Dissertation

Stationary quantum entanglement and the method of flow equations

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To my Mother and my Father with Heartfelt Gratitude.

Contents

Acknowledgements

"Nur Luft und Licht Und Freundeslieb! Ermude nicht, Wem dies noch blieb. J.W. von Goethe" This is true for me. I would like to thank my friends, who have generously given me support in various situations. I am greatly indebted to all of them for their kindness.

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Chapter 1

Introduction

Recently, a lot of efforts have been devoted to quantum information science in order to realize the fault-tolerant computation and communication. This, for example, can be seen in a success of the experimental realization of quantum teleportation between photons and atoms [1]. In the experiment, it is a crucial step to generate entanglement between polarization of light and cesium atoms by the use of the quantum-mechanical Faraday rotation. Entanglement between polarization of light and a rubidium atom was also generated in a more recent experiment [2].

Importance of photon-atom entanglement sheds fresh light on traditional quantum-optical systems. Majority of the discussions about photon-atom entanglement made in the literature may be concerned with nonstationary states with rapid temporal oscillations of entanglement [3,4], which are not so useful for quantum-information processing. Therefore, an important step for quantum-information

processing is to generate stationary entangled states in a coupled system of photons and atoms. Experimentally, this may require a series of "trial" operations on "initially" prepared product state in order for it to approach the exact stationary state due to non-trivial interaction between photons and atoms, in general. Accordingly, a question may naturally arise: what is an optimal procedure for it.

In this dissertation, we answer the above question by means of the method of flow equations introduced by Wegner [5-7]. This method is a powerful tool for diagonalizing a given quantum-mechanical Hamiltonian in analogy with the theory of renomalization group. It has widely been applied to a variety of problems in condensed matter physics [8-13], nuclear physics [14-16], particle physics [17], and quantum information [18]. The method of flow equations uses a continuous unitary transformation depending on a single parameter, $l \in [0, \infty)$, for the given Hamiltonian. Then, the transformed Hamiltonian tends to become diagonalized in the limit, $l \to \infty$. Here, generalizing Wegner's method and developing a new theory, we show that the generalized method can offer a geometrically optimal strategy for quantum-state engineering for realizing the stationary state in a wide class of systems.

Therefore, it is our opinion that the method of flow equations has physical meanings beyond just a tool for diagonalizing the Hamiltonian. In fact, it is known in mathematics [19] that the method is closely related to the Jacobi algorithm in matrix theory, which defines the steepest descent in the matrix space.

This suggests that if the trial operations are efficient, they are almost on a trajectory generated by the flow equations. Thus, it is meaningful to examine the flow equation approach for quantum-optical systems of photons and atoms.

Furthermore, in the analyses of several physical examples, we became aware of the fact that there seem to exist invariant quantities in the system of flow equations. This fact leads us to examine Noether's theorem [20-22] in the context of the flow equations.

Firstly, We consider the flow equation approach to establishing stationary entanglement between light and matter [18]. The unitary operator appearing in this approach is generically nonlocal one, which gives rise to the change of the degree of entanglement. To illustrate it, we employ the Jaynes-Cummings model of a two-level atom interacting with a monochromatic radiation field [23,24] and perform its complete flow equation analysis. Although it is well known that this particular model can be exactly solved by several other methods, the flow equation approach is clearly more general than those and in fact is applicable to a wide class of systems. In other words, the Jaynes-Cummings model we are going to treat here is nothing but a prototype example for demonstrating the approach in a simple manner. Also, we calculate the positive operator-valued measures (POVMs) for the photons and atom and analyze flow of the entanglement entropies, which evaluate the degree of correlation in the photon-atom system. In addition, we present a procedure for implementing the unitary operation appearing in the method.

Secondly, we develop a new theory for the method of flow equations, which generalizes the original Wegner's method [25]. Then, we consider the flow of a given quantum state through a submanifold of the projective Hilbert space composed of rays [26] in the general context. Then, we discover a condition, under which the corresponding flow becomes geodesic in the submanifold, independently of specific "initial" conditions at $l = 0$ [25]. This remarkable property is illustrated by analyzing some physical examples. Therefore, it is shown that the method of flow equations can provide us with the geometrically optimal procedure for quantum-state engineering for generating stationary states.

Finally, we note that there exist invariant quantities with respect to l-evolution in the class of exactly solvable systems. This may imply that the method of flow equations admits the underlying continuous symmetry. Now, as well known, if the action of a system is invariant under a continuous transformation of relevant variables up to the total derivative of a certain quantity, then there exists a conserved quantity (i.e., an invariant). Here, we study the invariant quantity in the time-dependent harmonic oscillator discovered by Lewis [27,28], following a manner presented by Lutzky [29,30], who has shown how to derive the invariant based on Noether's theorem. This issue is, however, nontrivial in connection with the flow equations, and needs further elaborations.

The present dissertation is organized as follows. The fundamental properties of the Jaynes-Cummings model is described in Chapter 2. In Chapter 3, the basics

of the method of flow equations is given, and some physical examples are analyzed. Then, the generation of stationary photon-atom entanglement is discussed by perfoming the flow equation analysis of the Jaynes-Cummings model in Chapter 4. There, POVMs and entanglement entropies for photons and atom are also calculated. Additionally, a comment is made on implementing the unitary operation in the method. In Chapter 5, generalizing Wegner'method, a condition, under which the flow of a quantum state becomes geodesic in the submanifold of the projective Hilbert space, independently of specific "initial" conditions at $l = 0$, is presented. This is illustrated by analyzing some physical examples. Finally, in Appendix A, we explain in detail how to derive the invariant in the time-dependent harmonic oscillator based on Noether'theorem.

Bibliography

[1] J.F. Sherson, H. Krauter, R.K. Olsson, B. Julsgaard, K. Hammerer, I. Cirac, E.S. Polzik, Nature (London) **443**, 557 (2006).

[2] T. Wilk, S.C. Webster, A. Kuhn, G. Rempe, Science **317**, 488 (2007).

[3] A.K. Rajagopal, K.L. Jensen, F.W. Cummings, Phys. Lett. A **259**, 285 (1999).

[4] S. Furuichi, M. Ohya, Lett. Math. Phys. **49**, 279 (1999).

[5] F. Wegner, Ann. Phys. (Leipzig) **3**, 77 (1994).

[6] F.J. Wegner, Phys. Rep. **348**, 77 (2001).

[7] F. Wegner, J. Phys. A: Math. Gen. **39**, 8221 (2006).

[8] S.K. Kehrein and A. Mielke, Ann. Phys. (NY) **252**,1 (1996).

[9] C. Raas, U. Low, G.S. Uhrig, and R.W. Kuhne, Phys. Rev. B **65**, 144438 (2002).

[10] V. Hankevych and F. Wegner, Eur. Phys. J. B **31**, 333 (2003).

[11] T. Stauber, Phys. Rev. B **67**, 205107 (2003).

[12] T. Domanski and J. Ranninger, Phys. Rev. B **70**, 184513 (2004).

[13] G. Aprea, C. Di Castro, M. Grilli,and J. Lorenzana, Nucl. Phys. B **744**, 277 (2006).

[14] A.B. Bylev and H.J. Pirner, Phys. Lett. B **428**, 329 (1998).

[15] S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004).

[16] S. Dusuel, J. Vidal, J.M. Arias, J. Dukelsky,and J.E. Garcia-Ramos, Phys. Rev. C **72**, 064332 (2005).

[17] S.D. Glazek, Nucl. Phys. B (Proc. Suppl.) **161**, 59 (2006).

[18] Y. Itto and S. Abe, Physica A **387**, 1063 (2008).

[19] M.T. Chu, Linear Algebra and Appl. **147**, 75 (1991).

[20] E. Noether, Nachr. v. d. Ges. d. Wiss. zu Gottingen 235 (1918).

[21] R.M. Marinho Jr., Eur. J. Phys. **28**, 37 (2007).

[22] S. Weinberg, The Quantum Theory of Fields Volume I (Cambridge: Cambridge University Press) 1995.

[23] E.T. Jaynes and F.W. Cummings, Proc. IEEE **51**, 89 (1963).

[24] C.C. Gerry and P.L. Knight, Introductory Quantum Optics, Cambridge University Press, Cambridge, 2005.

[25] Y. Itto and S. Abe, When is Wegner's flow geodesic?, e-print quant-ph/0806.4425.

[26] S. Abe, Phys. Rev. A **46**, 1667 (1992); S. Abe, Phys. Rev. A **48**, 4102 (1993).

[27] H.R. Lewis Jr., Phys. Rev. Lett. **18**, 510 (1967).

[28] H.R. Lewis Jr., 1968 J. Math. Phys. **9**, 1976 (1968).

[29] M. Lutzky, Phys. Lett. A **68**, 3 (1978).

[30] S. Abe, Y. Itto, and M. Matsunaga, Eur. J. Phys, **30**, 1337 (2009).

Chapter 2

Stationary photon-atom entanglement

2.1 The Jaynes-Cummings model

Let us consider the system of a two-level atom interacting with a monochromatic radiation field in the rotating wave approximation proposed by Jaynes and Cummings [1,2]. Although this is a highly simplified model of the coupled system of light and matter, remarkably it can experimentally be realized to generate single-atom maser or micromaser in a cavity [3]. The Hamiltonian H of the system is given by

$$H = \frac{\omega_0}{2}\sigma_3 + \omega a^\dagger a + \lambda(\sigma_+ a + \sigma_- a^\dagger). \tag{2.1}$$

σ_3, σ_+, and σ_- are the operators of the two-level atom, and are defined as

$$\sigma_3 =| e\rangle\langle e | - | g\rangle\langle g |, \quad \sigma_+ =| e\rangle\langle g |, \quad \sigma_- =| g\rangle\langle e |, \tag{2.2}$$

where $| g\rangle$ and $| e\rangle$ stand for the normalized ground and excited states of the atom, respectively. The operators in Eq. (2.2) fulfil the commutation relations:

$$[\sigma_+, \sigma_-] = \sigma_3, \quad [\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm. \tag{2.3}$$

Then, $a^\dagger$ and a describe the creation and annihilation operators of the photon, which satisfy the algebra:

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger]. \tag{2.4}$$

The physical coefficients ω_0, ω, and λ appearing in the Jaynes-Cummings Hamiltonian of Eq. (2.1) are the transition frequency between the ground and excited states of the atom, the frequency of the single-mode photon, and the real coupling constant between the photon and the atom, respectively. Here, Planck constant $\hbar$ is set equal to unity for the sake of simplicity. The Jaynes-Cummings Hamiltonian in Eq. (2.1) includes the two interaction terms: one is the term described by, $\sigma_+ a$, which mean that the atom in the ground state is excited by absorbing a photon, the other is the term, $\sigma_- a^\dagger$, which mean that the atom in the excited state is in the ground state by radiating a photon. The interaction is schematically illustrated in Fig. 1.

