき.

$$
\begin{aligned}
\tilde{S}_2(y)&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-1}{\beta}-\frac{1}{\beta}+\frac{1}{\beta^2}(-\beta+1)}\\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-2}{\beta}+\frac{1}{\beta^2}(-\beta+1)}\\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\beta-3}{\beta}+\frac{1}{\beta^2}(-\beta+1)}\\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\beta-3}{\beta}+\frac{1}{\beta^2}(-\beta+1)}\\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{2\beta^2+\beta(\beta-3)-\beta+1}\\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-4\beta+1}
\end{aligned}
$$

$$= \frac{\beta^2 + \beta\mathbb{I}_{[0,T(1))}(y) - \beta\mathbb{I}_{[0,T(0))}(y) + \mathbb{I}_{[0,T^2(1))}(y) - \mathbb{I}_{[0,T^2(0))}(y)}{(3\beta-1)(\beta-1)} \tag{5.53}$$

となるから，

$$\begin{aligned}
&\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \tilde{S}_2(y)\,d\beta \\
&\quad= \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\beta^2 + \beta\mathbb{I}_{[0,T(1))}(y) - \beta\mathbb{I}_{[0,T(0))}(y) + \mathbb{I}_{[0,T^2(1))}(y) - \mathbb{I}_{[0,T^2(0))}(y)}{(3\beta-1)(\beta-1)}\,d\beta \\
&\quad= \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\beta^2}{(3\beta-1)(\beta-1)}\,d\beta + \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\beta\mathbb{I}_{[0,T(1))}(y)}{(3\beta-1)(\beta-1)}\,d\beta \\
&\qquad - \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\beta\mathbb{I}_{[0,T(0))}(y)}{(3\beta-1)(\beta-1)}\,d\beta + \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\mathbb{I}_{[0,T^2(1))}(y)}{(3\beta-1)(\beta-1)}\,d\beta \\
&\qquad - \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cup[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{(3\beta-1)(\beta-1)}\,d\beta
\end{aligned} \tag{5.54}$$

である． $\frac{3}{2} \le \beta < \frac{1+\sqrt{5}}{2}$, $\sqrt{3} \le \beta < \frac{1+\sqrt{7}}{2}$ の場合で被積分関数 $\tilde{S}_2(y)$ は同じであるが，$T^2(1)$, $T^2(0)$ の値が違うので 2 つの区間に分けて積分する．

(ハ−1) $\frac{3}{2} \le \beta < \frac{1+\sqrt{5}}{2}$ のとき．

まず，式 (5.54) の第 1 項目の積分を計算する．

$$\begin{aligned}
&\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \frac{\beta^2}{(3\beta-1)(\beta-1)}\,d\beta \\
&\quad= \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \frac{1}{3} + \frac{\frac{4}{3}\beta - \frac{1}{3}}{(3\beta-1)(\beta-1)}\,d\beta \\
&\quad= \frac{1}{3}\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} 1 + \frac{4\beta-1}{(3\beta-1)(\beta-1)}\,d\beta \\
&\quad= \frac{1}{3}\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} 1 + \frac{-\frac{1}{2}}{3\beta-1} + \frac{\frac{3}{2}}{\beta-1}\,d\beta \\
&\quad= \frac{1}{3}\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} 1 - \frac{1}{2}\cdot\frac{1}{3\beta-1} + \frac{1}{2}\cdot\frac{3}{\beta-1}\,d\beta \\
&\quad= \frac{1}{3}\left[\beta - \frac{1}{6}\log|3\beta-1| + \frac{3}{2}\log|\beta-1|\right]_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \\
&\quad= \frac{1}{3}\Bigg[\frac{1+\sqrt{5}}{2} - \frac{1}{6}\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \frac{3}{2}\log|\frac{1+\sqrt{5}}{2} - 1| \\
&\qquad\quad - \left(\frac{3}{2} - \frac{1}{6}\log|3\cdot\frac{3}{2} - 1| + \frac{3}{2}\log|\frac{3}{2} - 1|\right)\Bigg] \\
&\quad= \frac{1}{3}\left[\frac{1+\sqrt{5}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{5}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{5}}{2} - \left(\frac{3}{2} - \frac{1}{6}\log\frac{7}{2} + \frac{3}{2}\log\frac{1}{2}\right)\right] \\
&\quad= \frac{1}{3}\left(\frac{1+\sqrt{5}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{5}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{5}}{2} - \frac{3}{2} + \frac{1}{6}\log\frac{7}{2} - \frac{3}{2}\log\frac{1}{2}\right) \\
&\quad= \frac{1}{3}\left(\frac{-2+\sqrt{5}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{5}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{5}}{2} + \frac{1}{6}\log\frac{7}{2} - \frac{3}{2}\log\frac{1}{2}\right).
\end{aligned} \tag{5.55}$$

次に，式 (5.54) の第 2 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \frac{\beta \mathbb{I}_{[0,T(1))}(y)}{(3\beta-1)(\beta-1)}\,d\beta \\
&\qquad = \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \left(\frac{-\frac{1}{2}}{3\beta-1}+\frac{\frac{1}{2}}{\beta-1}\right)\mathbb{I}_{[0,T(1))}(y)\,d\beta \\
&\qquad = \frac{1}{2}\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cap[\frac{y+3}{2},2)} \frac{-1}{3\beta-1}+\frac{1}{\beta-1}\,d\beta
\end{aligned}
$$

$y \le T(1) < 1$ をとくと，

$$
\begin{aligned}
y &\le 2\{\beta\}-1<1,\\
y+1 &\le 2\{\beta\}<2,\\
\frac{y+1}{2} &\le \{\beta\}<1,\\
\frac{y+3}{2} &\le \beta<2.
\end{aligned}
$$

$\frac{3}{2} \le \beta < \frac{1+\sqrt{5}}{2}$ だから，$\frac{y+3}{2} \ge \frac{1+\sqrt{5}}{2}$ の場合は，$[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cap[\frac{y+3}{2},2)=\emptyset$ だから、$\frac{y+3}{2} < \frac{1+\sqrt{5}}{2}$，つまり、$0 \le y < \sqrt{5}-2$ の場合だけ積分の計算する. 積分範囲は $\frac{y+3}{2} \le \beta < \frac{1+\sqrt{5}}{2}$ である.

$$
\begin{aligned}
&= \frac{1}{2}\int_{[\frac{y+3}{2},\frac{1+\sqrt{5}}{2})} \frac{-1}{3\beta-1}+\frac{1}{\beta-1}\,d\beta\,\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&= \frac{1}{2}\left[-\frac{1}{3}\log|3\beta-1|+\log|\beta-1|\right]_{[\frac{y+3}{2},\frac{1+\sqrt{5}}{2})}\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&= \frac{1}{2}\Bigg[-\frac{1}{3}\log|3\cdot\frac{1+\sqrt{5}}{2}-1|+\log|\frac{1+\sqrt{5}}{2}-1|\\
&\qquad -\left(-\frac{1}{3}\log|3\cdot\frac{y+3}{2}-1|+\log|\frac{y+3}{2}-1|\right)\Bigg]\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&= \frac{1}{2}\left[-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}-\left(-\frac{1}{3}\log\frac{3y+7}{2}+\log\frac{y+1}{2}\right)\right]\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&= \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y)
\end{aligned}
\tag{5.56}
$$

次に，式 (5.54) の第 3 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \frac{\beta \mathbb{I}_{[0,T(0))}(y)}{(3\beta-1)(\beta-1)}\,d\beta \\
&\qquad = \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \left(\frac{-\frac{1}{2}}{3\beta-1}+\frac{\frac{1}{2}}{\beta-1}\right)\mathbb{I}_{[0,T(0))}(y)\,d\beta \\
&\qquad = \frac{1}{2}\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cap[y+1,2)} \frac{-1}{3\beta-1}+\frac{1}{\beta-1}\,d\beta
\end{aligned}
$$

$y \leq T(0) < 1$ をとくと，

$$y \leq \{\beta\} < 1,$$
$$y \leq \beta - 1 < 1,$$
$$y + 1 \leq \beta < 2.$$

$\frac{3}{2} \leq \beta < \frac{1+\sqrt{5}}{2}$ だから，$y+1 \geq \frac{1+\sqrt{5}}{2}$ の場合は，$[\frac{3}{2}, \frac{1+\sqrt{5}}{2}) \cap [y+1, 2) = \emptyset$ だから、$y+1 < \frac{1+\sqrt{5}}{2}$，つまり、$0 \leq y < \frac{-1+\sqrt{5}}{2}$ の場合だけ積分の計算する. 積分範囲は $\max\{\frac{3}{2}, y+1\} \leq \beta < \frac{1+\sqrt{5}}{2}$ である.

$$\begin{aligned}
&= \frac{1}{2}\int_{[\frac{3}{2}, \frac{1+\sqrt{5}}{2})} \frac{-1}{3\beta-1} + \frac{1}{\beta-1}\, d\beta \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\int_{[y+1, \frac{1+\sqrt{5}}{2})} \frac{-1}{3\beta-1} + \frac{1}{\beta-1}\, d\beta \mathbb{I}_{[\frac{1}{2}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left[-\frac{1}{3}\log|3\beta-1| + \log|\beta-1|\right]_{[\frac{3}{2}, \frac{1+\sqrt{5}}{2})} \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\left[-\frac{1}{3}\log|3\beta-1| + \log|\beta-1|\right]_{[y+1, \frac{1+\sqrt{5}}{2})} \mathbb{I}_{[\frac{1}{2}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\Bigg[-\frac{1}{3}\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \log|\frac{1+\sqrt{5}}{2} - 1| \\
&\quad - \left(-\frac{1}{3}\log|3\cdot\frac{3}{2} - 1| + \log|\frac{3}{2} - 1|\right)\Bigg]\mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\Bigg[-\frac{1}{3}\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \log|\frac{1+\sqrt{5}}{2} - 1| \\
&\quad - \left(-\frac{1}{3}\log|3(y+1) - 1| + \log|(y+1) - 1|\right)\Bigg]\mathbb{I}_{[\frac{1}{2}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left[-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} - \left(-\frac{1}{3}\log\frac{7}{2} + \log\frac{1}{2}\right)\right]\mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\left[-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} - \left(-\frac{1}{3}\log(3y+2) + \log y\right)\right] \\
&\qquad \times \mathbb{I}_{[\frac{1}{2}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \frac{1}{3}\log\frac{7}{2} - \log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \frac{1}{3}\log(3y+2) - \log y\right) \\
&\qquad \times \mathbb{I}_{[\frac{1}{2}, \frac{-1+\sqrt{5}}{2})}(y).
\end{aligned} \tag{5.57}$$

次に，式 (5.54) の第 4 項目の積分の計算をする.

$$\int_{[\frac{3}{2}, \frac{1+\sqrt{5}}{2})} \frac{\mathbb{I}_{[0,T^2(1))}(y)}{3\beta^2 - 4\beta + 1}$$

$$
\begin{aligned}
&= \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \left(\frac{-\frac{3}{2}}{3\beta-1} + \frac{\frac{1}{2}}{\beta-1} \right) \mathbb{I}_{[0,T^2(1))}(y)\, d\beta \\
&= \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cap[\frac{1+\sqrt{2y+3}}{2},\frac{1+\sqrt{5}}{2})} \left(\frac{-\frac{3}{2}}{3\beta-1} + \frac{\frac{1}{2}}{\beta-1} \right) d\beta
\end{aligned}
$$

$$
\left(\begin{aligned}
&y \le T^2(1) < 1 \text{ をとくと，} \\
&\qquad\qquad y \le 2\beta^2 - 2\beta - 1 < 1, \\
&\qquad\qquad \frac{1+\sqrt{2y+3}}{2} \le \beta < \frac{1+\sqrt{5}}{2}. \\
&\text{積分範囲は，} \max\{\tfrac{3}{2}, \tfrac{1+\sqrt{2y+3}}{2}\} \le \beta < \tfrac{1+\sqrt{5}}{2}
\end{aligned}\right.
$$

$$
\begin{aligned}
&= \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \left(\frac{-\frac{3}{2}}{3\beta-1} + \frac{\frac{1}{2}}{\beta-1} \right) d\beta \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \int_{[\frac{1+\sqrt{2y+3}}{2},\frac{1+\sqrt{5}}{2})} \left(\frac{-\frac{3}{2}}{3\beta-1} + \frac{\frac{1}{2}}{\beta-1} \right) d\beta \mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2}\left[-\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \log|\frac{1+\sqrt{5}}{2} - 1| - \left(-\log|3\cdot\frac{3}{2} - 1| + \log|\frac{3}{2} - 1| \right) \right] \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\Bigg[-\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \log|\frac{1+\sqrt{5}}{2} - 1| \\
&\qquad - \left(-\log|3\cdot\frac{1+\sqrt{2y+3}}{2} - 1| + \log|\frac{1+\sqrt{2y+3}}{2} - 1| \right) \Bigg] \mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log\frac{7}{2} - \log\frac{1}{2} \right) \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log\frac{1+3\sqrt{2y+3}}{2} - \log\frac{-1+\sqrt{2y+3}}{2} \right) \\
&\qquad\qquad \times \mathbb{I}_{[\frac{1}{2},1)}(y) \\
&= \frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log 7 \right) \mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad + \frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log\frac{1+3\sqrt{2y+3}}{2} - \log\frac{-1+\sqrt{2y+3}}{2} \right) \\
&\qquad\qquad \times \mathbb{I}_{[\frac{1}{2},1)}(y).
\end{aligned} \tag{5.58}
$$