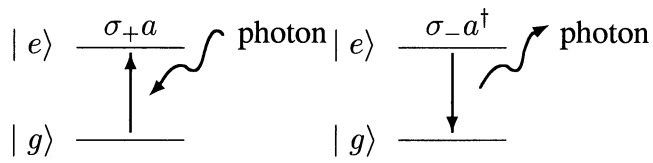

Fig. 1: The interaction of the Jaynes-Cummings Hamiltonian.

2.2 Stationary entangled states

In this section, we compute the stationary states of the Jaynes-Cummings model. Since the model is exactly solvable, it allows us to analytically discuss a variety of phenomena such as the Rabi oscillation and quantum collapses/revivals. We consider the Hamiltonian in Eq. (2.1) in matrix representation for obtaining the stationary states. In the bases $\mid e\rangle \mid n\rangle$ and $\mid g\rangle \mid n+1\rangle$ $(n = 0, 1, 2, \dots)$, the Hamiltonian in matrix representation, say $\mathbf{H}^{(n)}$, can be described as follows,

$$\mathbf{H}^{(n)} = \begin{pmatrix} n\omega + \frac{\omega_0}{2} & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & (n+1)\omega - \frac{\omega_0}{2} \end{pmatrix}. \tag{2.5}$$

The eigenvalues of the matrix in Eq. (2.5) are obtained as

$$E_{\pm}(n) = \omega\left(n + \frac{1}{2}\right) \pm \frac{\Omega_n}{2}, \tag{2.6}$$

where

$$\Delta \equiv \omega_0 - \omega, \tag{2.7}$$

$$\Omega_n \equiv \sqrt{\Delta^2 + 4\lambda^2(n+1)}, \tag{2.8}$$

are the detuning factor between the transition frequency of the atom and the frequency of the photon, and the Rabi frequency including the effect of the detuning

factor, respectively. The corresponding eigenstates $\mid n,+\rangle$ and $\mid n,-\rangle$ for the eigenvalues in Eq (2.6) are calculated to be

$$\mid n,+\rangle = u_n \mid e\rangle \mid n\rangle + v_n \mid g\rangle \mid n+1\rangle, \tag{2.9}$$

$$\mid n,-\rangle = -v_n \mid e\rangle \mid n\rangle + u_n \mid g\rangle \mid n+1\rangle \tag{2.10}$$

for $n = 0, 1, 2, \ldots$ with the probabilistic amplitude

$$u_n = \frac{1}{\sqrt{2}}\sqrt{1+\frac{\Delta}{\Omega_n}}, \tag{2.11}$$

$$v_n = \frac{1}{\sqrt{2}}\sqrt{1-\frac{\Delta}{\Omega_n}} \tag{2.12}$$

satisfying the condition of probability conservation

$$u_n^2 + v_n^2 = 1. \tag{2.13}$$

Particularly, in the limit of the exact resonance, $\Delta = 0$, the stationary states for $n = 0, 1, 2, \ldots$ become

$$\mid n,+\rangle_{\Delta=0} = \frac{1}{\sqrt{2}}(\mid e\rangle \mid n\rangle + \mid g\rangle \mid n+1\rangle), \tag{2.14}$$

$$\mid n,-\rangle_{\Delta=0} = \frac{1}{\sqrt{2}}(-\mid e\rangle \mid n\rangle + \mid g\rangle \mid n+1\rangle). \tag{2.15}$$

2.3 Experimental realization of the Jaynes-Cummings model

As mentioned above, the remarkable fact is that the Jaynes-Cummings model can experimentally be realized in a cavity. This is schematically illustrated in Fig. 2.

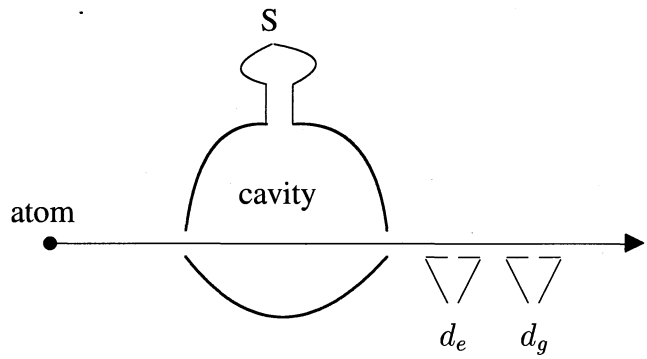

Fig. 2: Experimental setup for the realization the Jaynes-Cummings model. S is source of photon, whereas d_e and d_g are detector of $\mid e\rangle$ and $\mid g\rangle$, respectively. The Jaynes-Cummings model is realized when the atom goes through the cavity.

It may not be difficult to generate the single-mode photon. Then, the two-level atom can be found to be the circular Rydberg state. This is, for example, the exited state of valance electron of alkali metals with a sufficiently large principle quantum number. In the case of rubidium, the corresponding principle quantum number is about 50. The circular Rydberg state is defined as the state associated with the orbital angular momentum quantum number $l = n - 1$ and the magnetic quantum number $|m| = n - 1$ in the principle quantum number n. The atom in this state can transit from $n \leftrightarrow n - 1$ and $|m| \leftrightarrow |m| - 1$, then approximately can be seen to be the two-level atom. In fact, it is experimentally known that the cesium atom is approximately the two-level one [4].

Bibliography

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[2] C. C. Gerry and P. L. Knight, Introductory Quantum Optics (Cambridge University Press, Cambridge, 2005).

[3] H. Walther, Physica Scripta T **86**, 27 (2000).

[4] R. G. Hulet, E. S. Hilfer and D. Kleppner, Phys. Rev. Lett. **55**, 2137 (1985).

Chapter 3

The method of flow equations

In recent years, much attention has been focused on Wegner's method of flow equations [1] (for reviews, see [2,3]). This method offers a powerful tool for diagonalizing a given quantum-mechanical Hamiltonian, which has been devised in analogy with the theory of renormalization groups and has widely been applied to a variety of problems in condensed matter physics, nuclear and particle physics. Examples include the effective Hamiltonian of the Anderson impurity model [4], a Dirac particle in an external electromagnetic field [5], an effective spin-spin coupling arising from spin-phonon chains [6], the Hubbard model for high-temperature superconductivity [7], the Tomonaga-Luttinger model for interacting spinless electrons in one dimension [8], localized superfluidity [9], the Lipkin-Meshkov-Glick model in nuclear physics [10], quantum phase transition in the interacting boson model [11], light-cone quantum chromodynamics [12],

electron-electron and electron-phonon interactions in the Hubbard-Holstein model [13], and quantum information [14].

We devote this chapter to grasp the method of flow equations and analyze some physical examples.

Let us consider d-dimensional quantum system whose Hamiltonian is given by H. In general, the Hamiltonian is not diagonal for a naturally taken basis. To diagonalize (or block-diagonalize) it for the basis, the method of flow equations introduce a continuous unitary transformation $U(l)$, which satisfies $U(l)U^\dagger(l) = U^\dagger(l)U(l) = I$, parametrized by $l \in [0, \infty)$. This transforms the original Hamiltonian $H(0) = H$ to $H(l) = U(l)HU^\dagger(l)$. The derivative of the transformed Hamiltonian with respect to l leads to

$$\frac{dH(l)}{dl} = [\eta(l), H(l)]. \tag{3.1}$$

This is the flow equation. Here, in Eq. (3.1), the anti-Hermitian operator is introduced as

$$\eta(l) = \frac{dU(l)}{dl}U^\dagger(l), \tag{3.2}$$

which is called the generator. The differential equation in Eq. (3.1) determines the flow of physical parameters contained in the original Hamiltonian in analogy with the theory of renomalization group. Clearly, the generator should appropriately be chosen for diagonalizing (or block-diagonalizing) the original Hamiltonian.

3.1 Wegner's choice for the generator

As a choice for the generator, Wegner proposed

$$\eta(l) = [H_{\mathrm{d}}(l), H_{\text{o-d}}(l)], \tag{3.3}$$

where $H_{\mathrm{d}}(l)$ and $H_{\text{o-d}}(l)$ are the diagonal part and the off-diagonal part of the transformed Hamiltonian, respectively. In fact, this choice yields the following relation

$$\begin{aligned}\frac{d}{dl}\sum_{m,n=1,m\neq n}^{d} |\langle u_m \mid H(l) \mid u_n\rangle|^2 &= -\frac{d}{dl}\sum_{n=1}^{d}\epsilon_n^2(l) \\ &= -2\sum_{m,n=1}^{d}[\epsilon_m(l)-\epsilon_n(l)]^2|\langle u_m \mid H(l) \mid u_n\rangle|^2,\end{aligned} \tag{3.4}$$

where $\epsilon_n(l)$ is the nth eigenvalue of $H_{\mathrm{d}}(l)$ and $\mid u_n\rangle$ is nth eigenstate. This implies that the off-diagonal elements of the transformed Hamiltonian tends to decay as l increases due to the negativity of the right-hand side of Eq. (3.4) if $H_{\mathrm{d}}(l)$ is nondegenerate.

3.2 Physical examples

In this section, we see how the method of flow equations is successfully applied to quantum system by analyzing some examples.

Firstly, we consider the generalized harmonic oscillator. Hamiltonian H of the

system is given by

$$H = \omega a^\dagger a + \lambda {a^\dagger}^2 + \lambda^* a^2 + \nu, \tag{3.5}$$

where $a^\dagger$ and a are the creation and annihilation operators obeying the algebra: $[a, a^\dagger]$, $[a^\dagger, a^\dagger] = [a, a] = 0$. ω (> 0) and ν are real constants, whereas λ is complex. In particular, the condition, $\omega > 2|\lambda|$, is to be satisfied. Here and hereafter, $\hbar$ is set equal to unity. The unitary operator to be considered for this system is

$$U(l) = \exp \frac{1}{2}(\xi(l){a^\dagger}^2 - \xi^*(l)a^2), \tag{3.6}$$

where the complex coefficient $\xi(l)$ is parametrized as

$$\xi(l) = r(l)\mathrm{e}^{-i2\phi(l)} \quad (0 \leq r(l), 0 \leq \phi(l) < 2\pi\ ). \tag{3.7}$$

Thus, the transformed Hamiltonian is given by

$$\begin{aligned} H(l) =& U(l)HU^\dagger(l) \\ =& \omega(l)a^\dagger a + \lambda(l){a^\dagger}^2 + \lambda^*(l)a^2 + \nu(l), \end{aligned} \tag{3.8}$$

where the physical coefficients are defined as follows

$$\omega(l) = \omega \cosh(2r(l)) - (\lambda \mathrm{e}^{i2\phi(l)} + \lambda^* \mathrm{e}^{-i2\phi(l)}) \sinh(2r(l)), \tag{3.9}$$

$$\begin{aligned} \lambda(l) =& \frac{1}{2}(\lambda + \lambda^* \mathrm{e}^{-i4\phi(l)}) \cosh(2r(l)) \\ &+ \frac{1}{2}(\lambda - \lambda^* \mathrm{e}^{-i4\phi(l)}) - \frac{1}{2}\omega \mathrm{e}^{-i2\phi(l)} \sinh(2r(l)), \end{aligned} \tag{3.10}$$

$$\nu(l) = \omega \sinh^2(r(l)) - \frac{1}{2}(\lambda \mathrm{e}^{i2\phi(l)} + \lambda^* \mathrm{e}^{-i2\phi(l)}) \sinh(2r(l)) + \nu. \tag{3.11}$$

Accordingly, the diagonal part $H_{\rm d}(l)$ and the off-diagonal part $H_{\text{o-d}}(l)$ of the transformed Hamiltonian in Eq. (3.8) are

$$H_{\rm d}(l) = \omega(l)a^\dagger a + \nu(l), \tag{3.12}$$

$$H_{\text{o-d}}(l) = \lambda(l){a^\dagger}^2 + \lambda^*(l)a^2. \tag{3.13}$$

Using Eqs. (3.12) and (3.13), we immediately obtain $\eta(l)$ as

$$\begin{aligned}\eta(l) =&[H_{\rm d}(l), H_{\text{o-d}}(l)] \\ =&2\omega(l)[\lambda(l){a^\dagger}^2 - \lambda^*(l)a^2].\end{aligned} \tag{3.14}$$

Furthermore, from Eqs. (3.8) and (3.14), we get

$$\begin{aligned}[\eta(l), H(l)] =& -16\omega(l)|\ \lambda(l)\ |^2 a^\dagger a \\ &- 4\omega^2(l)\lambda(l){a^\dagger}^2 - 4\omega^2(l)\lambda^*(l)a^2 - 8\omega(l)|\ \lambda(l)\ |^2.\end{aligned} \tag{3.15}$$

Then, the derivative of $H(l)$ in Eq. (3.8) with respect to l leads to

$$\frac{dH(l)}{dl} = \frac{d\omega(l)}{dl}a^\dagger a + \frac{d\lambda(l)}{dl}{a^\dagger}^2 + \frac{d\lambda^*(l)}{dl}a^2 + \frac{d\nu(l)}{dl}. \tag{3.16}$$

Therefore, Eqs. (3.15) and (3.16) yield the following set of flow equations:

$$\frac{d\omega(l)}{dl} = -16\omega(l)|\ \lambda(l)\ |^2, \tag{3.17}$$

$$\frac{d\lambda(l)}{dl} = -4\omega^2(l)\lambda(l), \tag{3.18}$$

$$\frac{d\nu(l)}{dl} = -8\omega(l)|\ \lambda(l)\ |^2. \tag{3.19}$$

These flow equations determine the flow of the physical parameters $\omega(l)$, $\lambda(l)$, and $\nu(l)$. Here, it is noted that there exists conserved quantities with respect to l-evolution. Combining Eqs. (3.17) and (3.18), we obtain

$$\omega^2(l) - 4|\lambda(l)|^2 = \omega^2(0) - 4|\lambda(0)|^2, \tag{3.20}$$

or from Eqs. (3.17) and (3.19),

$$\omega(l) - 2\nu(l) = \omega(0) - 2\nu(0). \tag{3.21}$$

(These conserved quantities seem to be viewed as invariants in the method of flow equations. See Appendix A.) The solutions of Eqs. (3.17)-(3.19) are calculated to be

$$\omega(l) = \frac{\omega\Lambda}{\sqrt{\omega^2 - 4|\lambda|^2 \exp(-8\Lambda^2 l)}}, \tag{3.22}$$

$$\lambda(l) = \frac{\lambda\Lambda}{\sqrt{\omega^2 \exp(8\Lambda^2 l) - 4|\lambda|^2}}, \tag{3.23}$$

$$\nu(l) = \frac{1}{2}\frac{\omega\Lambda}{\sqrt{\omega^2 - 4|\lambda|^2 \exp(-8\Lambda^2 l)}} - \frac{\omega}{2} + \nu \tag{3.24}$$

with the definition of

$$\Lambda \equiv \sqrt{\omega^2 - 4|\lambda|^2}. \tag{3.25}$$

In the limit, $l \to \infty$, the physical coefficients in Eqs. (3.9)-(3.11) are given by

$$\omega(\infty) = \Lambda, \tag{3.26}$$

$$\lambda(\infty) = 0, \tag{3.27}$$

$$\nu(\infty) = \frac{\Lambda - \omega}{2} + \nu. \tag{3.28}$$

Thus, the original Hamiltonian in Eq. (3.5), in fact, can be diagonalized.

Secondly, we analyse a spin in a constant external magnetic field **B**. The Hamiltonian reads

$$H = \mathbf{S} \cdot \mathbf{B} \tag{3.29}$$

with the spin operator $\mathbf{S} = (S_x, S_y, S_z)$ in an appropriate unit. The unitary operator to be considered is

$$U(l) = \exp\left[\sigma(l)S_+ - \sigma^* S_-\right] \tag{3.30}$$

where $S_\pm \equiv S_x \pm iS_y$ and the complex coefficient $\sigma(l)$ is parametrized as $\sigma(l) = [\theta(l)/2]\mathrm{e}^{-i\varphi(l)}$ $(0 \leq \theta(l) < \pi, 0 \leq \varphi(l) < 2\pi)$ with $\theta(0) = 0$. The basic commutation relations satisfied by the spin operators are as follows: $[S_z, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_z$. The transformed Hamiltonian is found to be

$$\begin{aligned} H(l) &= U(l)HU^\dagger(l) \\ &= \beta_z(l)S_z + \beta(l)S_+ + \beta^*(l)S_-, \end{aligned} \tag{3.31}$$

where $\beta_z(l)$ and $\beta(l)$ are defined as

$$\begin{aligned} \beta_z(l) =& \frac{1}{2}(B_x - iB_y)\mathrm{e}^{i\varphi(l)} \sin\theta(l) \\ &+ \frac{1}{2}(B_x + iB_y)\mathrm{e}^{-i\varphi(l)} \sin\theta(l) + B_z \cos\theta(l), \end{aligned} \tag{3.32}$$

$$\begin{aligned}\beta(l) =& \frac{1}{4}(B_x - iB_y)(1 + \cos\theta(l)) \\ & - \frac{1}{4}(B_x + iB_y)(1 - \cos\theta(l))\mathrm{e}^{-i2\varphi(l)} \\ & - \frac{1}{2}B_z\mathrm{e}^{-i\varphi(l)}\sin\theta(l),\end{aligned} \tag{3.33}$$

respectively. Thus, the diagonal and the off-diagonal parts of the transformed Hamiltonian in Eq. (3.31) are

$$H_{\mathrm{d}}(l) = \beta_z(l)S_z \tag{3.34}$$

and

$$H_{\text{o-d}}(l) = \beta(l)S_+ + \beta^*(l)S_-. \tag{3.35}$$

The generator $\eta(l)$ for the present system is therefore calculated to be

$$\begin{aligned}\eta(l) &= [H_{\mathrm{d}}(l), H_{\text{o-d}}(l)] \\ &= \beta_z(l)[\beta(l)S_+ - \beta^*(l)S_-].\end{aligned} \tag{3.36}$$

Accordingly, the commutation relation between $\eta(l)$ in Eq. (3.36) and $H(l)$ in Eq. (3.31) is obtained as

$$[\eta(l), H(l)] = 4\beta_z(l)|\beta(l)|^2S_z - \beta_z^2(l)\beta(l)S_+ - \beta_z^2(l)\beta^*(l)S_-. \tag{3.37}$$

Then, the derivative of $H(l)$ in Eq. (3.31) with respect to l leads to

$$\frac{dH(l)}{dl} = \frac{d\beta_z(l)}{dl}S_z + \frac{d\beta(l)}{dl}S_+ + \frac{d\beta^*(l)}{dl}S_-. \tag{3.38}$$

Thus, we get the following set of flow equations:

$$\frac{d\beta_z(l)}{dl} = 4\beta_z(l)|\beta(l)|^2, \tag{3.39}$$

$$\frac{d\beta(l)}{dl} = -\beta_z^2(l)\beta(l). \tag{3.40}$$

These Eqs. (3.39) and (3.40) lead to the following conserved quantity,

$$\beta_z^2(l) + 4|\beta(l)|^2 = \beta_z^2(0) + 4|\beta(0)|^2. \tag{3.41}$$

(This is viewed as an invariant, again. See Appendix A.) The solutions of the flow equations in Eqs. (3.39) and (3.40) are found to be

$$\beta_z(l) = |\boldsymbol{B}|\sqrt{\frac{B_z^2}{(|\boldsymbol{B}| - B_z^2)\exp(-2|\boldsymbol{B}|^2 l) + B_z^2}} \tag{3.42}$$

and

$$|\beta(l)| = \frac{|\boldsymbol{B}|}{2}\sqrt{\frac{|\boldsymbol{B}|^2 - B_z^2}{|\boldsymbol{B}|^2 - B_z^2(1 - \exp(2|\boldsymbol{B}|^2 l))}}, \tag{3.43}$$

respectively. Therefore, in the limit, $l \to \infty$, these solutions become

$$\beta_z(l) = |\boldsymbol{B}|, \tag{3.44}$$

$$|\beta(\infty)| = 0. \tag{3.45}$$

Thus, the original Hamiltonian in Eq. (3.29) can be diagonalized again.

Bibliography

[1] F. Wegner, Ann. Phys. (Leipzig) **3**, 77 (1994).

[2] F.J. Wegner, Phys. Rep. **348**, 77 (2001).

[3] F. Wegner, J. Phys. A: Math. Gen. **39**, 8221 (2006).

[4] S.K. Kehrein and A. Mielke, Ann. Phys. (NY) **252**,1 (1996).

[5] A.B. Bylev and H.J. Pirner, Phys. Lett. B **428**, 329 (1998).

[6] C. Raas, U. Low, G.S. Uhrig, and R.W. Kuhne, Phys. Rev. B **65**, 144438 (2002).

[7] V. Hankevych and F. Wegner, Eur. Phys. J. B **31**, 333 (2003).

[8] T. Stauber, Phys. Rev. B **67**, 205107 (2003).

[9] T. Domanski and J. Ranninger, Phys. Rev. B **70**, 184513 (2004).

[10] S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004).

[11] S. Dusuel, J. Vidal, J.M. Arias, J. Dukelsky,and J.E. Garcia-Ramos, Phys. Rev. C **72**, 064332 (2005).

[12] S.D. Glazek, Nucl. Phys. B (Proc. Suppl.) **161**, 59 (2006).

[13] G. Aprea, C. Di Castro, M. Grilli,and J. Lorenzana, Nucl. Phys. B **744**, 277 (2006).

[14] Y. Itto and S. Abe, Physica A **387**, 1063 (2008).