次に，式 (5.54) の第 5 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2 - 4\beta + 1} d\beta \\
&\qquad = \int_{[\frac{3}{2},\frac{1+\sqrt{5}}{2})\cap[\sqrt{y+2},\sqrt{3})} \frac{1}{2}\left(-\frac{3}{3\beta-1} + \frac{1}{\beta-1} \right) d\beta
\end{aligned}
$$

$$\begin{cases} y \leq T^2(0) < 1 \text{ をとくと，} \\ \qquad\qquad y \leq \beta^2 - 2 < 1, \\ \qquad\qquad \sqrt{y+2} \leq \beta < \sqrt{3}. \\ \text{ここで，} \sqrt{y+2} \geq \frac{1+\sqrt{5}}{2} \text{ の場合は，} [\frac{3}{2}, \frac{1+\sqrt{5}}{2}) \cap [\sqrt{y+2}, \sqrt{3}) = \emptyset \text{ となり，積分の値が 0 になるため，} \\ \sqrt{y+2} < \frac{1+\sqrt{5}}{2} \text{ の場合，つまり } 0 \leq y < \frac{-1+\sqrt{5}}{2} \text{ の場合だけ積分する．積分範囲は，} \max\{\frac{3}{2}, \sqrt{y+2}\} \leq \beta < \frac{1+\sqrt{5}}{2} \end{cases}$$

$$\begin{aligned}
&= \int_{[\frac{3}{2}, \frac{1+\sqrt{5}}{2})} \frac{1}{2}\left(-\frac{3}{3\beta-1} + \frac{1}{\beta-1}\right) d\beta \mathbb{I}_{[0,\frac{1}{4})}(y) \\
&\qquad + \int_{[\sqrt{y+2}, \frac{1+\sqrt{5}}{2})} \frac{1}{2}\left(-\frac{3}{3\beta-1} + \frac{1}{\beta-1}\right) d\beta \mathbb{I}_{[\frac{1}{4}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left[-\log|3\beta-1| + \log|\beta-1|\right]_{[\frac{3}{2}, \frac{1+\sqrt{5}}{2})} \mathbb{I}_{[0,\frac{1}{4})}(y) \\
&\qquad + \frac{1}{2}\left[-\log|3\beta-1| + \log|\beta-1|\right]_{[\sqrt{y+2}, \frac{1+\sqrt{5}}{2})} \mathbb{I}_{[\frac{1}{4}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left[-\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \log|\frac{1+\sqrt{5}}{2} - 1| - \left(-\log|3\cdot\frac{3}{2} - 1| + \log|\frac{3}{2} - 1|\right)\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,\frac{1}{4})}(y) \\
&\qquad + \frac{1}{2}\Bigg[-\log|3\cdot\frac{1+\sqrt{5}}{2} - 1| + \log|\frac{1+\sqrt{5}}{2} - 1| \\
&\qquad - \left(-\log|3\cdot\sqrt{y+2} - 1| + \log|\sqrt{y+2} - 1|\right)\Bigg] \mathbb{I}_{[\frac{1}{4}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log\frac{7}{2} - \log\frac{1}{2}\right) \mathbb{I}_{[0,\frac{1}{4})}(y) \\
&\qquad + \frac{1}{2}\Bigg(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} \\
&\qquad + \log(3\sqrt{y+2} - 1) - \log(\sqrt{y+2} - 1)\Bigg) \mathbb{I}_{[\frac{1}{4}, \frac{-1+\sqrt{5}}{2})}(y) \\
&= \frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log 7\right) \mathbb{I}_{[0,\frac{1}{4})}(y) \\
&\qquad + \frac{1}{2}\Bigg(-\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} \\
&\qquad + \log(3\sqrt{y+2} - 1) - \log(\sqrt{y+2} - 1)\Bigg) \mathbb{I}_{[\frac{1}{4}, \frac{-1+\sqrt{5}}{2})}(y).
\end{aligned} \tag{5.59}$$

よって，$\frac{3}{2} \leq \beta \leq \frac{1+\sqrt{5}}{2}$ のとき，式 (5.55), (5.56), (5.57), (5.58), (5.59) より，

$$\begin{aligned}
&\int_{[\frac{3}{2}, \frac{1+\sqrt{5}}{2})} \tilde{S}_2 \, d\beta \\
&= \frac{1}{3}\left(\frac{-2+\sqrt{5}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{5}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{5}}{2} + \frac{1}{6}\log\frac{7}{2} - \frac{3}{2}\log\frac{1}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&-\left(\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{7}{2}-\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y)\right.\\
&\qquad+\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log(3y+2)-\log y\right)\\
&\qquad\qquad\left.\times\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y)\right)\\
&+\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log 7\right)\mathbb{I}_{[0,\frac{1}{2})}(y)\\
&\qquad+\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log\frac{1+3\sqrt{2y+3}}{2}-\log\frac{-1+\sqrt{2y+3}}{2}\right)\\
&\qquad\qquad\times\mathbb{I}_{[\frac{1}{2},1)}(y)\\
&-\left(\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log 7\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\right.\\
&\qquad+\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log(3\sqrt{y+2}-1)-\log(\sqrt{y+2}-1)\right)\\
&\qquad\qquad\left.\times\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y)\right)\\
=&\ \frac{1}{3}\left(\frac{-2+\sqrt{5}}{2}-\frac{1}{6}\log\frac{1+3\sqrt{5}}{2}+\frac{3}{2}\log\frac{-1+\sqrt{5}}{2}+\frac{1}{6}\log\frac{7}{2}-\frac{3}{2}\log\frac{1}{2}\right)\\
&+\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&-\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{7}{2}-\log\frac{1}{2}\right)\mathbb{I}_{[0,\frac{1}{2})}(y)\\
&\qquad-\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log(3y+2)-\log y\right)\\
&\qquad\qquad\times\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y)\\
&+\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log 7\right)\mathbb{I}_{[0,\frac{1}{2})}(y)\\
&\qquad+\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log\frac{1+3\sqrt{2y+3}}{2}-\log\frac{-1+\sqrt{2y+3}}{2}\right)\\
&\qquad\qquad\times\mathbb{I}_{[\frac{1}{2},1)}(y)\\
&-\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log 7\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&\qquad-\frac{1}{2}\left(-\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\log(3\sqrt{y+2}-1)-\log(\sqrt{y+2}-1)\right)\\
&\qquad\qquad\times\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{3}\left(\frac{-2+\sqrt{5}}{2} - \frac{5}{3}\log\frac{1+3\sqrt{5}}{2} + 3\log\frac{-1+\sqrt{5}}{2} + \frac{1}{6}\log\frac{7}{2} - \frac{3}{2}\log\frac{1}{2}\right) \\
&\quad + \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \frac{1}{3}\log\frac{3y+7}{2} - \log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad + \left(\frac{2}{3}\log\frac{1+3\sqrt{5}}{2} - \log\frac{-1+\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) + \frac{1}{3}\log\frac{7}{2}\mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+3}}{2} - \log\frac{-1+\sqrt{2y+3}}{2}\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&\quad - \frac{1}{2}\log 7\mathbb{I}_{[0,\frac{1}{4})}(y) - \frac{1}{2}\left(\log(3\sqrt{y+2}-1) - \log(\sqrt{y+2}-1)\right)\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y).
\end{aligned} \tag{5.60}
$$

(ハ $-$**2**) $\sqrt{3} \leq \beta < \frac{1+\sqrt{7}}{2}$ とき.

まず式 (5.54) の第 1 項目の積分の計算を行う.

$$
\begin{aligned}
&\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\beta^2}{3\beta^2-4\beta+1}\,d\beta \\
&= \frac{1}{3}\left[\beta - \frac{1}{6}\log|3\beta-1| + \frac{3}{2}\log|\beta-1|\right]_{[\sqrt{3},\frac{1+\sqrt{7}}{2})} \\
&= \frac{1}{3}\Bigg[\frac{1+\sqrt{7}}{2} - \frac{1}{6}\log\left|\frac{3(1+\sqrt{7})}{2}-1\right| + \frac{3}{2}\log\left|\frac{1+\sqrt{7}}{2}-1\right| \\
&\qquad - \left(\sqrt{3} - \frac{1}{6}\log|3\sqrt{3}-1| + \frac{3}{2}\log|\sqrt{3}-1|\right)\Bigg] \\
&= \frac{1}{3}\Bigg[\frac{1+\sqrt{7}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{7}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{7}}{2} \\
&\qquad - \left(\sqrt{3} - \frac{1}{6}\log(3\sqrt{3}-1) + \frac{3}{2}\log(\sqrt{3}-1)\right)\Bigg] \\
&= \frac{1}{3}\Bigg(\frac{1+\sqrt{7}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{7}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{7}}{2} \\
&\qquad - \sqrt{3} + \frac{1}{6}\log(3\sqrt{3}-1) - \frac{3}{2}\log(\sqrt{3}-1)\Bigg).
\end{aligned} \tag{5.61}
$$

次に，式 (5.54) の第 2 項目の積分を計算する.

$$
\begin{aligned}
&\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})} \frac{\beta\mathbb{I}_{[0,T(1))}}{(3\beta-1)(\beta-1)}\,d\beta \\
&= \frac{1}{2}\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})} \left(-\frac{1}{3\beta-1} + \frac{1}{\beta-1}\right)\mathbb{I}_{[0,T(1))}\,d\beta \\
&= \frac{1}{2}\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})\cap[\frac{y+3}{2},2)} \left(-\frac{1}{3\beta-1} + \frac{1}{\beta-1}\right)\,d\beta
\end{aligned}
$$

$$
\begin{cases}
y \leq T(1) < 1 \text{ をとくと，} \\
\qquad \begin{aligned} y &\leq 2\{\beta\} - 1 < 1, \\ \frac{y+1}{2} &\leq \{\beta\} < 1, \\ \frac{y+1}{2} &\leq \beta - 1 < 1 \\ \frac{y+3}{2} &\leq \beta < 2. \end{aligned} \\
\text{ここで，} \frac{y+3}{2} \geq \frac{1+\sqrt{7}}{2} \text{ の場合は，} [\sqrt{3}, \frac{1+\sqrt{7}}{2}) \cap [\frac{y+3}{2}, 2) = \emptyset \\
\text{となり，積分の値が } 0 \text{ になるため，} \frac{y+3}{2} < \frac{1+\sqrt{7}}{2} \text{ の場合，つ} \\
\text{まり } 0 \leq y < \sqrt{7} - 2 \text{ の場合だけ積分する．積分範囲は，} \\
\max\{\frac{y+3}{2}, \sqrt{3}\} \leq \beta < \frac{1+\sqrt{7}}{2} \text{ である．}
\end{cases}
$$