Chapter 4

Generation of stationary photon-atom entanglement

As mentioned in Chapter 1, a crucial step for quantum-information processing in photon-atom system is to generate stationary entangled states in the system. Mathematically, this is nothing more than diagonalizing the system Hamiltonian. Experimentally, however, given an initial state, it is generally hard to realize the quantum operation that can immediately transform such a state of the coupled system to the exact stationary state. Therefore, in reality, it is necessary to inquire into a series of "trial" operations in order to approach the stationary state. Then, the question is what constitutes a desirable strategy toward this end.

In this chapter, we answer the above question by means of the method of flow equations [1-3] to establishing stationary entanglement between light and matter

[4]. The unitary transformation operator appearing in this approach is generically nonlocal, changing the degree of entanglement. Our idea is to use this method as an optimal technique for quantum-state engineering. To illustrate it, we employ the Jaynes-Cummings model of a two-level atom interacting with a monochromatic radiation field [5,6] and perform its complete flow equation analysis. The Jaynes-Cummings model we are going to treat here is nothing but a prototype example for demonstrating the approach in a simple manner. We also calculate the positive operator-valued measures (POVMs) for the photons and atom, evaluate the degree of stationarity, and analyze flow of the entanglement entropies. In addition, we present a procedure for implementing the unitary operation appearing in the method.

4.1 The flow equation analysis of the Jaynes-Cummings model

Let us apply the method of flow equations to the Jaynes-Cummings model. The Hamiltonian reads

$$H = \frac{\omega_0}{2}\sigma_3 + \omega a^\dagger a + \lambda(\sigma_+ a + \sigma_- a^\dagger) \tag{4.1}$$

The symbols appearing in Eq. (4.1) are defined in the same way as in Chapter 2. $\hbar = 1$ is set equal to unity for the sake of simplicity. In the present work, we limit

our analysis to the off-resonant case. Therefore, the detuning factor, $\Delta \equiv \omega_0 - \omega$, is nonvanishing, and here it is assumed to be positive:

$$\Delta = \omega_0 - \omega > 0. \tag{4.2}$$

Here, it should be recalled that although this is a highly simplified model of the coupled system of light and matter, it can experimentally be realized to generate single-atom maser or micromaser in a cavity [7]. Since the model is exactly solvable, it allows to analytically discuss a variety of phenomena such as the Rabi oscillation and quantum collapses/revivals. For the Jaynes-Cummings Hamiltonian in Eq. (4.1), we have constructed the following operator of the continuous unitary transformation:

$$\begin{aligned} U(l) = \sum_{n=0}^{\infty} \big[& \alpha_n(l)\sigma_+\sigma_- \mid n\rangle\langle n \mid + \beta_{n-1}(l)\sigma_-\sigma_+ \mid n\rangle\langle n \mid \\ & + \gamma_n(l)\sigma_+ \mid n\rangle\langle n+1 \mid + \delta_n(l)\sigma_- \mid n+1\rangle\langle n \mid \big]. \end{aligned} \tag{4.3}$$

Here, $\beta_{-1}(l) = \mathrm{e}^{i\chi(l)}$ [which turns out to be unity, later: see Eq. (4.28)] and $\mid n\rangle = (n!)^{-1/2}(a^\dagger)^n \mid 0\rangle$ denotes the n-photon state with the normalized vacuum state $\mid 0\rangle$, i.e., $a \mid 0\rangle = 0$. The unitarity condition, $U(l)U^\dagger(l) = U^\dagger(l)U(l)$, leads to the following set of equations ($n = 0, 1, 2, \ldots$):

$$|\alpha_n(l)|^2 + |\gamma_n(l)|^2 = |\beta_n(l)|^2 + |\delta_n(l)|^2 = 1, \tag{4.4}$$

$$|\alpha_n(l)|^2 + |\delta_n(l)|^2 = |\beta_n(l)|^2 + |\gamma_n(l)|^2 = 1, \tag{4.5}$$

$$\alpha_n(l)\delta_n^*(l) + \beta_n^*(l)\gamma_n(l) = 0, \quad \alpha_n(l)\gamma_n^*(l) + \beta_n^*(l)\delta_n(l) = 0, \tag{4.6}$$

from which, it follows that

$$|\alpha_n(l)|^2 = |\beta_n(l)|^2, \quad |\gamma_n(l)|^2 = |\delta_n(l)|^2 \tag{4.7}$$

The "initial" conditions are

$$\alpha_n(0) = \beta_n(0) = 1, \quad \gamma_n(0) = \delta_n(0) = 0, \tag{4.8}$$

and $\beta_{-1}(0) = 1$. The transformed Hamiltonian is given by

$$\begin{aligned} H(l) = \sum_{n=0}^{\infty} [&A_n(l)\sigma_+\sigma_- \mid n\rangle\langle n \mid + B_{n-1}(l)\sigma_-\sigma_+ \mid n\rangle\langle n \mid \\ &C_n(l)\sigma_+ \mid n\rangle\langle n+1 \mid + C_n^*(l)\sigma_- \mid n+1\rangle\langle n \mid] \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} A_n(l) =& \frac{\omega_0}{2}[|\alpha_n(l)|^2 - |\gamma_n(l)|^2] + \omega[n + |\gamma_n(l)|^2] \\ &+ \lambda\sqrt{n+1}[\alpha_n(l)\gamma_n^*(l) + \alpha_n^*(l)\gamma_n(l)], \end{aligned} \tag{4.10}$$

$$\begin{aligned} B_n(l) =& \frac{\omega_0}{2}[|\delta_n(l)|^2 - |\beta_n(l)|^2] + \omega[n + |\beta_n(l)|^2] \\ &+ \lambda\sqrt{n+1}[\beta_n(l)\delta_n^*(l) + \beta_n^*(l)\delta_n(l)], \end{aligned} \tag{4.11}$$

$$\begin{aligned} C_n(l) =& \frac{\omega_0}{2}[\alpha_n(l)\delta_n^*(l) - \beta_n^*(l)\gamma_n(l)] + \omega\beta_n^*(l)\gamma_n(l) \\ &+ \lambda\sqrt{n+1}[\alpha_n(l)\beta_n^*(l) + \gamma_n(l)\delta_n^*(l)], \end{aligned} \tag{4.12}$$

for $n = 0, 1, 2, \ldots$, and $B_{-1}(l) \equiv -\omega_0/2$, provided that Eqs. (4.4)-(4.6) have been used. Note that both $A_n(l)$ and $B_n(l)$ are real. The "initial" conditions are

$$A_n(0) = \frac{\omega_0}{2} + \omega n, \tag{4.13}$$

$$B_n(0) = -\frac{\omega_0}{2} + \omega(n+1), \tag{4.14}$$

$$C_n(0) = \lambda\sqrt{n+1}. \tag{4.15}$$

The diagonal part in Eq. (4.9) is given by

$$H_{\rm d}(l) = \sum_{n=0}^{\infty}[A_n(l)\sigma_+\sigma_- \mid n\rangle\langle n \mid + B_{n-1}(l)\sigma_-\sigma_+ \mid n\rangle\langle n \mid]. \tag{4.16}$$

Accordingly, Wegner's choice for the generator is calculated to be

$$\begin{aligned}\eta(l) = \sum_{n=0}^{\infty}&[A_n(l) - B_n(l)] \\ &[C_n(l)\sigma_+ \mid n\rangle\langle n+1 \mid -C_n^*(l)\sigma_- \mid n+1\rangle\langle n \mid]\end{aligned} \tag{4.17}$$

leading to

$$\begin{aligned}[\eta(l), H(l)] = \sum_{n=0}^{\infty}&[A_n(l) - B_n(l)] \\ &\{2|C_n(l)|^2(\sigma_+\sigma_- \mid n\rangle\langle n \mid -\sigma_-\sigma_+ \mid n+1\rangle\langle n+1 \mid)\} \\ &- [A_n(l) - B_n(l)][\sigma_+C_n(l) \mid n\rangle\langle n+1 \mid +\sigma_-C_n^*(l) \mid n+1\rangle\langle n \mid]\}.\end{aligned} \tag{4.18}$$

Therefore, $dH(l)/dl = [\eta(l), H(l)]$ yields the following set of flow equations ($n = 0, 1, 2, \ldots,$):

$$\frac{dA_n(l)}{dl} = 2\big[A_n(l) - B_n(l)\big]|C_n(l)|^2, \tag{4.19}$$

$$\frac{dB_n(l)}{dl} = -2\big[A_n(l) - B_n(l)\big]|C_n(l)|^2, \tag{4.20}$$

$$\frac{dC_n(l)}{dl} = -\big[A_n(l) - B_n(l)\big]^2 C_n(l). \tag{4.21}$$

From Eq. (4.21), it is obvious that the off-diagonal matrix elements in fact decay as l grows (note that because of the off-resonant condition in Eq. (4.2)). From Eqs. (4.15) and (4.21) as well as the realities of $A_n(l)$ and $B_n(l)$, $C_n(l)$ also turns out to be real. Here, it is noted that there exist conserved quantities with respect to l-evolution. Eqs. (4.19) and (4.20) lead to

$$A_n(l) + B_n(l) = A_n(0) + B_n(0), \tag{4.22}$$

and Eqs. (4.19)-(4.21) give

$$(A_n(l) - B_n(l))^2 + 4C_n^2(l) = (A_n(0) - B_n(0))^2 + 4C_n^2(0), \tag{4.23}$$

for $n = 0, 1, 2, \ldots$. (In the present solvable system, invariants quantities exist, as expected. See Appendix A.) The exact solutions to these equations under the "initial" conditions in Eqs. (4.13)-(4.15) are

$$A_n(l) = \omega\Big(n + \frac{1}{2}\Big) + \frac{\Delta}{2} L_n(l), \tag{4.24}$$

$$B_n(l) = \omega\Big(n + \frac{1}{2}\Big) - \frac{\Delta}{2} L_n(l), \tag{4.25}$$

$$C_n(l) = \frac{\Omega_n}{2}\sqrt{1 - \frac{\Delta^2}{\Omega_n^2} L_n^2(l)}, \tag{4.26}$$

for $n = 0, 1, 2, \ldots$, where $\Omega_n = \sqrt{\Delta^2 + 4\lambda^2(n+1)}$ is the Rabi frequency, and $L_n(l)$ is given by

$$L_n(l) = \frac{1}{\sqrt{\frac{\Delta^2}{\Omega_n^2} + \Big(1 - \frac{\Delta^2}{\Omega_n^2}\Big)\exp(-2\Omega_n^2 l)}}. \tag{4.27}$$

We note that without losing generality $A_n(l)$ is taken to be larger than $B_n(l)$. (If Δ were taken negative, then the case $A_n(l) < B_n(l)$ could be realized. This point does not affect the subsequent discussion.)

Furthermore, the equalities, $\eta(l) = [dU(l)/dl]U^\dagger(l)$ and $dU(l)/dl = \eta(l)U(l)$, together with Eqs. (4.4)-(4.8) and (4.17), lead to the fact that $\alpha_n(l)$, $\beta_n(l)$,$\gamma_n(l)$, and $\delta_n(l)$ are all real and satisfy the relations

$$\alpha_n(l) = \beta_n(l), \quad \gamma_n(l) = -\delta_n(l), \quad \beta_{-1}(l) = 1. \tag{4.28}$$

From these relations as well as Eqs. (4.4)-(4.7) and (4.28), the coefficients appearing in the unitary operator in Eq. (4.3) are found to be given as follows ($n = 0, 1, 2, \ldots$):

$$\alpha_n(l) = \beta_n(l) = \sqrt{\frac{1 + \widetilde{L}_n(l)}{2}}, \quad \gamma_n(l) = -\delta_n(l) = \sqrt{\frac{1 - \widetilde{L}_n(l)}{2}}, \tag{4.29}$$

where

$$\widetilde{L}_n(l) = \frac{\Delta^2}{\Omega_n^2} L_n(l) + \sqrt{\left(1 - \frac{\Delta^2}{\Omega_n^2}\right)\left(1 - \frac{\Delta^2}{\Omega_n^2} L_n^2(l)\right)}. \tag{4.30}$$

Recalling the off-resonant condition in Eq. (4.2), Eqs. (4.24)-(4.26) respectively converge, in the limit $l \to \infty$, to

$$A_n(\infty) = \omega\left(n + \frac{1}{2}\right) + \frac{\Omega_n}{2} \equiv E_+(n), \tag{4.31}$$

$$B_n(\infty) = \omega\left(n + \frac{1}{2}\right) - \frac{\Omega_n}{2} \equiv E_-(n), \tag{4.32}$$

$$C_n(\infty) = 0, \tag{4.33}$$

with the energy eigenvalues, $E_{\pm}(n)$, corresponding to $\mid E_{+}\rangle = U^{\dagger}(\infty) \mid e\rangle \mid n\rangle$ and $\mid E_{-}\rangle = U^{\dagger}(\infty) \mid g\rangle \mid n+1\rangle$. Eq. (4.33) explicitly shows that the Hamiltonian is in fact exactly diagonalizable.

Closing this section, we point out that although the present flow equation approach assumes the off-resonant case as in Eq. (4.2), the limiting quantities, $A_n(\infty)$ and $B_n(\infty)$ (which are the energy eigenvalues in the states, $\mid e\rangle \mid n\rangle$ and $\mid g\rangle \mid n+1\rangle$, respectively) converge in the resonant limit, $\Delta \to 0$, to

$$A_n(\infty) \to \omega\left(n+\frac{1}{2}\right)+\lambda\sqrt{n+1}, \tag{4.34}$$

$$B_n(\infty) \to \omega\left(n+\frac{1}{2}\right)-\lambda\sqrt{n+1}, \tag{4.35}$$

which are the correct results obtained by other methods [5, 6].

4.2 Positive operator-valued measures and subsystems

In this section, we discuss the effects of the unitary operation by $U(l)$ on the subsystems, i.e., the photons and atom. We shall explicitly calculate the positive operator-valued measures (POVMs).

Consider $\mid \Psi\rangle = \mid e\rangle \mid n\rangle$ with $(n \neq 0)$, which is a natural "initial"state to be prepared since in a realistic experimental situation using a cavity the photons and atom may initially be uncorrelated. The corresponding density matrix of the total

system reads

$$\rho_{\text{total}} = \mid e\rangle\langle e \mid \otimes \mid n\rangle\langle n \mid \tag{4.36}$$

Recalling the operation of $U(l)$ on the Hamiltonian, $H(l) = U(l)HU^{\dagger}(l)$, we see the corresponding operation on the state to be

$$\mid \Psi(l)\rangle = U^{\dagger}(l) \mid \Psi\rangle \tag{4.37}$$

Therefore, the above density matrix is transformed as

$$\rho_{\text{total}}(l) = U^{\dagger}(l)\rho_{\text{total}}U(l). \tag{4.38}$$

The reduced density matrices of the subsystems are given by $\rho_{\text{photon}}(l) = \text{Tr}_{\text{atom}}\rho_{\text{total}}(l)$ and $\rho_{\text{atom}}(l) = \text{Tr}_{\text{photon}}\rho_{\text{total}}(l)$. The form in Eq. (4.36) allows us to obtain the following Kraus representations [8]:

$$\rho_{\text{photon}}(l) = \sum_{i=g,e} A_i(l) \mid n\rangle\langle n \mid A_i^{\dagger}(l), \tag{4.39}$$

$$\rho_{\text{atom}}(l) = \sum_{m=0}^{\infty} B_m(l) \mid e\rangle\langle e \mid B_m^{\dagger}(l). \tag{4.40}$$

Here, $\{A_i(l)\}_{i=g,e}$ and $\{B_m(l)\}_{m=0,1,2,\ldots}$ form POVMs

$$\sum_{i=g,e} A_i^{\dagger}(l)A_i(l) = I_{\text{photon}}, \tag{4.41}$$

$$\sum_{m=0}^{\infty} B_m^{\dagger}(l)B_m(l) = I_{\text{atom}} \tag{4.42}$$

with I's being the identity operators. Using Eqs. (4.3) and (4.36), they are explicitly calculated as follows:

$$\begin{aligned} A_i(l) &= \langle i \mid U^\dagger(l) \mid e \rangle \\ &= \sum_{n=0}^{\infty} [\alpha_n(l)\delta_{i,e} \mid n\rangle\langle n \mid + \gamma_n(l)\delta_{i,g} \mid n+1\rangle\langle n \mid]. \end{aligned} \tag{4.43}$$

$$\begin{aligned} B_m(l) &= \langle m \mid U^\dagger(l) \mid n \rangle \\ &= \delta_{n-1}\sigma_+\delta_{m,n-1} + [\alpha_n(l)\sigma_+\sigma_- + \beta_{n-1}(l)\sigma_-\sigma_+]\delta_{m,n} + \gamma_n(l)\sigma_-\delta_{m,n+1}. \end{aligned} \tag{4.44}$$

These operators do not satisfy the unital conditions: $\sum_{i=g,e} A_i(l)A_i^\dagger(l) \neq I_{\text{photon}}$, $\sum_{m=0}^{\infty} B_m(l)B_m^\dagger(l) \neq I_{\text{atom}}$. Accordingly, entropy, which will be discussed in the next section, does not increase in general under the operations of these POVMs [9-11]: that is, whether entropy increases by the operations depends on choice of a state to be transformed.

4.3 Entanglement entropies

As mentioned earlier, our idea is to use the method of flow equations as an optimal technique for quantum-state engineering. This means that it is possible to measure how quantum-state engineering makes a given "initial" state close to the stationary state.

As well as the previous section, taking into account a realistic experimental

situation using a cavity, a natural "initial" state is a product state. Since $U(l)$ in Eq. (4.3) is a nonlocal unitary operator, it necessarily creates entanglement between the photons and atom. Then, it is of interest to quantify the degree of entanglement in the engineered state as well as its closeness to the exact stationary state. Here, we discuss this issue by analyzing the entanglement entropies.

The entanglement entropies are defined by the von Neumann entropies of the subsystems:

$$S_{\text{photon}}(l) = -\text{Tr}_{\text{photon}}(l)[\rho_{\text{photon}}\ln\rho_{\text{photon}}(l)], \tag{4.45}$$

$$S_{\text{atom}}(l) = -\text{Tr}_{\text{atom}}(l)[\rho_{\text{atom}}\ln\rho_{\text{atom}}(l)], \tag{4.46}$$

where $\rho_{\text{photon}}(l)$ and $\rho_{\text{atom}}(l)$ are the reduced density matrices defined in the previous section. These are nonzero because $|\ \Psi(l)\rangle$ in Eq. (4.37) is an entangled state for $l > 0$. From the Araki-Lieb inequalities [12] in the present case,

$$|S_{\text{photon}}(l) - S_{\text{atom}}(l)| \leq S_{\text{total}}(l) \leq S_{\text{photon}}(l) + S_{\text{atom}}(l), \tag{4.47}$$

it follows that the entanglement entropies of the photons and atom are identical

$$S_{\text{photon}}(l) = S_{\text{atom}}(l) \tag{4.48}$$

at each value of l, since $S_{\text{total}}(l)$ vanishes because of purity of the total state in Eq. (4.38). Therefore, henceforth we consider only $S_{\text{atom}}(l)$.

An explicit calculation yields

$$S_{\rm atom}(l) = -s_+(l)\ln s_+(l) - s_-(l)\ln s_-(l), \tag{4.49}$$

$$s_\pm(l) = \frac{1 \pm \widetilde{L}_n(l)}{2} \tag{4.50}$$

$$\rightarrow \quad \frac{1}{2}\left[1 \pm \frac{\Delta}{\Omega_n}\right] \quad (l \rightarrow \infty), \tag{4.51}$$

where $\widetilde{L}_n(l)$ is given in Eq. (4.30).

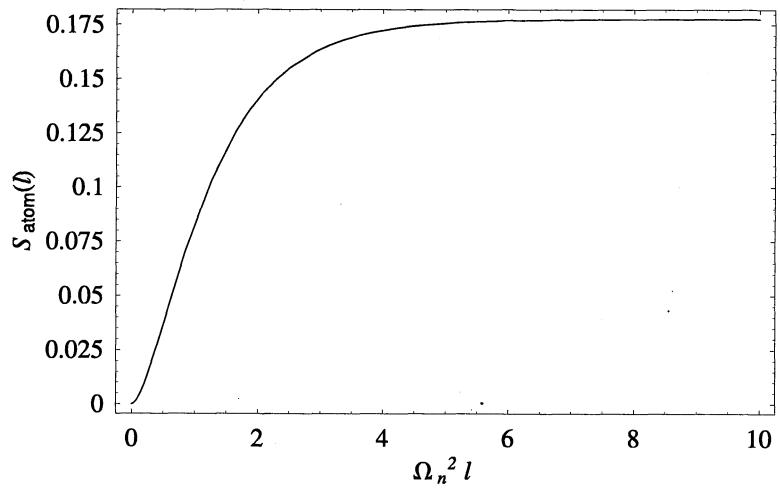

Fig. 1: Flow of the dimensionless entanglement entropy of the atom $S_{\rm atom}(l)$ in terms of the dimensionless variable $\Omega_n^2 l$.The values of the physical parameters used are $n = 1$, $\Delta = 1$ [MHz], $\lambda = 50\pi$ [kHz].

In Fig. 1, we show the flow of $S_{\rm atom}(l)$. There, the values of the physical parameters used are taken from real experiments [13,14]. One clearly appreciates a monotonic behavior, exhibiting how the degree of entanglement increases as the "initial" product state approaches the exact stationary state. It is of interest to see that the convergence to the maximum value, $S_{\rm atom}(\infty)$, is fast. This might

be understood as the fact that the flow equation offers the optimal strategy for realizing the stationary state. The flow of $S_{\rm atom}(l)$ can be also used for quantifying the deviation of an engineered state from the stationary state.