$$
\begin{aligned}
&= \frac{1}{2} \int_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \left(-\frac{1}{3\beta - 1} + \frac{1}{\beta - 1} \right) d\beta \mathbb{I}_{[0, 2\sqrt{3}-3)}(y) \\
&\qquad + \frac{1}{2} \int_{[\frac{y+3}{2}, \frac{1+\sqrt{7}}{2})} \left(-\frac{1}{3\beta - 1} + \frac{1}{\beta - 1} \right) d\beta \mathbb{I}_{[2\sqrt{3}-3, \sqrt{7}-2)}(y) \\
&= \frac{1}{2} \left[-\frac{1}{3} \log |3\beta - 1| + \log |\beta - 1| \right]_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \mathbb{I}_{[0, 2\sqrt{3}-3)}(y) \\
&\qquad + \frac{1}{2} \left[-\frac{1}{3} \log |3\beta - 1| + \log |\beta - 1| \right]_{[\frac{y+3}{2}, \frac{1+\sqrt{7}}{2})} \mathbb{I}_{[2\sqrt{3}-3, \sqrt{7}-2)}(y) \\
&= \frac{1}{2} \Bigg[-\frac{1}{3} \log |3 \cdot \frac{1+\sqrt{7}}{2} - 1| + \log |\frac{1+\sqrt{7}}{2} - 1| \\
&\qquad - \left(-\frac{1}{3} \log |3\sqrt{3} - 1| + \log |\sqrt{3} - 1| \right) \Bigg] \mathbb{I}_{[0, 2\sqrt{3}-3)}(y) \\
&\qquad + \frac{1}{2} \Bigg[-\frac{1}{3} \log |3 \cdot \frac{1+\sqrt{7}}{2} - 1| + \log |\frac{1+\sqrt{7}}{2} - 1| \\
&\qquad - \left(-\frac{1}{3} \log |3 \cdot \frac{y+3}{2} - 1| + \log |\frac{y+3}{2} - 1| \right) \Bigg] \mathbb{I}_{[2\sqrt{3}-3, \sqrt{7}-2)}(y) \\
&= \frac{1}{2} \left(-\frac{1}{3} \log \frac{1+3\sqrt{7}}{2} + \log \frac{-1+\sqrt{7}}{2} + \frac{1}{3} \log(3\sqrt{3} - 1) - \log(\sqrt{3} - 1) \right) \\
&\qquad\qquad \times \mathbb{I}_{[0, 2\sqrt{3}-3)}(y) \\
&\qquad + \frac{1}{2} \left(-\frac{1}{3} \log \frac{1+3\sqrt{7}}{2} + \log \frac{-1+\sqrt{7}}{2} + \frac{1}{3} \log \frac{3y+7}{2} - \log \frac{y+1}{2} \right) \\
&\qquad\qquad \times \mathbb{I}_{[2\sqrt{3}-3, \sqrt{7}-2)}(y).
\end{aligned} \tag{5.62}
$$

次に，式 (5.54) の第 3 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \frac{\beta \mathbb{I}_{[0, T(0))}(y)}{(3\beta - 1)(\beta - 1)} \, d\beta \\
&\quad = \frac{1}{2} \int_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \left(-\frac{1}{3\beta - 1} + \frac{1}{\beta - 1} \right) \mathbb{I}_{[0, T(0))}(y) \, d\beta
\end{aligned}
$$

$$
= \frac{1}{2} \int_{[\sqrt{3}, \frac{1+\sqrt{7}}{2}) \cap [y+1, 2)} -\frac{1}{3\beta - 1} + \frac{1}{\beta - 1} \, d\beta
$$

$y \leq T(0) < 1$ をとくと，

$$
\begin{aligned}
&y \leq \{\beta\} < 1, \\
&y \leq \beta - 1 < 1, \\
&y + 1 \leq \beta < 2.
\end{aligned}
$$

ここで，$y+1 \geq \frac{1+\sqrt{7}}{2}$ の場合は，$[\sqrt{3}, \frac{1+\sqrt{7}}{2}) \cap [y+1, 2) = \emptyset$ となり，積分の値が 0 になるため，$y + 1 < \frac{1+\sqrt{7}}{2}$ の場合，つまり $0 \leq y < \frac{-1+\sqrt{7}}{2}$ の場合だけ積分する．積分範囲は，$\max\{y + 1, \sqrt{3}\} \leq \beta < \frac{1+\sqrt{7}}{2}$ である．

$$
\begin{aligned}
&= \frac{1}{2} \left[-\frac{1}{3} \log |3\beta - 1| + \log |\beta - 1| \right]_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \mathbb{I}_{[0, \sqrt{3}-1)}(y) \\
&\quad = \frac{1}{2} \left[-\frac{1}{3} \log |3\beta - 1| + \log |\beta - 1| \right]_{[y+1, \frac{1+\sqrt{7}}{2})} \mathbb{I}_{[\sqrt{3}-1, \frac{-1+\sqrt{7}}{2})}(y) \\
&= \frac{1}{2} \Bigg[-\frac{1}{3} \log \left|3 \cdot \frac{1 + \sqrt{7}}{2} - 1\right| + \log \left|\frac{1 + \sqrt{7}}{2} - 1\right| \\
&\qquad - \left(-\frac{1}{3} \log |3 \cdot \sqrt{3} - 1| + \log |\sqrt{3} - 1| \right) \Bigg] \mathbb{I}_{[0, \sqrt{3}-1)}(y) \\
&\qquad + \frac{1}{2} \Bigg[-\frac{1}{3} \log \left|3 \cdot \frac{1 + \sqrt{7}}{2} - 1\right| + \log \left|\frac{1 + \sqrt{7}}{2} - 1\right| \\
&\qquad - \left(-\frac{1}{3} \log |3 \cdot (y + 1) - 1| + \log |(y + 1) - 1| \right) \Bigg] \mathbb{I}_{[\sqrt{3}-1, \frac{-1+\sqrt{7}}{2})}(y) \\
&= \frac{1}{2} \left[-\frac{1}{3} \log \frac{1 + 3\sqrt{7}}{2} + \log \frac{-1 + \sqrt{7}}{2} - \left(-\frac{1}{3} \log(3\sqrt{3} - 1) + \log(\sqrt{3} - 1) \right) \right] \\
&\qquad\qquad \times \mathbb{I}_{[0, \sqrt{3}-1)}(y) \\
&\qquad + \frac{1}{2} \left[-\frac{1}{3} \log \frac{1 + 3\sqrt{7}}{2} + \log \frac{-1 + \sqrt{7}}{2} - \left(-\frac{1}{3} \log(3y + 2) + \log y \right) \right] \\
&\qquad\qquad \times \mathbb{I}_{[\sqrt{3}-1, \frac{-1+\sqrt{7}}{2})}(y) \\
&= \frac{1}{2} \left(-\frac{1}{3} \log \frac{1 + 3\sqrt{7}}{2} + \log \frac{-1 + \sqrt{7}}{2} + \frac{1}{3} \log(3\sqrt{3} - 1) - \log(\sqrt{3} - 1) \right) \\
&\qquad\qquad \times \mathbb{I}_{[0, \sqrt{3}-1)}(y) \\
&\qquad + \frac{1}{2} \left(-\frac{1}{3} \log \frac{1 + 3\sqrt{7}}{2} + \log \frac{-1 + \sqrt{7}}{2} + \frac{1}{3} \log(3y + 2) - \log y \right) \\
&\qquad\qquad \times \mathbb{I}_{[\sqrt{3}-1, \frac{-1+\sqrt{7}}{2})}(y).
\end{aligned} \tag{5.63}
$$

次に，式 (5.54) の第 4 項目の積分を計算する．

$$
\int_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \frac{\mathbb{I}_{[0, T^2(1))}(y)}{(3\beta - 1)(\beta - 1)} \, d\beta
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})} \left(-\frac{3}{3\beta-1}+\frac{1}{\beta-1}\right)\mathbb{I}_{[0,T^2(1))}(y)\,d\beta \\
&= \frac{1}{2}\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})\cap[\frac{1+\sqrt{2y+5}}{2},\frac{1+\sqrt{7}}{2})} -\frac{3}{(3\beta-1)}+\frac{1}{(\beta-1)}\,d\beta
\end{aligned}
$$

$\left(\begin{array}{l} y \le T^2(1) < 1 \text{ をとくと，} \\ \qquad\qquad y \le 2\beta^2 - 2\beta - 2 < 1, \\ \qquad\qquad \dfrac{1+\sqrt{2y+5}}{2} \le \beta < \dfrac{1+\sqrt{7}}{2} \\ \text{となり，積分範囲は，} \max\{\frac{1+\sqrt{2y+5}}{2},\sqrt{3}\} \le \beta < \frac{1+\sqrt{7}}{2} \text{ で} \\ \text{ある．} \end{array}\right.$

$$
\begin{aligned}
&= \frac{1}{2}\left[-\log|3\beta-1|+\log|\beta-1|\right]_{[\sqrt{3},\frac{1+\sqrt{7}}{2})}\mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\qquad + \frac{1}{2}\left[-\log|3\beta-1|+\log|\beta-1|\right]_{[\frac{1+\sqrt{2y+5}}{2},\frac{1+\sqrt{7}}{2})}\mathbb{I}_{[4-2\sqrt{3},1)}(y) \\
&= \frac{1}{2}\Bigg[-\log|3\cdot\frac{1+\sqrt{7}}{2}-1|+\log|\frac{1+\sqrt{7}}{2}-1| \\
&\qquad - \Big(-\log|3\cdot\sqrt{3}-1|+\log|\sqrt{3}-1|\Big)\Bigg]\mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\qquad + \frac{1}{2}\Bigg[-\log|3\cdot\frac{1+\sqrt{7}}{2}-1|+\log|\frac{1+\sqrt{7}}{2}-1| \\
&\qquad - \Bigg(-\log|3\cdot\frac{1+\sqrt{2y+5}}{2}-1|+\log|\frac{1+\sqrt{2y+5}}{2}-1|\Bigg)\Bigg]\mathbb{I}_{[4-2\sqrt{3},1)}(y) \\
&= \frac{1}{2}\Bigg(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\Bigg) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\qquad + \frac{1}{2}\Bigg(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log\frac{1+3\sqrt{2y+5}}{2}-\log\frac{-1+\sqrt{2y+5}}{2}\Bigg) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\mathbb{I}_{[4-2\sqrt{3},1)}(y).
\end{aligned}
\tag{5.64}
$$

次に，式 (5.54) の第 5 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})}\frac{\mathbb{I}_{[0,T^2(0))}(y)}{(3\beta-1)(\beta-1)}\,d\beta \\
&\quad= \frac{1}{2}\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})}\left(-\frac{3}{3\beta-1}+\frac{1}{\beta-1}\right)\mathbb{I}_{[0,T^2(0))}(y)\,d\beta \\
&\quad= \frac{1}{2}\int_{[\sqrt{3},\frac{1+\sqrt{7}}{2})\cap[\sqrt{y+3},2)} -\frac{3}{3\beta-1}+\frac{1}{\beta-1}\,d\beta
\end{aligned}
$$

$$
\begin{cases}
y \leq T^2(0) < 1 \text{ をとくと，} \\
\qquad \begin{aligned} y &\leq \beta^2 - 3 < 1, \\ \sqrt{y+3} &\leq \beta < 2. \end{aligned} \\
\sqrt{y+3} \geq \frac{1+\sqrt{7}}{2} \text{ の場合は，} [\sqrt{3}, \frac{1+\sqrt{7}}{2}) \cap [\sqrt{y+3}, 2) = \emptyset \text{ と} \\
\text{なるため，積分の値は } 0 \text{ になるため，} \sqrt{y+3} < \frac{1+\sqrt{7}}{2} \text{ の場} \\
\text{合，つまり } 0 \leq y < \frac{-2+\sqrt{7}}{2} \text{ の場合だけ積分の値を計算す} \\
\text{る．積分範囲は，} \sqrt{y+3} \leq \beta < \frac{1+\sqrt{7}}{2} \text{ である．}
\end{cases}
$$

$$
\begin{aligned}
&= \frac{1}{2} \left[-\log|3\beta - 1| + \log|\beta - 1| \right]_{[\sqrt{y+3}, \frac{1+\sqrt{7}}{2})} \mathbb{I}_{[0, \frac{-2+\sqrt{7}}{2})}(y) \\
&= \frac{1}{2} \Bigg[-\log|3 \cdot \frac{1+\sqrt{7}}{2} - 1| + \log|\frac{1+\sqrt{7}}{2} - 1| \\
&\qquad - \Big(-\log|3\sqrt{y+3} - 1| + \log|\sqrt{y+3} - 1| \Big) \Bigg] \mathbb{I}_{[0, \frac{-2+\sqrt{7}}{2})}(y) \\
&= \frac{1}{2} \Bigg(-\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \log(3\sqrt{y+3} - 1) - \log(\sqrt{y+3} - 1) \Bigg) \\
&\qquad \times \mathbb{I}_{[0, \frac{-2+\sqrt{7}}{2})}(y)
\end{aligned} \tag{5.65}
$$