Since $\mid \Psi(l)\rangle$ with a finite value of l is not the stationary state, its time evolution makes the entanglement entropies oscillatory. To see this, we have calculated the time evolution of $S_{\rm atom}(l,t)$ for $\mid \Psi(l,t) \equiv {\rm e}^{-iHt} \mid \Psi(l)\rangle$. The result is

$$S_{\rm atom}(l,t) = -s_+(l,t){\rm ln}s_+(l,t) - s_-(l,t){\rm ln}s_-(l,t), \tag{4.52}$$

$$\begin{aligned} s_\pm(l,t) = \frac{1}{2}\Bigg\{ & 1 \pm \widetilde{L}_n(l)\cos\Omega_n t \\ & \pm (1-\cos\Omega_n t)\frac{\Delta}{\Omega_n}\left[\frac{\Delta}{\Omega_n} + \sqrt{(1-\widetilde{L}_n^2(l))\left(1-\frac{\Delta^2}{\Omega_n^2}\right)}\right]\Bigg\}. \end{aligned} \tag{4.53}$$

In Fig. 2, we present the plot of $S_{\rm atom}(l,t)$. There, one sees how the oscillation decays as l increases.

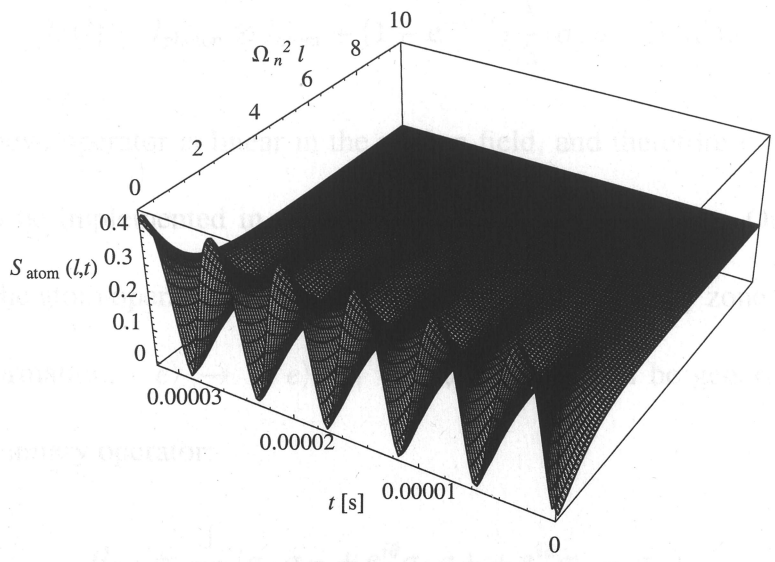

Fig. 2: Time dependence of the dimensionless entanglement entropy $S_{\text{atom}}(l,t)$ for each value of the dimensionless variable $\Omega_n^2 l$.The values of the physical parameters used are the same as those in Fig. 1.

4.4 Comment on implementation of the unitary operation

Finally, we discuss how to approximately implement the unitary operation appearing in the method of flow equation for the Jaynes-Cummings model.

In a real experimental condition [13], $\lambda/\Delta = O(10^{-3})$. Therefore, expanding the coefficients, α 's, β 's, γ's, and δ 's in Eq. (4.3) with respect to λ/Δ, we obtain the following approximate expression:

$$U(l) \sim I_{\text{photon}} \otimes I_{\text{atom}} + (1 - \mathrm{e}^{-\Delta^2 l})\frac{\lambda}{\Delta}(\sigma_+ a - \sigma_- a^\dagger). \tag{4.54}$$

The above operator is linear in the photon field, and therefore it may not be difficult to be implemented in terms of the quadrature operators. On the other hand, for the atom operators, we note that the use of the Ramsey zone [15] yields the transformation, $\mid e\rangle \rightarrow (\mid e\rangle + \mid g\rangle)/\sqrt{2}$, which can be generated by the following unitary operator:

$$R_{\theta,\phi} = \frac{1}{\sqrt{2}}(\sigma_+\sigma_- + \mathrm{e}^{i\theta}\sigma_-\sigma_+ + \mathrm{e}^{i\phi}\sigma_+ + \sigma_-), \tag{4.55}$$

where the phases satisfy the condition, $\theta - \psi = \pi \mathrm{mod}(2\pi)$. Therefore, we find the atom operators to be given as follows:

$$\sigma_+ = \frac{1}{2}[\sqrt{2}(R_{\pi,0} - R_{0,\pi}) - R_{\pi,0}R_{0,\pi} + I_{\mathrm{atom}}], \tag{4.56}$$

$$\sigma_- = \frac{1}{2}[\sqrt{2}(R_{\pi,0} + R_{0,\pi}) - R_{\pi,0}R_{0,\pi} - I_{\mathrm{atom}}]. \tag{4.57}$$

Therefore, the unitary transformation of the flow equation can approximately be implemented by making use of the quadrature operators of the maser photon field and the Ramsey zone.

Bibliography

[1] F. Wegner, Ann. Phys. (Leipzig) **3**, 77 (1994).

[2] F.J. Wegner, Phys. Rep. **348** 77 (2001).

[3] F. Wegner, J. Phys. A: Math. Gen. **39**, 8221 (2006).

[4] Y. Itto and S. Abe, Physica A **387**, 1063 (2008).

[5] E.T. Jaynes and F.W. Cummings, Proc. IEEE **51**, 89 (1963).

[6] C.C. Gerry and P.L. Knight, Introductory Quantum Optics, Cambridge University Press, Cambridge, 2005.

[7] H. Walther, Physica Scripta T **86**, 27 (2000).

[8] K. Kraus, States, Effects and Operations: Fundamental Notions of Quantum Theory, Springer-Verlag, New York, 1983.

[9] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.

[10] A. Chefles, Phys. Rev. A **65**, 052314 (2002).

[11] S. Abe and A.K. Rajagopal, Phys. Rev. Lett. **91**, 120601 (2003).

[12] H. Araki, E. H. Lieb, Commun. Math. Phys. **18**, 160 (1970).

[13] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. **76**, 1800 (1996).

[14] H. Walther, Fortschr. Phys. **54**, 617 (2006).

[15] P.B. Monteiro, A. Aragao, A.T. Avelar,and B. Baseia, Physica A **354**, 235 (2005); and the references therein.

Chapter 5

Optimal strategy for quantum-state engineering

Wegner's method [1-3] has been applied to many problems in quantum physics [4-14] as mentioned in Chapter 3. Its inherent properties, however, do not seem to be explored well. In this chapter, an interesting geometric property hidden behind Wegner's flow equations is revealed. It is our opinion that this method is more than just a mathematical tool for diagonalizing a Hamiltonian. Firstly, we develop a new theory for the method of flow equations, which generalize Wegner's method itself [15]. The original method of Wegner is a special case within this new framework. Next, we consider the flow of a given quantum state through a submanifold of the projective Hilbert space composed of rays explained below. Then, we discover a condition, under which the flow becomes geodesic in the submanifold,

independently of specific "initial" conditions at $l = 0$ [15]. This remarkable property is illustrated by analyzing some physical examples. In this context the unitary transformation in Wegner's method is now shown to offer the optimal strategy for quantum-state engineering in a wide class of systems (satisfying the condition discovered here).

5.1 New theory for the method of flow equations

Let us recall the original method of flow equations. Wegner's method employs a continuous unitary transformation represented by the operator, $U(l)$, where $l \in [0, \infty)$ is referred to as the flow parameter. $U(l)$ transforms the original Hamiltonian $H = H(0)$ to $H(l) = U(l)HU^{\dagger}(l)$, which satisfies $dH(l)/dl = [\eta(l), H(l)]$, where $\eta(l)$ is the anti-Hermitian generator given by $\eta(l) = [dU(l)/dl]U^{\dagger}(l)$. Wegner's choice for $\eta(l)$ to diagonalize (or, block-diagonalize) the Hamiltonian reads

$$\eta(l) = \eta^{W}(l) \equiv [H_{\rm d}(l), H_{\text{o-d}}(l)], \tag{5.1}$$

where $H_{\rm d}(l)$ and $H_{\text{o-d}}(l)$ stand for the diagonal and off-diagonal parts of $H(l)$, respectively.

Now, we consider a quantum system in a d-dimensional Hilbert space, where d is either finite or infinite. Its unitary-transformed Hamiltonian, $H(l)$, is decom-

posed into two parts: $H(l) = H_{\rm d}(l) + H_{\rm o\text{-}d}(l)$, where $H_{\rm o\text{-}d}(l)$ is expressed as

$$H_{\rm o\text{-}d}(l) = \sum_{a=1}^{A} H_{\rm o\text{-}d}^{(a)}(l) \quad (1 \leq A \leq d-1). \tag{5.2}$$

Using the complete set of the normalized eigenstates of $H_{\rm d}(l)$, $\mid u_n\rangle_{n=1,2,\ldots,d}$, we write them in the following forms:

$$H_{\rm d}(l) = \sum_{n=1}^{d} \epsilon_n(l) \mid u_n\rangle\langle u_n \mid, \tag{5.3}$$

$$H_{\rm o\text{-}d}^{(a)}(l) = \sum_{n=1}^{d} [C_{n+i_a}^{(a)}(l) \mid u_{n+i_a}\rangle\langle u_n \mid + C_{n+i_a}^{(a)*}(l) \mid u_n\rangle\langle u_{n+i_a} \mid], \tag{5.4}$$

where $\epsilon(l)$ is the nth eigenvalue of $H_{\rm d}(l)$ and $C_{n+i_a}^{(a)}(l)$'s are the complex expansion coefficients. The nonzero index i_a describes the off-diagonality and is ordered without loss of generality as follows: $0 < i_1 < 1_2 < \cdots < i_A$. It is understood that $C_{n+i_a}^{(a)}(l) = 0$ ($C_{n-i_a}^{(a)}(l) = 0$), if $n + i_a > d$ ($n - i_a \leq 1$).