よって，$\sqrt{3} \leq \beta \leq \frac{1+\sqrt{7}}{2}$ のとき，式 (5.61), (5.62), (5.63), (5.64), (5.65) より，

$$
\begin{aligned}
&\int_{[\sqrt{3}, \frac{1+\sqrt{7}}{2})} \tilde{S}_2(y)\, d\beta \\
&= \frac{1}{3} \Bigg(\frac{1+\sqrt{7}}{2} - \frac{1}{6}\log\frac{1+3\sqrt{7}}{2} + \frac{3}{2}\log\frac{-1+\sqrt{7}}{2} \\
&\qquad - \sqrt{3} + \frac{1}{6}\log(3\sqrt{3} - 1) - \frac{3}{2}\log(\sqrt{3} - 1) \Bigg) \\
&\qquad + \frac{1}{2} \left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \frac{1}{3}\log(3\sqrt{3} - 1) - \log(\sqrt{3} - 1) \right) \\
&\qquad\qquad \times \mathbb{I}_{[0, 2\sqrt{3}-3)}(y) \\
&\qquad + \frac{1}{2} \left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \frac{1}{3}\log\frac{3y+7}{2} - \log\frac{y+1}{2} \right) \\
&\qquad\qquad \times \mathbb{I}_{[2\sqrt{3}-3, \sqrt{7}-2)}(y) \\
&\qquad - \Bigg(\frac{1}{2} \left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \frac{1}{3}\log(3\sqrt{3} - 1) - \log(\sqrt{3} - 1) \right) \\
&\qquad\qquad \times \mathbb{I}_{[0, \sqrt{3}-1)}(y) \\
&\qquad + \frac{1}{2} \left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \frac{1}{3}\log(3y+2) - \log y \right) \\
&\qquad\qquad \times \mathbb{I}_{[\sqrt{3}-1, \frac{-1+\sqrt{7}}{2})}(y) \Bigg) \\
&\qquad + \frac{1}{2} \left(-\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \log(3\sqrt{3} - 1) - \log(\sqrt{3} - 1) \right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&+\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log\frac{1+3\sqrt{2y+5}}{2}-\log\frac{-1+\sqrt{2y+5}}{2}\right)\\
&\qquad\qquad \times \mathbb{I}_{[4-2\sqrt{3},1)}(y)\\
&-\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{y+3}-1)-\log(\sqrt{y+3}-1)\right)\\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y).\\
=\ &\frac{1}{3}\Bigg(\frac{1+\sqrt{7}}{2}-\frac{1}{6}\log\frac{1+3\sqrt{7}}{2}+\frac{3}{2}\log\frac{-1+\sqrt{7}}{2}\\
&-\sqrt{3}+\frac{1}{6}\log(3\sqrt{3}-1)-\frac{3}{2}\log(\sqrt{3}-1)\Bigg)\\
&+\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\frac{1}{3}\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)\\
&\qquad\qquad \times \mathbb{I}_{[0,2\sqrt{3}-3)}(y)\\
&+\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\\
&\qquad\qquad \times \mathbb{I}_{[2\sqrt{3}-3,\sqrt{7}-2)}(y)\\
&-\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\frac{1}{3}\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)\\
&\qquad\qquad \times \mathbb{I}_{[0,\sqrt{3}-1)}(y)\\
&-\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\frac{1}{3}\log(3y+2)-\log y\right)\\
&\qquad\qquad \times \mathbb{I}_{[\sqrt{3}-1,\frac{-1+\sqrt{7}}{2})}(y)\\
&+\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)\\
&\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&+\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log\frac{1+3\sqrt{2y+5}}{2}-\log\frac{-1+\sqrt{2y+5}}{2}\right)\\
&\qquad\qquad \times \mathbb{I}_{[4-2\sqrt{3},1)}(y)\\
&-\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{y+3}-1)-\log(\sqrt{y+3}-1)\right)\\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y)\\
=\ &\frac{1}{3}\left(\frac{1+\sqrt{7}}{2}-\frac{5}{3}\log\frac{1+3\sqrt{7}}{2}+3\log\frac{-1+\sqrt{7}}{2}+\frac{1}{6}\log(3\sqrt{3}-1)-\frac{3}{2}\log(\sqrt{3}-1)\right)\\
&+\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)\mathbb{I}_{[0,2\sqrt{3}-3)}(y)\\
&+\frac{1}{2}\left(\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\mathbb{I}_{[2\sqrt{3}-3,\sqrt{7}-2)}(y)\\
&-\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y)\\
&-\frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)\mathbb{I}_{[0,\sqrt{3}-1)}(y)\\
&-\frac{1}{2}\left(\frac{1}{3}\log(3y+2)-\log y\right)\mathbb{I}_{[\sqrt{3}-1,\frac{-1+\sqrt{7}}{2})}(y)\\
&+\frac{1}{2}\left(\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)\mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&+\frac{1}{2}\left(\log\frac{1+3\sqrt{2y+5}}{2}-\log\frac{-1+\sqrt{2y+5}}{2}\right)\mathbb{I}_{[4-2\sqrt{3},1)}(y)\\
&-\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{y+3}-1)-\log(\sqrt{y+3}-1)\right)\\
&\qquad\qquad\qquad\qquad\times\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y)
\end{aligned}
\tag{5.66}
$$

（ニ）$\frac{1+\sqrt{5}}{2}\leq\beta<\sqrt{3},\ \frac{1+\sqrt{7}}{2}\leq\beta<2$ のとき．

$$
\begin{aligned}
\tilde{S}_2(y)&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-\mathbb{I}_{[\frac{3}{2},2)}(\beta)}{\beta}-\frac{1}{\beta}+\frac{1}{\beta^2}(-\beta)}\\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\{\beta\}-1}{\beta}-\frac{1}{\beta}-\frac{1}{\beta}}\\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\beta-1-1}{\beta}-\frac{1}{\beta}-\frac{1}{\beta}}\\
&=\frac{1+\frac{1}{\beta}\mathbb{I}_{[0,T(1))}(y)+\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(1))}(y)-\frac{1}{\beta}\mathbb{I}_{[0,T(0))}(y)-\frac{1}{\beta^2}\mathbb{I}_{[0,T^2(0))}(y)}{2+\frac{\beta-4}{\beta}}\\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{2\beta^2+\beta(\beta-4)}\\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{3\beta^2-4\beta}\\
&=\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{\beta(3\beta-4)}
\end{aligned}
\tag{5.67}
$$

となるから，

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}\tilde{S}_2(y)\,d\beta\\
&\qquad=\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}\frac{\beta^2+\beta\mathbb{I}_{[0,T(1))}(y)-\beta\mathbb{I}_{[0,T(0))}(y)+\mathbb{I}_{[0,T^2(1))}(y)-\mathbb{I}_{[0,T^2(0))}(y)}{\beta(3\beta-4)}\,d\beta\\
&\qquad=\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)}\frac{\beta^2}{\beta(3\beta-4)}\,d\beta
\end{aligned}
$$

$$
+\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)} \frac{\beta\mathbb{I}_{[0,T(1))}(y)}{\beta(3\beta-4)}\,d\beta - \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)} \frac{\beta\mathbb{I}_{[0,T(0))}(y)}{\beta(3\beta-4)}\,d\beta
$$
$$
+\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)} \frac{\mathbb{I}_{[0,T^2(1))}(y)}{\beta(3\beta-4)}\,d\beta - \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cup[\frac{1+\sqrt{7}}{2},2)} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{\beta(3\beta-4)}\,d\beta \tag{5.68}
$$

$\frac{1+\sqrt{5}}{2} \le \beta < \sqrt{3}$, $\frac{1+\sqrt{7}}{2} \le \beta < 2$ の場合で被積分関数 $\tilde{S}_2$ は同じであるが，$T^2(1)$, $T^2(0)$ の値が違うので 2 つの区間に分けて積分する．

(ニ −1) $\frac{1+\sqrt{5}}{2} \le \beta < \sqrt{3}$ のとき．

まず，式 (5.68) の第 1 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{\beta^2}{\beta(3\beta-4)}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{\beta}{3\beta-4}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{1}{3} + \frac{\frac{4}{3}}{3\beta-4}\,d\beta \\
&\quad= \frac{1}{3}\left[\beta + \frac{4}{3}\log|3\beta-4|\right]_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \\
&\quad= \frac{1}{3}\left[\sqrt{3} + \frac{4}{3}\log|3\sqrt{3}-4| - \left(\frac{1+\sqrt{5}}{2} + \frac{4}{3}\log|3\cdot\frac{1+\sqrt{5}}{2}-4|\right)\right] \\
&\quad= \frac{1}{3}\left(\sqrt{3} + \frac{4}{3}\log(3\sqrt{3}-4) - \frac{1+\sqrt{5}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{5}}{2}\right).
\end{aligned} \tag{5.69}
$$

次に，式 (5.68) の第 2 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{\beta\mathbb{I}_{[0,T(1))}(y)}{\beta(3\beta-4)}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{\mathbb{I}_{[0,T(1))}(y)}{3\beta-4}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cap[\frac{y+3}{2},2)} \frac{1}{3\beta-4}\,d\beta
\end{aligned}
$$

$\Biggl($ $y \le T(1) < 1$ をとくと，

$$
y \le 2\{\beta\} - 1 < 1,
$$
$$
\frac{y+3}{2} \le \beta < 2
$$

となる．$\frac{y+3}{2} \ge \sqrt{3}$ の場合は，$[\frac{1+\sqrt{5}}{2},\sqrt{3}) \cap [\frac{y+3}{2},2) = \emptyset$ となり，積分の値が 0 になるから $\frac{y+3}{2} < \sqrt{3}$，つまり $0 \le y < 2\sqrt{3}-3$ の場合だけ積分の値を計算する．積分範囲は，$\max\{\frac{y+3}{2}, \frac{1+\sqrt{5}}{2}\} \le \beta < \sqrt{3}$ である．

$$
= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{1}{3\beta-4}\,d\beta\,\mathbb{I}_{[0,\sqrt{5}-2)}(y) + \int_{[\frac{y+3}{2},\sqrt{3})} \frac{1}{3\beta-4}\,d\beta\,\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y)
$$

$$
\begin{aligned}
&= \left[\frac{1}{3}\log|3\beta-4|\right]_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \mathbb{I}_{[0,\sqrt{5}-2)}(y) + \left[\frac{1}{3}\log|3\beta-4|\right]_{[\frac{y+3}{2},\sqrt{3})} \mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y) \\
&= \left[\frac{1}{3}\log|3\sqrt{3}-4| - \frac{1}{3}\log\left|3\cdot\frac{1+\sqrt{5}}{2}-4\right|\right]\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\qquad + \left[\frac{1}{3}\log|3\sqrt{3}-4| - \frac{1}{3}\log\left|3\cdot\frac{y+3}{2}-4\right|\right]\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y) \\
&= \left(\frac{1}{3}\log(3\sqrt{3}-4) - \frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\qquad + \left(\frac{1}{3}\log(3\sqrt{3}-4) - \frac{1}{3}\log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y). \qquad (5.70)
\end{aligned}
$$

次に，式 (5.68) の第 3 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{\beta\mathbb{I}_{[0,T(0))}(y)}{\beta(3\beta-4)}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{\mathbb{I}_{[0,T(0))}(y)}{3\beta-4}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cap[y+1,2)} \frac{1}{3\beta-4}\,d\beta
\end{aligned}
$$

$y \leq T(0) < 1$ をとくと，

$$
\begin{aligned}
y &\leq \{\beta\} < 1, \\
y &\leq \beta - 1 < 1, \\
y + 1 &\leq \beta < 2
\end{aligned}
$$

となる．$y+1 \geq \sqrt{3}$ の場合は，$[\frac{1+\sqrt{5}}{2},\sqrt{3})\cap[y+1,2)=\emptyset$ となり，積分の値が 0 になるから $y+1<\sqrt{3}$，つまり $0 \leq y < \sqrt{3}-1$ の場合だけ積分の値を計算する．積分範囲は，$\max\{y+1,\frac{1+\sqrt{5}}{2}\} \leq \beta < \sqrt{3}$ である．

$$
\begin{aligned}
&= \int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \frac{1}{3\beta-4}\,d\beta\,\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) + \int_{[y+1,\sqrt{3})} \frac{1}{3\beta-4}\,d\beta\,\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&= \left[\frac{1}{3}\log|3\beta-4|\right]_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) + \left[\frac{1}{3}\log|3\beta-4|\right]_{[y+1,\sqrt{3})} \mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&= \left[\frac{1}{3}\log|3\sqrt{3}-4| - \frac{1}{3}\log\left|3\cdot\frac{1+\sqrt{5}}{2}-4\right|\right]\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\qquad + \left[\frac{1}{3}\log|3\sqrt{3}-4| - \frac{1}{3}\log|3(y+1)-4|\right]\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&= \left(\frac{1}{3}\log(3\sqrt{3}-4) - \frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\qquad + \left(\frac{1}{3}\log(3\sqrt{3}-4) - \frac{1}{3}\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&= \frac{1}{3}\left(\log(3\sqrt{3}-4) - \log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y)
\end{aligned}
$$

$$+\frac{1}{3}\left(\log(3\sqrt{3}-4)-\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y). \tag{5.71}$$

次に，式 (5.68) の第 4 項目の積分を計算する．

$$\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})}\frac{\mathbb{I}_{[0,T^2(1))}(y)}{\beta(3\beta-4)}\,d\beta\\
&\quad=\frac{1}{4}\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})}\left(-\frac{1}{\beta}+\frac{3}{3\beta-4}\right)\mathbb{I}_{[0,T^2(1))}(y)\,d\beta\\
&\quad=\frac{1}{4}\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cap[\frac{1+\sqrt{2y+5}}{2},\frac{1+\sqrt{7}}{2})}-\frac{1}{\beta}+\frac{3}{3\beta-4}\,d\beta
\end{aligned}$$

$y\leq T^2(1)<1$ をとくと，

$$y\leq 2\beta^2-2\beta-2<1,$$
$$\frac{1+\sqrt{2y+5}}{2}\leq\beta<\frac{1+\sqrt{7}}{2}$$

となる．$\frac{1+\sqrt{2y+5}}{2}\geq\sqrt{3}$ の場合は，$[\frac{1+\sqrt{5}}{2},\sqrt{3})\cap[\frac{1+\sqrt{2y+5}}{2},\frac{1+\sqrt{7}}{2})=\emptyset$ となり積分の値は 0 になるから，$\frac{1+\sqrt{2y+5}}{2}<\sqrt{3}$，つまり $0\leq\beta<4-2\sqrt{3}$ の場合だけ積分の値を計算する．積分範囲は，$\frac{1+\sqrt{2y+5}}{2}\leq\beta<\sqrt{3}$ である．

$$\begin{aligned}
&\quad=\frac{1}{4}\int_{[\frac{1+\sqrt{2y+5}}{2},\sqrt{3})}-\frac{1}{\beta}+\frac{3}{3\beta-4}\,d\beta\mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&\quad=\frac{1}{4}\left[-\log|\beta|+\log|3\beta-4|\right]_{[\frac{1+\sqrt{2y+5}}{2},\sqrt{3})}\mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&\quad=\frac{1}{4}\Bigg[-\log|\sqrt{3}|+\log|3\sqrt{3}-4|\\
&\qquad-\left(-\log|\frac{1+\sqrt{2y+5}}{2}|+\log|3\cdot\frac{1+\sqrt{2y+5}}{2}-4|\right)\Bigg]\mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&\quad=\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{2y+5}}{2}-\log\frac{-5+3\sqrt{2y+5}}{2}\right)\\
&\qquad\times\mathbb{I}_{[0,4-2\sqrt{3})}(y).
\end{aligned} \tag{5.72}$$

次に，式 (5.68) の第 5 項目の積分を計算する．

$$\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})}\frac{\mathbb{I}_{[0,T^2(0))}(y)}{\beta(3\beta-4)}\,d\beta\\
&\quad=\frac{1}{4}\int_{[\frac{1*\sqrt{5}}{2},\sqrt{3})}\left(-\frac{1}{\beta}+\frac{3}{3\beta-4}\right)\mathbb{I}_{[0,T^2(0))}(y)\,d\beta\\
&\quad=\frac{1}{4}\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})\cap[\sqrt{y+2},\sqrt{3})}-\frac{1}{\beta}+\frac{3}{3\beta-4}\,d\beta
\end{aligned}$$

$$
\begin{cases}
y \le T^2(0) < 1 \text{ をとくと，} \\
\qquad\qquad y \le \beta^2 - 2 < 1, \\
\qquad\qquad \sqrt{y+2} \le \beta < \sqrt{3} \\
\text{となる．積分範囲は，} \max\{\sqrt{y+2}, \frac{1+\sqrt{5}}{2}\} \le \beta < \sqrt{3} \text{ で} \\
\text{ある．}
\end{cases}
$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta \mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{4}\int_{[\sqrt{y+2},\sqrt{3})} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta \mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y) \\
&= \frac{1}{4}\left[-\log|\beta| + \log|3\beta-4|\right]_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{4}\left[-\log|\beta| + \log|3\beta-4|\right]_{[\sqrt{y+2},\sqrt{3})} \mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y) \\
&= \frac{1}{4}\left[-\log|\sqrt{3}| + \log|3\sqrt{3}-4| - \left(-\log|\frac{1+\sqrt{5}}{2}| + \log|3\cdot\frac{1+\sqrt{5}}{2} - 4|\right)\right] \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{4}\left[-\log|\sqrt{3}| + \log|3\sqrt{3}-4| - \left(-\log|\sqrt{y+2}| + \log|3\sqrt{y+2}-4|\right)\right] \\
&\qquad\qquad \times \mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y) \\
&= \frac{1}{4}\left(-\log\sqrt{3} + \log(3\sqrt{3}-4) + \log\frac{1+\sqrt{5}}{2} - \log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{4}\left(-\log\sqrt{3} + \log(3\sqrt{3}-4) + \log\sqrt{y+2} - \log(3\sqrt{y+2}-4)\right) \\
&\qquad\qquad \times \mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y).
\end{aligned} \tag{5.73}
$$

$\frac{1+\sqrt{5}}{2} \le \beta < \sqrt{3}$ のとき，式 (5.69), (5.70), (5.71), (5.72), (5.73) より，

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{5}}{2},\sqrt{3})} \tilde{S}_2(y)\,d\beta \\
&= \frac{1}{3}\left(\sqrt{3} + \frac{4}{3}\log(3\sqrt{3}-4) - \frac{1+\sqrt{5}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{5}}{2}\right) \\
&\quad + \frac{1}{3}\left(\log(3\sqrt{3}-4) - \log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad + \frac{1}{3}\left(\log(3\sqrt{3}-4) - \log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y) \\
&\quad - \left(\frac{1}{3}\left(\log(3\sqrt{3}-4) - \log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y)\right. \\
&\quad \left. + \frac{1}{3}\left(\log(3\sqrt{3}-4) - \log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y)\right) \\
&\quad + \frac{1}{4}\left(-\log\sqrt{3} + \log(3\sqrt{3}-4) + \log\frac{1+\sqrt{2y+5}}{2} - \log\frac{-5+3\sqrt{2y+5}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad - \left(\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{5}}{2}-\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y)\right. \\
&\quad \left.+\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\sqrt{y+2}-\log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y)\right) \\
&= \frac{1}{3}\left(\sqrt{3}+\frac{4}{3}\log(3\sqrt{3}-4)-\frac{1+\sqrt{5}}{2}-\frac{4}{3}\log\frac{-5+3\sqrt{5}}{2}\right) \\
&\quad +\frac{1}{3}\left(\log(3\sqrt{3}-4)-\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad +\frac{1}{3}\left(\log(3\sqrt{3}-4)-\log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y) \\
&\quad -\frac{1}{3}\left(\log(3\sqrt{3}-4)-\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad -\frac{1}{3}\left(\log(3\sqrt{3}-4)-\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&\quad +\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{2y+5}}{2}-\log\frac{-5+3\sqrt{2y+5}}{2}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad -\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{5}}{2}-\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad -\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\sqrt{y+2}-\log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y). \\
&= \frac{1}{3}\left(\sqrt{3}+\frac{7}{12}\log(3\sqrt{3}-4)-\frac{1+\sqrt{5}}{2}-\frac{4}{3}\log\frac{-5+3\sqrt{5}}{2}+\frac{3}{4}\log\sqrt{3}\right) \\
&\quad +\frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,2\sqrt{3}-3)}(y)-\frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad -\frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y)-\frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,\sqrt{3}-1)}(y) \\
&\quad -\frac{1}{3}\left(-\log(3y-1)\right))\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&\quad +\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{2y+5}}{2}-\log\frac{-5+3\sqrt{2y+5}}{2}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad +\left(-\frac{1}{4}\log\frac{1+\sqrt{5}}{2}+\frac{7}{12}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad -\frac{1}{4}\left(\log\sqrt{y+2}-\log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y). \qquad (5.74)
\end{aligned}
$$

(**ニ −2**) $\frac{1+\sqrt{7}}{2} \leq \beta < 2$ のとき．

まず，式 (5.68) の第 1 項目の積分を計算する．

$$
\int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\beta^2}{\beta(3\beta-4)}\,d\beta
$$

$$
\begin{aligned}
&= \int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\beta}{3\beta-4}\,d\beta \\
&= \frac{1}{3}\left[\beta + \frac{4}{3}\log|3\beta-4|\right]_{[\frac{1+\sqrt{7}}{2},2)} \\
&= \frac{1}{3}\left[2 + \frac{4}{3}\log 2 - \left(\frac{1+\sqrt{7}}{2} + \frac{4}{3}\log|3\cdot\frac{1+\sqrt{7}}{2} - 4|\right)\right] \\
&= \frac{1}{3}\left(2 + \frac{4}{3}\log 2 - \frac{1+\sqrt{7}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right).
\end{aligned} \tag{5.75}
$$

次に，式 (5.68) の第 2 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\beta\mathbb{I}_{[0,T(1))}(y)}{\beta(3\beta-4)}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\mathbb{I}_{[0,T(1))}(y)}{3\beta-4}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{7}}{2},2)\cap[\frac{y+3}{2},2)} \frac{1}{3\beta-4}\,d\beta
\end{aligned}
$$

$$
\left(\begin{array}{l}
y \le T(1) < 1 \text{ をとくと，} \\
\qquad\qquad\begin{aligned}
&y \le 2\{\beta\} - 1 < 1, \\
&\frac{y+1}{2} \le \{\beta\} < 1, \\
&\frac{y+1}{2} \le \beta - 1 < 1, \\
&\frac{y+3}{2} \le \beta < 2.
\end{aligned} \\
\text{となる．積分範囲は，} \max\{\frac{y+3}{2}, \frac{1+\sqrt{7}}{2}\} \le \beta < 2 \text{ である．}
\end{array}\right.
$$

$$
\begin{aligned}
&= \int_{[\frac{1+\sqrt{7}}{2},2)} \frac{1}{3\beta-4}\,d\beta\mathbb{I}_{[0,\sqrt{7}-2)}(y) + \int_{[\frac{y+3}{2},2)} \frac{1}{3\beta-4}\,d\beta\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&= \left[\frac{1}{3}\log|3\beta-4|\right]_{[\frac{1+\sqrt{7}}{2},2)} \mathbb{I}_{[0,\sqrt{7}-2)}(y) + \left[\frac{1}{3}\log|3\beta-4|\right]_{[\frac{y+3}{2},2)} \mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&= \left[\frac{1}{3}\log 2 - \frac{1}{3}\log|3\cdot\frac{1+\sqrt{7}}{2} - 4|\right]\mathbb{I}_{[0,\sqrt{7}-2)}(y) \\
&\qquad + \left[\frac{1}{3}\log 2 - \frac{1}{3}\log|3\cdot\frac{y+3}{2} - 4|\right]\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&= \left(\frac{1}{3}\log 2 - \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) + \left(\frac{1}{3}\log 2 - \frac{1}{3}\log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&= \frac{1}{3}\left(\log 2 - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) + \frac{1}{3}\left(\log 2 - \log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{7}-2,1)}(y).
\end{aligned} \tag{5.76}
$$

次に，式 (5.68) の第 3 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\mathbb{I}_{[0,T(0))}(y)}{3\beta-4}\,d\beta \\
&\qquad = \int_{[\frac{1+\sqrt{7}}{2},2)\cap[y+1,2)} \frac{1}{3\beta-4}\,d\beta
\end{aligned}
$$

$$
\left(\begin{array}{l}
y \leq T(0) < 1 \text{ をとくと，} \\
\qquad\qquad\qquad y \leq \{\beta\} < 1, \\
\qquad\qquad\qquad y \leq \beta - 1 < 1, \\
\qquad\qquad\qquad y + 1 \leq \beta < 2. \\
\text{となる．積分範囲は，} \max\{y+1, \frac{1+\sqrt{7}}{2}\} \leq \beta < 2 \text{ である．}
\end{array}\right.
$$

$$
\begin{aligned}
&= \int_{[\frac{1+\sqrt{7}}{2},2)} \frac{1}{3\beta-4}\,d\beta \mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) + \int_{[y+1,2)} \frac{1}{3\beta-4}\,d\beta \mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y) \\
&= \left[\frac{1}{3}\log|3\beta-4|\right]_{[\frac{1+\sqrt{7}}{2},2)} \mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) + \left[\frac{1}{3}\log|3\beta-4|\right]_{[y+1,2)} \mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y) \\
&= \left(\frac{1}{3}\log 2 - \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\right) \mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) \\
&\qquad + \left(\frac{1}{3}\log 2 - \frac{1}{3}\log(3y-1)\right) \mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y).
\end{aligned} \tag{5.77}
$$