Here, we generalize Wegner's method as follows. Instead of taking the whole of $H_{\rm o\text{-}d}(l)$, we employ only a single term, say $H_{\rm o\text{-}d}^{(a)}(l)$:

$$\eta^{(a)}(l) = [H_{\rm d}(l), H_{\rm o\text{-}d}^{(a)}(l)]. \tag{5.5}$$

Clearly, Wegner's choice is

$$\eta^{W}(l) = \sum_{a=1}^{A} \eta^{(a)}(l). \tag{5.6}$$

It can be found after a straightforward calculation using Eqs. (5.3)-(5.5) that the corresponding generalized flow equation

$$\frac{dH(l)}{dl} = [\eta^{(a)}(l), H(l)] \tag{5.7}$$

gives rise to

$$\begin{aligned}\frac{d}{dl}\sum_{m,n=1,m\neq n}^{d}|\langle u_m \mid H(l) \mid u_n\rangle|^2 &= -\frac{d}{dl}\sum_{n=1}^{d}\epsilon_n^2(l) \\ &= -4\sum_{n=1}^{d}[\epsilon_{n+i_a}(l)-\epsilon_n(l)]^2|C^{(a)}_{n+i_a}(l)|^2,\end{aligned} \tag{5.8}$$

implying that the off-diagonal elements of $H(l)$ tend to decay as l increases. The analysis based on $\eta^{(a)}(l)$ can yield the results obtained using $\eta^W(l)$ by taking the sum over the index a. The reverse is, however, not possible. In this sense, the present method is regarded as a generalization of Wegner's.

5.2 The flow of a quantum state on a submanifold of the projective Hilbert space

Next, let us translate the flow of the Hamiltonian into the flow of a state. Given a normalized state $\mid \Psi\rangle$, the stationary Schrödinger equation reads $H \mid \Psi(\infty)\rangle = E \mid \Psi(\infty)\rangle$, where $\mid \Psi(l)\rangle = U^\dagger(l) \mid \Psi\rangle$. As well known, two states different from each other only by total phases are equivalent in quantum mechanics, and therefore a physical state is represented by a ray. Accordingly, the quantum-state space is the projective Hilbert space that is generically a curved space. Eq. (5.7) determines the flows of the physical coefficients contained in the Hamiltonian, which depend on the parameters appearing in the unitary operator. Let us explicitly write as follows: $U = U(\alpha)$, $\mid \Psi(\alpha) = U^\dagger(\alpha) \mid \Psi\rangle$, where $\alpha \equiv (\alpha^1, \alpha^2, \ldots, \alpha^k)$. The

set of parameters, α, defines a local coordinate on the submanifold of the projective Hilbert space. This submanifold is referred to as the quantum evolution submanifold [16], on which a quantum state flows. Then, the Fubini-Study metric [16-19] induced on this submanifold is, up to the second-order infinitesimals, given by

$$ds^2 = 1 - |\langle \Psi(\alpha) \mid \Psi(\alpha + d\alpha) \rangle|^2 \equiv g_{ij}(\alpha) d\alpha^i d\alpha^j, \tag{5.9}$$

where the metric tensor is expressed in terms of the anti-Hermitian operator $G_i(\alpha) = [\partial_i U(\alpha)]U^\dagger(\alpha)$ ($\partial_i = \partial/\partial\alpha^i; i = 1, 2, \ldots, k$) as follows:

$$\begin{aligned} g_{ij}(\alpha) = -\frac{1}{2}\langle \Psi \mid G_i(\alpha)G_j(\alpha) + G_j(\alpha)G_i(\alpha) \mid \Psi \rangle \\ + \langle \Psi \mid G_i(\alpha) \mid \Psi \rangle \langle \Psi \mid G_j(\alpha) \mid \Psi \rangle. \end{aligned} \tag{5.10}$$

Here and hereafter, Einstein's convention is understood for the repeated indices. The equation for a geodesic curve parametrized by the arc length, $\alpha(s)$, is given by $d^2\alpha^h/ds^2 + \Gamma^h_{ij}(d\alpha^i/ds)(d\alpha^j/ds) = 0$, where Γ^h_{ij} is Christoffel's symbol defined by $\Gamma_{hij} = g_{hl}\Gamma^l_{ij} = (1/2)(\partial_i g_{hj} + \partial_j g_{ih} - \partial_h g_{ij})$.

Let us parametrize the curve by the flow parameter, instead of the arc length, i.e., $\alpha(l)$, and consider the functional

$$S[\alpha] = \int_{l_1}^{l_2} dl L(\alpha, \dot{\alpha}), \tag{5.11}$$

where

$$L(\alpha, \dot{\alpha}) = \frac{ds}{dl} = \sqrt{g_{ij}(\alpha)\dot{\alpha}^i\dot{\alpha}^j} \tag{5.12}$$

and $\dot{\alpha}^i \equiv d\alpha^i(l)/dl$. The quantity in Eq. (5.11) is the arc length in the interval $[l_1, l_2]$. The variation of S with respect to $\alpha(l)$ is calculated to be

$$\frac{\delta S[\alpha]}{\delta \alpha^i} = -\frac{1}{4L^3} X_i, \tag{5.13}$$

where

$$\begin{aligned} X_i = & \left(2\langle \Psi \mid \eta \mid \Psi \rangle \langle \Psi \mid \frac{\partial \eta}{\partial \dot{\alpha}^i} \mid \Psi \rangle - \langle \Psi \mid \frac{\partial \eta^2}{\partial \dot{\alpha}^i} \mid \Psi \rangle \right) \\ & \frac{d}{dl} \left(\langle \Psi \mid \eta^2 \mid \Psi \rangle - \langle \Psi \mid \eta \mid \Psi \rangle^2 \right) \\ & + 2 \Bigg(\langle \Psi \mid \left\{ \frac{d\eta}{dl}, \frac{\partial \eta}{\partial \alpha^i} \right\} + \left[\eta^2, \frac{\partial \eta}{\partial \dot{\alpha}^i} \right] \mid \Psi \rangle - 2 \langle \Psi \mid \frac{d\eta}{dl} \mid \Psi \rangle \langle \Psi \mid \frac{\partial \eta}{\partial \dot{\alpha}^i} \mid \Psi \rangle \\ & - 2 \langle \Psi \mid \eta \mid \Psi \rangle \langle \Psi \mid \left[\eta, \frac{\partial \eta}{\partial \dot{\alpha}^i} \right] \mid \Psi \rangle \Bigg) \left(\langle \Psi \mid \eta^2 \mid \Psi \rangle - \langle \Psi \mid \eta \mid \Psi \rangle^2 \right) \end{aligned} \tag{5.14}$$

with

$$\eta = \eta(\alpha(l)) = \frac{dU(\alpha(l))}{dl} U^\dagger(\alpha(l)) = G_i(\alpha(l)) \dot{\alpha}^i, \tag{5.15}$$

provided that the following relation has been used:

$$\frac{d}{dl} \left(\frac{\partial \eta}{\partial \dot{\alpha}^i} \right) = \frac{\partial \eta}{\partial \alpha^i} + \left[\eta, \frac{\partial \eta}{\partial \dot{\alpha}^i} \right]. \tag{5.16}$$

We wish to prove that under a certain condition the generalized flow equation in Eq. (5.7) defines a geodesic curve associated with the "initial state" $\mid \Psi \rangle = \mid u_n \rangle$, which is an eigenstate of $H_{\rm d}(l)$. (This is the state of relevance, because we are considering the flow to an exact stationary state.) That is, we are going to show that, under the condition found later, the above X_i vanishes for $\eta = \eta^{(a)}$ in Eq. (5.5).

Clearly, $\langle u_n \mid \eta^{(a)} \mid u_n \rangle$ and its derivatives with respect to l and $\dot{\alpha}^i$ vanish. Furthermore, it can be shown by using Eqs. (5.3)-(5.5) that $\langle u_n \mid \left[{\eta^{(a)}}^2, \partial\eta^{(a)}/\partial\dot{\alpha}^i \right] \mid u_n \rangle$ also vanishes. Therefore, X_i for $\mid \Psi \rangle = \mid u_n \rangle$ and $\eta = \eta^{(a)}$ is reduced to

$$\begin{aligned} X_i =& 2\langle u_n \mid {\eta^{(a)}}^2 \mid u_n \rangle \langle u_n \mid \left\{ \frac{d\eta^{(a)}}{dl}, \frac{\partial\eta^{(a)}}{\partial\dot{\alpha}^i} \right\} \mid u_n \rangle \\ & - \langle u_n \mid \frac{d{\eta^{(a)}}^2}{dl} \mid u_n \rangle \langle u_n \mid \frac{\partial{\eta^{(a)}}^2}{\partial\dot{\alpha}^i} \mid u_n \rangle. \end{aligned} \tag{5.17}$$

This equation can be rewritten by using Eqs. (5.3)-(5.5) again as follows:

$$\begin{aligned} X_i =& 4 \left(|D^{(a)}_{n+i_a}|^2 + |D^{(a)}_n|^2 \sum_{m=1}^{d} \delta_{n,m+i_a} \right) \\ & \mathrm{Re} \left[\frac{dD^{(a)}_{n+i_a}}{dl} \frac{\partial D^{(a)*}_{n+i_a}}{\partial\dot{\alpha}^i} + \frac{dD^{(a)}_n}{dl} \frac{\partial D^{(a)*}_n}{\partial\dot{\alpha}^i} \sum_{m=1}^{d} \delta_{n,m+i_a} \right] \\ & - \left(\frac{d|D^{(a)}_{n+i_a}|^2}{dl} + \frac{d|D^{(a)}_n|^2}{dl} \sum_{m=1}^{d} \delta_{n,m+i_a} \right) \left(\frac{\partial|D^{(a)}_{n+i_a}|^2}{\partial\dot{\alpha}^i} + \frac{\partial|D^{(a)}_n|^2}{\partial\dot{\alpha}^i} \sum_{m=1}^{d} \delta_{n,m+i_a} \right), \end{aligned} \tag{5.18}$$

where $D^{(a)}_{n+i_a} \equiv (\epsilon_{n+i_a}(l) - \epsilon_n(l))\, C^{(a)}_{n+i_a}(l)$. The l-derivatives appearing in this expression should be calculated from the generalized flow equation in Eq. (5.7). It is also noted that α^i and $\dot{\alpha}^i$ are not independent in Eqs. (5.17) and (5.18) any more, since Eq. (5.5) has already been used.