次に，式 (5.68) の第 4 項目の積分を計算する．

$$
\begin{aligned}
&\int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\mathbb{I}_{[0,T^2(1))}(y)}{\beta(3\beta-4)}\,d\beta \\
&\qquad = \int_{[\frac{1+\sqrt{7}}{2},2)} \left(\frac{-\frac{1}{4}}{\beta} + \frac{\frac{3}{4}}{3\beta-4}\right) \mathbb{I}_{[0,T^2(1))}(y)\,d\beta \\
&\qquad = \frac{1}{4}\int_{[\frac{1+\sqrt{7}}{2},2)} \left(-\frac{1}{\beta} + \frac{3}{3\beta-4}\right) \mathbb{I}_{[0,T^2(1))}(y)\,d\beta \\
&\qquad = \frac{1}{4}\int_{[\frac{1+\sqrt{7}}{2},2)\cap[\frac{1+\sqrt{2y+7}}{2},2)} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta
\end{aligned}
$$

$$
\left(\begin{array}{l}
y \leq T^2(1) < 1 \text{ をとくと，} \\
\qquad\qquad\qquad y \leq 2\beta^2 - 2\beta - 3 < 1, \\
\qquad\qquad\qquad \dfrac{1+\sqrt{2y+7}}{2} \leq \beta < 2. \\
\text{となる．積分範囲は，} \frac{1+\sqrt{2y+7}}{2} \leq \beta < 2 \text{ である．}
\end{array}\right.
$$

$$
\begin{aligned}
&\qquad = \frac{1}{4}\int_{[\frac{1+\sqrt{2y+7}}{2},2)} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta \\
&\qquad = \frac{1}{4}\left[-\log|\beta| + \log|3\beta-4|\right]_{[\frac{1+\sqrt{2y+7}}{2},2)} \\
&\qquad = \frac{1}{4}\left[-\log 2 + \log 2 - \left(-\log\frac{1+\sqrt{2y+7}}{2} + \log|3\cdot\frac{1+\sqrt{2y+7}}{2} - 4|\right)\right]
\end{aligned}
$$

$$= \frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2} - \log\frac{-5+3\sqrt{2y+7}}{2}\right). \tag{5.78}$$

次に，式 (5.68) の第 5 項目の積分を計算する．

$$\begin{aligned}
&\int_{[\frac{1+\sqrt{7}}{2},2)} \frac{\mathbb{I}_{[0,T^2(0))}(y)}{\beta(3\beta-4)}\,d\beta \\
&\quad= \int_{[\frac{1+\sqrt{7}}{2},2)} \left(\frac{-\frac{1}{4}}{\beta} + \frac{\frac{3}{4}}{3\beta-4}\right)\mathbb{I}_{[0,T^2(0))}(y)\,d\beta \\
&\quad= \frac{1}{4}\int_{[\frac{1+\sqrt{7}}{2},2)} \left(-\frac{1}{\beta} + \frac{3}{3\beta-4}\right)\mathbb{I}_{[0,T^2(0))}(y)\,d\beta \\
&\quad= \frac{1}{4}\int_{[\frac{1+\sqrt{7}}{2},2)\cap[\sqrt{y+3},2)} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta
\end{aligned}$$

$$\begin{cases}
y \le T^2(0) < 1 \text{ をとくと，} \\
\qquad y \le \beta^2 - 3 < 1, \\
\qquad \sqrt{y+3} \le \beta < 2. \\
\text{となる．積分範囲は，} \max\{\sqrt{y+3}, \frac{1+\sqrt{7}}{2}\} \le \beta < 2 \text{ である．}
\end{cases}$$

$$\begin{aligned}
&\quad= \frac{1}{4}\int_{[\frac{1+\sqrt{7}}{2},2)} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})} + \frac{1}{4}\int_{[\sqrt{y+3},2)} -\frac{1}{\beta} + \frac{3}{3\beta-4}\,d\beta\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)} \\
&\quad= \frac{1}{4}\left[-\log|\beta| + \log|3\beta-4|\right]_{[\frac{1+\sqrt{7}}{2},2)}\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})} \\
&\qquad + \frac{1}{4}\left[-\log|\beta| + \log|3\beta-4|\right]_{[\sqrt{y+3},2)}\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)} \\
&\quad= \frac{1}{4}\left[-\log 2 + \log 2 - \left(-\log\frac{1+\sqrt{7}}{2} + \log\left|3\cdot\frac{1+\sqrt{7}}{2} - 4\right|\right)\right]\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})} \\
&\qquad + \frac{1}{4}\left[-\log 2 + \log 2 - \left(-\log\sqrt{y+3} + \log(3\sqrt{y+3}-4)\right)\right]\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)} \\
&\quad= \frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2} - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \\
&\qquad + \frac{1}{4}\left(\log\sqrt{y+3} - \log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y).
\end{aligned} \tag{5.79}$$

式 (5.75), (5.76), (5.77), (5.78), (5.79) より，$\frac{1+\sqrt{7}}{2} \le \beta < 2$ のとき，

$$\begin{aligned}
&\int_{[\frac{1+\sqrt{7}}{2},2)} \tilde{S}_2(y)\,d\beta \\
&\quad= \frac{1}{3}\left(2 + \frac{4}{3}\log 2 - \frac{1+\sqrt{7}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right) \\
&\qquad + \frac{1}{3}\left(\log 2 - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) + \frac{1}{3}\left(\log 2 - \log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&\qquad - \Bigg(\left(\frac{1}{3}\log 2 - \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) \\
&\qquad + \left(\frac{1}{3}\log 2 - \frac{1}{3}\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2} - \log\frac{-5+3\sqrt{2y+7}}{2}\right) \\
&\quad - \left(\frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2} - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \right.\\
&\quad \left. + \frac{1}{4}\left(\log\sqrt{y+3} - \log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y)\right) \\
&= \frac{1}{3}\left(2 + \frac{4}{3}\log 2 - \frac{1+\sqrt{7}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right) \\
&\quad + \frac{1}{3}\left(\log 2 - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) + \frac{1}{3}\left(\log 2 - \log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&\quad - \left(\frac{1}{3}\log 2 - \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) \\
&\quad - \left(\frac{1}{3}\log 2 - \frac{1}{3}\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y) \\
&\quad + \frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2} - \log\frac{-5+3\sqrt{2y+7}}{2}\right) \\
&\quad - \frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2} - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \\
&\quad - \frac{1}{4}\left(\log\sqrt{y+3} - \log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y). \\
&= \frac{1}{3}\left(2 + \frac{4}{3}\log 2 - \frac{1+\sqrt{7}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right) \\
&\quad + \frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2} - \log\frac{-5+3\sqrt{2y+7}}{2}\right) \\
&\quad - \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\sqrt{7}-2)}(y) - \frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&\quad + \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})} + \frac{1}{3}\log(3y-1)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y) \\
&\quad - \frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2} - \log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \\
&\quad - \frac{1}{4}\left(\log\sqrt{y+3} - \log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y). \qquad (5.80)
\end{aligned}
$$

(イ)∼(ニ) より，

$$
\begin{aligned}
&\int_{(1,2)} \tilde{S}_2(y)\, d\beta \\
&\quad = \frac{1}{3}\left(\frac{-3+\sqrt{17}}{4} + \frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right) \\
&\qquad - \frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\frac{1+\sqrt{8y+9}}{4} + \log\frac{-5+3\sqrt{8y+9}}{4}\right) \\
&\qquad + \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log\frac{3y+2}{2}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4}-\log(3y+1)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)\\
&\quad +\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\sqrt{y+1}+\log(3\sqrt{y+1}-2)\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\\
&+\frac{1}{3}\left(\frac{3}{2}+\frac{5}{3}\log\frac{5}{2}-\sqrt{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)-\frac{1}{2}\log\frac{3}{5}\\
&\quad -\frac{1}{3}\log\frac{5}{2}\mathbb{I}_{[0,\frac{1}{2})}(y)-\frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,2\sqrt{2}-2)}-\frac{1}{3}\log\frac{3y+2}{2}\mathbb{I}_{[2\sqrt{2}-2,1)}(y)\\
&\quad +\frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,\sqrt{2}-1)}(y)+\frac{1}{3}\log(3y+1)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y)\\
&\quad -\frac{1}{2}\left(-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad -\frac{1}{2}\left(-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y)\\
&\quad +\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&+\frac{1}{12}\left(4\sqrt{2}+\frac{8}{3}\log(3\sqrt{2}-1)-1-\sqrt{17}-3\log\frac{-3+\sqrt{17}}{4}+\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right)\\
&\quad +\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y)\\
&\quad +\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y)\\
&\quad +\frac{1}{4}\left(-\frac{1}{3}\log\frac{3y+8}{2}-\log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y)\\
&\quad -\frac{1}{4}\left(\log(3\sqrt{2}-1)+\log(\sqrt{2}-1)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y)\\
&\quad -\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)\\
&\quad -\frac{1}{4}\left(-\frac{1}{3}\log(3y+4)-\log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y)\\
&\quad +\frac{1}{4}\left(-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)+\log\frac{7+3\sqrt{8y+17}}{4}-\log\frac{-3+\sqrt{8y+17}}{4}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\quad -\frac{1}{4}\left(\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\\
&\quad -\frac{1}{4}\left(\log(3\sqrt{y+1}+1)-\log(\sqrt{y+1}-1)\right)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y)\\
&+\frac{1}{3}\left(\frac{-2+\sqrt{5}}{2}-\frac{5}{3}\log\frac{1+3\sqrt{5}}{2}+3\log\frac{-1+\sqrt{5}}{2}+\frac{1}{6}\log\frac{7}{2}-\frac{3}{2}\log\frac{1}{2}\right)\\
&\quad +\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&\quad +\left(\frac{2}{3}\log\frac{1+3\sqrt{5}}{2}-\log\frac{-1+\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y)+\frac{1}{3}\log\frac{7}{2}\mathbb{I}_{[0,\frac{1}{2})}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+3}}{2} - \log\frac{-1+\sqrt{2y+3}}{2}\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&\quad - \frac{1}{2}\log 7\mathbb{I}_{[0,\frac{1}{4})}(y) - \frac{1}{2}\left(\log(3\sqrt{y+2}-1) - \log(\sqrt{y+2}-1)\right)\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y) \\
&+ \frac{1}{3}\left(\frac{1+\sqrt{7}}{2} - \frac{5}{3}\log\frac{1+3\sqrt{7}}{2} + 3\log\frac{-1+\sqrt{7}}{2} - \sqrt{3} + \frac{1}{6}\log(3\sqrt{3}-1) - \frac{3}{2}\log(\sqrt{3}-1)\right) \\
&\quad + \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) \\
&\quad + \frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,2\sqrt{3}-3)}(y) \\
&\quad + \frac{1}{2}\left(\frac{1}{3}\log\frac{3y+7}{2} - \log\frac{y+1}{2}\right)\mathbb{I}_{[2\sqrt{3}-3,\sqrt{7}-2)}(y) \\
&\quad - \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) \\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,\sqrt{3}-1)}(y) \\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\sqrt{3}-1,\frac{-1+\sqrt{7}}{2})}(y) \\
&\quad + \frac{1}{2}\left(\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+5}}{2} - \log\frac{-1+\sqrt{2y+5}}{2}\right)\mathbb{I}_{[4-2\sqrt{3},1)}(y) \\
&\quad - \frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \log(3\sqrt{y+3}-1) - \log(\sqrt{y+3}-1)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \\
&+ \frac{1}{3}\left(\sqrt{3} + \frac{7}{12}\log(3\sqrt{3}-4) - \frac{1+\sqrt{5}}{2} - \frac{4}{3}\log\frac{-5+3\sqrt{5}}{2} + \frac{3}{4}\log\sqrt{3}\right) \\
&\quad + \frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,2\sqrt{3}-3)}(y) - \frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad - \frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y) - \frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,\sqrt{3}-1)}(y) \\
&\quad - \frac{1}{3}\left(-\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&\quad + \frac{1}{4}\left(-\log\sqrt{3} + \log(3\sqrt{3}-4) + \log\frac{1+\sqrt{2y+5}}{2} - \log\frac{-5+3\sqrt{2y+5}}{2}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad + \left(-\frac{1}{4}\log\frac{1+\sqrt{5}}{2} - \frac{7}{12}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad - \frac{1}{4}\left(\log\sqrt{y+2} - \log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\left(2+\frac{4}{3}\log 2-\frac{1+\sqrt{7}}{2}-\frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\\
&\qquad+\frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2}-\log\frac{-5+3\sqrt{2y+7}}{2}\right)\\
&\qquad-\frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\sqrt{7}-2)}(y)-\frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{7}-2,1)}(y)\\
&\qquad+\frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}+\frac{1}{3}\log(3y-1)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y)\\
&\qquad-\frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2}-\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y)\\
&\qquad-\frac{1}{4}\left(\log\sqrt{y+3}-\log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y)\\
&=\frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right)\\
&\qquad-\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\frac{1+\sqrt{8y+9}}{4}+\log\frac{-5+3\sqrt{8y+9}}{4}\right)\\
&\qquad+\frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4}-\log\frac{3y+2}{2}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y)\\
&\qquad-\frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4}-\log(3y+1)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)\\
&\qquad+\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\sqrt{y+1}+\log(3\sqrt{y+1}-2)\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\\
&+\frac{1}{3}\left(\frac{5}{3}\log\frac{5}{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)-\frac{1}{2}\log\frac{3}{5}\\
&\qquad-\frac{1}{3}\log\frac{5}{2}\mathbb{I}_{[0,\frac{1}{2})}(y)-\frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,2\sqrt{2}-2)}-\frac{1}{3}\log\frac{3y+2}{2}\mathbb{I}_{[2\sqrt{2}-2,1)}(y)\\
&\qquad+\frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,\sqrt{2}-1)}(y)+\frac{1}{3}\log(3y+1)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y)\\
&\qquad-\frac{1}{2}\left(-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y)\\
&\qquad-\frac{1}{2}\left(-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y)\\
&\qquad+\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y)\\
&+\frac{1}{12}\left(\frac{8}{3}\log(3\sqrt{2}+1)-3\log\frac{-3+\sqrt{17}}{4}+\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right)\\
&\qquad+\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y)\\
&\qquad+\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y)\\
&\qquad+\frac{1}{4}\left(-\frac{1}{3}\log\frac{3y+8}{2}-\log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y) \\
&\quad - \frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\quad - \frac{1}{4}\left(-\frac{1}{3}\log(3y+4) - \log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\quad + \frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) + \log\frac{7+3\sqrt{8y+17}}{4} - \log\frac{-3+\sqrt{8y+17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\quad - \frac{1}{4}\left(\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\quad - \frac{1}{4}\left(\log(3\sqrt{y+1}+1) - \log(\sqrt{y+1}-1)\right)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&+ \frac{1}{3}\left(-\frac{5}{3}\log\frac{1+3\sqrt{5}}{2} + 3\log\frac{-1+\sqrt{5}}{2} + \frac{1}{6}\log\frac{7}{2} - \frac{3}{2}\log\frac{1}{2}\right) \\
&\quad + \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \log\frac{3y+7}{2} - \log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad + \left(\frac{2}{3}\log\frac{1+3\sqrt{5}}{2} - \log\frac{-1+\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) + \frac{1}{3}\log\frac{7}{2}\mathbb{I}_{[0,\frac{1}{2})}(y) \\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+3}}{2} - \log\frac{-1+\sqrt{2y+3}}{2}\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&\quad - \frac{1}{2}\log 7\mathbb{I}_{[0,\frac{1}{4})}(y) - \frac{1}{2}\left(\log(3\sqrt{y+2}-1) - \log(\sqrt{y+2}-1)\right)\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y) \\
&+ \frac{1}{3}\left(-\frac{5}{3}\log\frac{1+3\sqrt{7}}{2} + 3\log\frac{-1+\sqrt{7}}{2} + \frac{1}{6}\log(3\sqrt{3}-1) - \frac{3}{2}\log(\sqrt{3}-1)\right) \\
&\quad + \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) \\
&\quad + \frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,2\sqrt{3}-3)}(y) \\
&\quad + \frac{1}{2}\left(\frac{1}{3}\log\frac{3y+7}{2} - \log\frac{y+1}{2}\right)\mathbb{I}_{[2\sqrt{3}-3,\sqrt{7}-2)}(y) \\
&\quad - \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) \\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,\sqrt{3}-1)}(y) \\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\sqrt{3}-1,\frac{-1+\sqrt{7}}{2})}(y) \\
&\quad + \frac{1}{2}\left(\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+5}}{2} - \log\frac{-1+\sqrt{2y+5}}{2}\right)\mathbb{I}_{[4-2\sqrt{3},1)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{y+3}-1)-\log(\sqrt{y+3}-1)\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \\
&+\frac{1}{3}\left(\frac{7}{12}\log(3\sqrt{3}-4)-\frac{4}{3}\log\frac{-5+3\sqrt{5}}{2}+\frac{3}{4}\log\sqrt{3}\right) \\
&\quad +\frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,2\sqrt{3}-3)}(y)-\frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&\quad -\frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y)-\frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,\sqrt{3}-1)}(y) \\
&\quad -\frac{1}{3}\left(-\log(3y-1)\right)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y) \\
&\quad +\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{2y+5}}{2}-\log\frac{-5+3\sqrt{2y+5}}{2}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&\quad +\left(-\frac{1}{4}\log\frac{1+\sqrt{5}}{2}+\frac{7}{12}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&\quad -\frac{1}{4}\left(\log\sqrt{y+2}-\log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y) \\
&+\frac{1}{3}\left(\frac{4}{3}\log 2-\frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right) \\
&\quad +\frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2}-\log\frac{-5+3\sqrt{2y+7}}{2}\right) \\
&\quad -\frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\sqrt{7}-2)}(y)-\frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{7}-2,1)}(y) \\
&\quad +\frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}+\frac{1}{3}\log(3y-1)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y) \\
&\quad -\frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2}-\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y) \\
&\quad -\frac{1}{4}\left(\log\sqrt{y+3}-\log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y) \\
&=\frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right) \\
&\quad -\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\frac{1+\sqrt{8y+9}}{4}+\log\frac{-5+3\sqrt{8y+9}}{4}\right) \\
&\quad +\frac{1}{3}\left(\frac{5}{3}\log\frac{5}{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)-\frac{1}{2}\log\frac{3}{5} \\
&\quad +\frac{1}{12}\left(\frac{8}{3}\log(3\sqrt{2}+1)-3\log\frac{-3+\sqrt{17}}{4}+\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right) \\
&\quad +\frac{1}{3}\left(-\frac{5}{3}\log\frac{1+3\sqrt{5}}{2}+3\log\frac{-1+\sqrt{5}}{2}+\frac{1}{6}\log\frac{7}{2}-\frac{3}{2}\log\frac{1}{2}\right) \\
&\quad +\frac{1}{3}\left(-\frac{5}{3}\log\frac{1+3\sqrt{7}}{2}+3\log\frac{-1+\sqrt{7}}{2}+\frac{1}{6}\log(3\sqrt{3}-1)-\frac{3}{2}\log(\sqrt{3}-1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{3}\left(\frac{7}{12}\log(3\sqrt{3}-4) - \frac{4}{3}\log\frac{-5+3\sqrt{5}}{2} + \frac{3}{4}\log\sqrt{3}\right) \\
&\quad + \frac{1}{3}\left(\frac{4}{3}\log 2 - \frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right) \\
&\quad + \frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2} - \log\frac{-5+3\sqrt{2y+7}}{2}\right) \\
&+ \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log\frac{3y+2}{2}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\quad - \frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4} - \log(3y+1)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4} - \log\frac{-5+3\sqrt{17}}{4} - \log\sqrt{y+1} + \log(3\sqrt{y+1}-2)\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&- \frac{1}{3}\log\frac{5}{2}\mathbb{I}_{[0,\frac{1}{2})}(y) - \frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,2\sqrt{2}-2)} - \frac{1}{3}\log\frac{3y+2}{2}\mathbb{I}_{[2\sqrt{2}-2,1)}(y) \\
&\quad + \frac{1}{3}\log(3\sqrt{2}-2)\mathbb{I}_{[0,\sqrt{2}-1)}(y) + \frac{1}{3}\log(3y+1)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y) \\
&\quad - \frac{1}{2}\left(-\log\sqrt{2} + \log(3\sqrt{2}-2)\right)\mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\quad - \frac{1}{2}\left(-\log\frac{1+\sqrt{8y+17}}{4} + \log\frac{-5+3\sqrt{8y+17}}{4}\right)\mathbb{I}_{[2-\sqrt{2},1)}(y) \\
&\quad + \frac{1}{2}\left(\log\frac{3}{5} - \log\sqrt{y+2} + \log(3\sqrt{y+2}-2)\right)\mathbb{I}_{[0,\frac{1}{4})}(y) \\
&+ \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y) \\
&\quad + \frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y) \\
&\quad + \frac{1}{4}\left(-\frac{1}{3}\log\frac{3y+8}{2} - \log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y) \\
&\quad - \frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1) + \log(\sqrt{2}-1)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y) \\
&\quad - \frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y) \\
&\quad - \frac{1}{4}\left(-\frac{1}{3}\log(3y+4) - \log y\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&\quad + \frac{1}{4}\left(-\log(3\sqrt{2}+1) + \log(\sqrt{2}-1) + \log\frac{7+3\sqrt{8y+17}}{4} - \log\frac{-3+\sqrt{8y+17}}{4}\right) \\
&\qquad\qquad \times \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&\quad - \frac{1}{4}\left(\log\frac{7+3\sqrt{17}}{4} - \log\frac{-3+\sqrt{17}}{4}\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y) \\
&\quad - \frac{1}{4}\left(\log(3\sqrt{y+1}+1) - \log(\sqrt{y+1}-1)\right)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&+ \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2} + \log\frac{-1+\sqrt{5}}{2} + \frac{3y+7}{2} - \log\frac{y+1}{2}\right)\mathbb{I}_{[0,\sqrt{5}-2)}(y)
\end{aligned}
$$

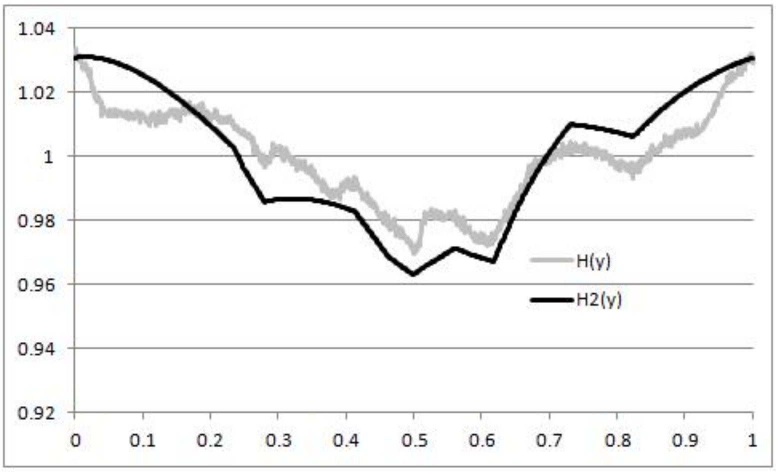
$$
\begin{aligned}
&\quad + \left(\frac{2}{3}\log\frac{1+3\sqrt{5}}{2} - \log\frac{-1+\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) + \frac{1}{3}\log\frac{7}{2}\mathbb{I}_{[0,\frac{1}{2})}(y)\\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y)\\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+3}}{2} - \log\frac{-1+\sqrt{2y+3}}{2}\right)\mathbb{I}_{[\frac{1}{2},1)}(y)\\
&\quad - \frac{1}{2}\log 7\mathbb{I}_{[0,\frac{1}{4})}(y) - \frac{1}{2}\left(\log(3\sqrt{y+2}-1) - \log(\sqrt{y+2}-1)\right)\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y)\\
&+ \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y)\\
&\quad + \frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,2\sqrt{3}-3)}(y)\\
&\quad + \frac{1}{2}\left(\frac{1}{3}\log\frac{3y+7}{2} - \log\frac{y+1}{2}\right)\mathbb{I}_{[2\sqrt{3}-3,\sqrt{7}-2)}(y)\\
&\quad - \frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y)\\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,\sqrt{3}-1)}(y)\\
&\quad - \frac{1}{2}\left(\frac{1}{3}\log(3y+2) - \log y\right)\mathbb{I}_{[\sqrt{3}-1,\frac{-1+\sqrt{7}}{2})}(y)\\
&\quad + \frac{1}{2}\left(\log(3\sqrt{3}-1) - \log(\sqrt{3}-1)\right)\mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&\quad + \frac{1}{2}\left(\log\frac{1+3\sqrt{2y+5}}{2} - \log\frac{-1+\sqrt{2y+5}}{2}\right)\mathbb{I}_{[4-2\sqrt{3},1)}(y)\\
&\quad - \frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2} + \log\frac{-1+\sqrt{7}}{2} + \log(3\sqrt{y+3}-1) - \log(\sqrt{y+3}-1)\right)\\
&\qquad \times \mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y)\\
&+ \frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,2\sqrt{3}-3)}(y) - \frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\mathbb{I}_{[0,\sqrt{5}-2)}(y)\\
&\quad - \frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y) - \frac{1}{3}\log(3\sqrt{3}-4)\mathbb{I}_{[0,\sqrt{3}-1)}(y)\\
&\quad - \frac{1}{3}\left(-\log(3y-1)\right))\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y)\\
&\quad + \frac{1}{4}\left(-\log\sqrt{3} + \log(3\sqrt{3}-4) + \log\frac{1+\sqrt{2y+5}}{2} - \log\frac{-5+3\sqrt{2y+5}}{2}\right)\\
&\qquad \times \mathbb{I}_{[0,4-2\sqrt{3})}(y)\\
&\quad + \left(-\frac{1}{4}\log\frac{1+\sqrt{5}}{2} + \frac{7}{12}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y)\\
&\quad - \frac{1}{4}\left(\log\sqrt{y+2} - \log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y)\\
&\quad - \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\sqrt{7}-2)}(y) - \frac{1}{3}\log\frac{3y+1}{2}\mathbb{I}_{[\sqrt{7}-2,1)}(y)\\
&\quad + \frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})} + \frac{1}{3}\log(3y-1)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2}-\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y)\\
&\quad -\frac{1}{4}\left(\log\sqrt{y+3}-\log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y)\\
&=\frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}\log\frac{3\sqrt{17}-5}{4}\right)\\
&\quad -\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\frac{1+\sqrt{8y+9}}{4}+\log\frac{-5+3\sqrt{8y+9}}{4}\right)\\
&\quad +\frac{1}{3}\left(\frac{5}{3}\log\frac{5}{2}-\frac{2}{3}\log(3\sqrt{2}-2)\right)-\frac{1}{2}\log\frac{3}{5}\\
&\quad +\frac{1}{12}\left(\frac{8}{3}\log(3\sqrt{2}+1)-3\log\frac{-3+\sqrt{17}}{4}+\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}\right)\\
&\quad +\frac{1}{3}\left(-\frac{5}{3}\log\frac{1+3\sqrt{5}}{2}+3\log\frac{-1+\sqrt{5}}{2}+\frac{1}{6}\log\frac{7}{2}-\frac{3}{2}\log\frac{1}{2}\right)\\
&\quad +\frac{1}{3}\left(-\frac{5}{3}\log\frac{1+3\sqrt{7}}{2}+3\log\frac{-1+\sqrt{7}}{2}+\frac{1}{6}\log(3\sqrt{3}-1)-\frac{3}{2}\log(\sqrt{3}-1)\right)\\
&\quad +\frac{1}{3}\left(\frac{7}{12}\log(3\sqrt{3}-4)-\frac{4}{3}\log\frac{-5+3\sqrt{5}}{2}+\frac{3}{4}\log\sqrt{3}\right)\\
&\quad +\frac{1}{3}\left(\frac{4}{3}\log 2-\frac{4}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\\
&\quad +\frac{1}{4}\left(\log\frac{1+\sqrt{2y+7}}{2}-\log\frac{-5+3\sqrt{2y+7}}{2}\right)\\
&+\left(\frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4}-\log\frac{3y+2}{2}\right)+\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{2})}(y)\\
&+\left(-\frac{1}{3}\left(\log\frac{-5+3\sqrt{17}}{4}-\log(3y+1)\right)-\frac{1}{4}\left(-\frac{1}{3}\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\right)\mathbb{I}_{[0,\frac{-3+\sqrt{17}}{4})}(y)\\
&+\left(\frac{1}{2}\left(\log\frac{1+\sqrt{17}}{4}-\log\frac{-5+3\sqrt{17}}{4}-\log\sqrt{y+1}+\log(3\sqrt{y+1}-2)\right)\right.\\
&\quad \left.-\frac{1}{4}\left(\log\frac{7+3\sqrt{17}}{4}-\log\frac{-3+\sqrt{17}}{4}\right)\right)\mathbb{I}_{[0,\frac{1+\sqrt{17}}{8})}(y)\\
&+\left(\frac{1}{3}\log\frac{7}{5}\right)\mathbb{I}_{[0,\frac{1}{2})}(y)\\
&+\left(-\frac{1}{3}\log(3\sqrt{2}-2)+\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\right)\right)\mathbb{I}_{[0,2\sqrt{2}-2)}(y)\\
&-\frac{1}{3}\log\frac{3y+2}{2}\mathbb{I}_{[2\sqrt{2}-2,1)}(y)\\
&+\left(\frac{1}{3}\log(3\sqrt{2}-2)-\frac{1}{4}\left(\frac{1}{3}\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)\right)\right)\mathbb{I}_{[0,\sqrt{2}-1)}(y)\\
&+\frac{1}{3}\log(3y+1)\mathbb{I}_{[\sqrt{2}-1,\frac{1}{2})}(y)\\
&+\left(-\frac{1}{2}\left(-\log\sqrt{2}+\log(3\sqrt{2}-2)\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{4}\left(-\log(3\sqrt{2}+1)+\log(\sqrt{2}-1)+\log\frac{7+3\sqrt{8y+17}}{4}-\log\frac{-3+\sqrt{8y+17}}{4}\right)\Bigg) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,2-\sqrt{2})}(y) \\
&+\left(-\frac{1}{2}\left(-\log\frac{1+\sqrt{8y+17}}{4}+\log\frac{-5+3\sqrt{8y+17}}{4}\right)\right)\mathbb{I}_{[2-\sqrt{2},1)}(y) \\
&+\frac{1}{2}\left(\log\frac{3}{5}-\log\sqrt{y+2}+\log(3\sqrt{y+2}-2)-\log 7\right)\mathbb{I}_{[0,\frac{1}{4})}(y) \\
&+\frac{1}{4}\left(-\frac{1}{3}\log\frac{3y+8}{2}-\log\frac{y}{2}\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{2},2\sqrt{2}-2)}(y) \\
&+\left(-\frac{1}{4}\left(-\frac{1}{3}\log(3y+4)-\log y\right)\right)\mathbb{I}_{[\frac{-3+\sqrt{17}}{4},\sqrt{2}-1)}(y) \\
&+\left(-\frac{1}{4}\left(\log(3\sqrt{y+1}+1)-\log(\sqrt{y+1}-1)\right)\right)\mathbb{I}_{[\frac{1+\sqrt{17}}{8},1)}(y) \\
&+\left(\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{5}}{2}+\log\frac{-1+\sqrt{5}}{2}+\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)-\frac{1}{3}\log\frac{-5+3\sqrt{5}}{2}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{I}_{[0,\sqrt{5}-2)}(y) \\
&+\left(\frac{2}{3}\log\frac{1+3\sqrt{5}}{2}-\log\frac{-1+\sqrt{5}}{2}-\frac{1}{4}\log\frac{1+\sqrt{5}}{2}+\frac{7}{12}\log\frac{-5+3\sqrt{5}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{5}}{2})}(y) \\
&-\frac{1}{2}\left(\frac{1}{3}\log(3y+2)-\log y\right)\mathbb{I}_{[\frac{1}{2},\frac{-1+\sqrt{5}}{2})}(y) \\
&+\frac{1}{2}\left(\log\frac{1+3\sqrt{2y+3}}{2}-\log\frac{-1+\sqrt{2y+3}}{2}\right)\mathbb{I}_{[\frac{1}{2},1)}(y) \\
&-\frac{1}{2}\left(\log(3\sqrt{y+2}-1)-\log(\sqrt{y+2}-1)\right)\mathbb{I}_{[\frac{1}{4},\frac{-1+\sqrt{5}}{2})}(y) \\
&+\left(\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}\right)-\frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\sqrt{7}-2)}(y) \\
&+\left(\frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)+\frac{1}{3}\log(3\sqrt{3}-4)\right)\mathbb{I}_{[0,2\sqrt{3}-3)}(y) \\
&+\frac{1}{2}\left(\frac{1}{3}\log\frac{3y+7}{2}-\log\frac{y+1}{2}\right)\mathbb{I}_{[2\sqrt{3}-3,\sqrt{7}-2)}(y) \\
&+\left(-\frac{1}{2}\left(-\frac{1}{3}\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}\right)+\frac{1}{3}\log\frac{-5+3\sqrt{7}}{2}\right)\mathbb{I}_{[0,\frac{-1+\sqrt{7}}{2})}(y) \\
&-\left(\frac{1}{2}\left(\frac{1}{3}\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right)+\frac{1}{3}\log(3\sqrt{3}-4)\right)\mathbb{I}_{[0,\sqrt{3}-1)}(y) \\
&-\frac{1}{2}\left(\frac{1}{3}\log(3y+2)-\log y\right)\mathbb{I}_{[\sqrt{3}-1,\frac{-1+\sqrt{7}}{2})}(y) \\
&+\Bigg(\frac{1}{2}\left(\log(3\sqrt{3}-1)-\log(\sqrt{3}-1)\right) \\
&\quad+\frac{1}{4}\left(-\log\sqrt{3}+\log(3\sqrt{3}-4)+\log\frac{1+\sqrt{2y+5}}{2}-\log\frac{-5+3\sqrt{2y+5}}{2}\right)\Bigg)\mathbb{I}_{[0,4-2\sqrt{3})}(y) \\
&+\frac{1}{2}\left(\log\frac{1+3\sqrt{2y+5}}{2}-\log\frac{-1+\sqrt{2y+5}}{2}\right)\mathbb{I}_{[4-2\sqrt{3},1)}(y)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{1}{2}\left(-\log\frac{1+3\sqrt{7}}{2}+\log\frac{-1+\sqrt{7}}{2}+\log(3\sqrt{y+3}-1)-\log(\sqrt{y+3}-1)\right)\right.\\
&\qquad\left.+\frac{1}{4}\left(\log\frac{1+\sqrt{7}}{2}-\log\frac{-5+3\sqrt{7}}{2}\right)\right)\mathbb{I}_{[0,\frac{-2+\sqrt{7}}{2})}(y)\\
&-\left(\frac{1}{3}\log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{5}-2,2\sqrt{3}-3)}(y)\\
&+\left(\frac{1}{3}\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},\sqrt{3}-1)}(y)\\
&-\frac{1}{4}\left(\log\sqrt{y+2}-\log(3\sqrt{y+2}-4)\right)\mathbb{I}_{[\frac{-1+\sqrt{5}}{2},1)}(y)\\
&-\left(\frac{1}{3}\log\frac{3y+1}{2}\right)\mathbb{I}_{[\sqrt{7}-2,1)}(y)\\
&+\left(\frac{1}{3}\log(3y-1)\right)\mathbb{I}_{[\frac{-1+\sqrt{7}}{2},1)}(y)\\
&-\frac{1}{4}\left(\log\sqrt{y+3}-\log(3\sqrt{y+3}-4)\right)\mathbb{I}_{[\frac{-2+\sqrt{7}}{2},1)}(y).
\end{aligned}
\tag{5.81}
$$

求められた $H_2(y)=\int_{[1,2)}\tilde{S}_2(y)\,d\beta$ の式 (5.81) のグラフを図示すると，



となる．図には，$M_\beta^n(w_0)\,(n=1,2,\cdots,1<\beta<2)$ の分布も重ねて図示してある．これによると，式 (5.81) は $k=1$ の場合の式 (5.20) $H_1(y)=\int_{[1,2)}\tilde{S}_1(y)\,d\beta$ よりも近似の精度は相当上がっているが，式自体はかなり複雑である．

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