5.3 Condition for the geodesic flow

In this section, we present our discovery. If the labels a and a' satisfying

$$i_{a'} \neq 2i_a, 3i_a \quad (1 \leq a' \leq A) \tag{5.19}$$

can be taken for $\eta^{(a)}(l)$, then sandwiching the flow equation in Eq. (5.7) by three pairs, $\langle u_n \mid$ and $\mid u_{n-i_a}\rangle$, $\langle u_{n+i_a} \mid$ and $\mid u_n\rangle$, $\langle u_{n+i_a} \mid$ and $\mid u_{n-i_a}\rangle$, we have

$$\langle u_n \mid \left[\eta^{(a)}(l), \sum_{a'=1}^{A} H_{\text{o-d}}^{(a')}(l)\right] \mid u_{n-i_a}\rangle = 0, \tag{5.20}$$

$$\langle u_{n+i_a} \mid \left[\eta^{(a)}(l), \sum_{a'=1}^{A} H_{\text{o-d}}^{(a')}(l)\right] \mid u_n\rangle = 0, \tag{5.21}$$

$$\langle u_{n+i_a} \mid \left[\eta^{(a)}(l), \sum_{a'=1,a'\neq a}^{A} H_{\text{o-d}}^{(a')}(l)\right] \mid u_{n-i_a}\rangle = 0, \tag{5.22}$$

respectively. And, correspondingly, we obtain the following sandwiched flow equations:

$$\frac{dC_n^{(a)}(l)}{dl} = -\left(\epsilon_n(l) - \epsilon_{n-i_a}(l)\right)^2 C_n^{(a)}(l), \tag{5.23}$$

$$\frac{dC_{n+i_a}^{(a)}(l)}{dl} = -\left(\epsilon_{n+i_a}(l) - \epsilon_n(l)\right)^2 C_{n+i_a}^{(a)}(l), \tag{5.24}$$

$$\left(\epsilon_{n+i_a}(l) + \epsilon_{n-i_a}(l) - 2\epsilon_n(l)\right) C_n^{(a)}(l) C_{n+i_a}^{(a)}(l) = 0. \tag{5.25}$$

It is mentioned that, from Eqs. (5.23) and (5.24), the phases of $C_n^{(a)}(l)$ and $C_{n+i_a}^{(a)}(l)$ are found to be independent of l. Case-A: If $C_n^{(a)}(l) = C_{n+i_a}^{(a)}(l) = 0$, then X_i in Eq. (5.18) obviously vanishes. Case-B: If $C_n^{(a)}(l) = 0$ and $C_{n+i_a}^{(a)}(l) \neq 0$ (or, $C_n^{(a)}(l) = C_{n+i_a}^{(a)}(l) \neq 0$ and $C_{n+i_a}^{(a)}(l) = 0$), then X_i vanishes, due to the fact that the phases of $D_n^{(a)}(l)$ and $D_{n+i_a}^{(a)}(l)$ are independent of l. Case-C: If both $C_n^{(a)}(l)$ and $C_{n+i_a}^{(a)}(l)$ are nonzero, then Eq. (5.25) yields $\epsilon_{n+i_a}(l) + \epsilon_{n-i_a}(l) - 2\epsilon_n(l) = 0$. Combining the relation in Case-C with Eqs. (5.23) and (5.24), we obtain a crucial result that $|C_{n+i_a}^{(a)}(l)|$ *is proportional to* $|C_n^{(a)}(l)|$. And, this makes X_i vanish again.

Therefore, we conclude that the flow equation with the generator $\eta^{(a)}(l)$ with the label a satisfying the condition in Eq. (5.19) gives rise to the geodesic flow of $\mid u_n\rangle$, independently of a specific initial condition. This is the main result of the present dissertation.

5.4 Physical Examples

In what follows, we illustrate our result by analyzing some physical examples.

The first example we consider is the generalized harmonic oscillator. The Hamiltonian reads

$$H = \omega a^\dagger a + \lambda {a^\dagger}^2 + \lambda^* a^2 + \mu a^\dagger + \mu^* a + \nu. \tag{5.26}$$

Here, $a^\dagger$ and a are the creation and annihilation operators obeying the algebra: $[a, a^\dagger]$, $[a^\dagger, a^\dagger] = [a, a] = 0$. ω (> 0) and ν are real constants, whereas λ and μ are complex. In particular, the condition, $\omega > 2|\lambda|$, is to be satisfied. Here and hereafter, $\hbar$ is set equal to unity. We analyze the following two cases: (i) $\mu \neq 0$, $\lambda = 0$ and (ii) $\mu = 0$, $\lambda \neq 0$. In case (i), the Hamiltonian is transformed by the displacement operator

$$U(l) = \exp\left[z(l)a^\dagger - z^*(l)a\right] \tag{5.27}$$

with the complex $z(l)$ satisfying $z(0) = 0$. For the normalized ground state, $\mid 0\rangle$, satisfying $a \mid 0\rangle = 0$, as the "initial state" at $l = 0$, the flow of the state is along

the coherent state. The corresponding Fubini-Study metric is Euclidean [16]:

$$ds^2 = \frac{1}{2}(dx^2 + dp^2) \tag{5.28}$$

with the parametrization, $z = (x + ip)/\sqrt{2}$. Then, the condition in Eq. (5.19) is satisfied, and the flow is, in fact, geodesic: a straight line in the space with a global coordinate (x, p). In case (ii), the squeezing operator

$$U(l) = \exp\left[\frac{1}{2}\left(\xi(l)a^{\dagger 2} - \xi^*(l)a^2\right)\right] \tag{5.29}$$

is to be considered, where the complex coefficient $\xi(l)$ is parametrized as $\xi(l) = r(l)\mathrm{e}^{-2i\phi(l)}$ $(0 \leq r(l), 0 \leq \phi(l) < 2\pi)$. Accordingly, $(\alpha^1, \alpha^2) \equiv (r, \phi)$. The condition $U(0) = I$ (with the identity operator I) leads to $r(0) = 0$. The initial state is taken to be the number state, $\mid n\rangle = (n!)^{-1/2}a^{\dagger n} \mid 0\rangle$. The corresponding metric is [16]:

$$ds^2 = \frac{1}{2}(n^2 + n + 1)\left[dr^2 + \left(\sinh^2 2r\right)d\phi^2\right], \tag{5.30}$$

which shows that the manifold is the Lobachevsky space. The transformed Hamiltonian is written as follows:

$$H(l) = \omega(l)a^\dagger a + \lambda(l)a^{\dagger 2} + \lambda^*(l)a^2 + \nu(l). \tag{5.31}$$

The coefficients appearing here depend not only on their original values at $l = 0$ but also on $\xi(l)$. In this case,

$$H_{\text{o-d}}(l) = \lambda(l)a^{\dagger 2} + \lambda^*(l)a^2 \tag{5.32}$$

is the one and only off-diagonal part. Therefore, the generator in Eq. (5.5) is identical to Wegner's choice

$$\eta^W(l) = 2\omega(l)\left[\lambda(l)a^{\dagger 2} - \lambda^*(l)a^2\right], \tag{5.33}$$

and the condition in Eq. (5.19) is automatically fulfilled. The flow equation for $\lambda(l)$ is given by

$$\frac{d\lambda(l)}{dl} = -4\omega^2(l)\lambda(l), \tag{5.34}$$

showing that the phase of $\lambda(l)$ does not depend on l. Then, one can explicitly find that Eq. (5.17) indeed vanishes. Comparing $[dU(l)/dl]U^\dagger(l)$ with $\eta^W(l)$, we obtain $\phi(l) = \text{const}$, which in fact turns out to make both $\delta S/\delta r(l)$ and $\delta S/\delta\phi(l)$ vanish, where S is the arc-length functional defined in terms of the above metric. We also mention that the spectrum of $H_{\rm d}(l) = \dot{\omega}(l)a^\dagger a + \nu(l)$ is equally spaced, and accordingly Case-C in the preceding section is realized. Thus, Wegner's flow is geodesic.

The second example is a spin-s, $\mathbf{S} = (S_x, S_y, S_z)$, in a constant external magnetic field $\mathbf{B}$. The Hamiltonian reads

$$H = \mathbf{S}\cdot\mathbf{B} \tag{5.35}$$

in an appropriate unit. The unitary operator to be considered is

$$U(l) = \exp\left[\sigma(l)S_+ - \sigma^* S_-\right] \tag{5.36}$$

where $S_\pm \equiv S_x \pm iS_y$ and the complex coefficient $\sigma(l)$ is parametrized as $\sigma(l) = [\theta(l)/2]\mathrm{e}^{-i\varphi(l)}$ $(0 \leq \theta(l) < \pi, 0 \leq \varphi(l) < 2\pi)$ with $\theta(0) = 0$. The basic commutation relations satisfied by the spin operators are as follows: $[S_z, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_z$. The local coordinate is given by $(\alpha^1, \alpha^2) \equiv (\theta, \varphi)$. The initial state is taken to be $\mid m\rangle_s$ $(m = -s, -s+1, \ldots, 0, \ldots, s-1, s)$, which satisfies $S_z \mid m\rangle_s = m \mid m\rangle_s$. The metric is found to be given by [16]:

$$ds^2 = \frac{1}{2}(s^2 + s - m^2)\left[d\theta^2 + \left(\sin^2\theta\right) d\varphi^2\right], \tag{5.37}$$

which is of a sphere. The transformed Hamiltonian is written as

$$H(l) = \beta_z(l)S_z + \beta(l)S_+ + \beta^*(l)S_-. \tag{5.38}$$

$\beta_z(l)$ is real, whereas $\beta(l)$ is complex. They depend not only on **B** but also on $\sigma(l)$. The one and only off-diagonal part is

$$H_{\text{o-d}}(l) = \beta(l)S_+ + \beta^*(l)S_-, \tag{5.39}$$

and so Wegner's choice

$$\eta^W(l) = \beta_z(l)[\beta(l)S_+ - \beta^*(l)S_-] \tag{5.40}$$

is employed. Therefore, clearly the condition in Eq. (5.19) is fulfilled. The flow equation for $\beta(l)$ is

$$\frac{d\beta(l)}{dl} = -\beta_z^2(l)\beta(l), \tag{5.41}$$

from which the phase of $\beta(l)$ is seen to be independent of l, and accordingly Eq. (5.17) vanishes. Comparison of the above generator, $\eta^W(l)$, with $[dU(l)/dl]U^\dagger(l)$ yields $\varphi(l) = \text{const}$, which leads to the fact that the variations of the arc-length functional (S, calculated using the above metric) with respect to $\theta(l)$ and $\varphi(l)$ vanish. Also, the spectrum of $H_{\rm d}(l) = \beta_z(l)S_z$ with $s \geq 1$ is equally spaced, and so Case-C in the preceding section is realized. Thus, Wegner's flow is geodesic (i.e., the great circle).

The third example is the Jaynes-Cummings model [20], which describes a two-level atom interacting with a single-mode radiation field. The Hamiltonian is given by

$$H = \frac{1}{2}\omega_0\sigma_3 + \omega a^\dagger a + \kappa(\sigma_+ a + \sigma_- a^\dagger). \tag{5.42}$$

Here, ω_0, ω, and κ are the transition frequency of the atom, the frequency of the radiation, and the coupling constant, respectively. $a^\dagger$ and a are the creation and annihilation operators of the radiation field satisfying the same algebra as in the first example. σ's are the operators of the atom, which are given in terms of the orthonormal basis, the ground state $\mid g\rangle$ and the excited state $\mid e\rangle$, as follows: $\sigma_+ = \mid e\rangle\langle g \mid$, $\sigma_- = \mid g\rangle\langle e \mid$, $\sigma_3 = \mid e\rangle\langle e \mid - \mid g\rangle\langle g \mid$. This model is known to be exactly solvable. The associated unitary operator is

$$\begin{aligned} U(l) = \sum_{n=0}^{\infty} \big[& \alpha_n(l)\sigma_+\sigma_- \mid n\rangle\langle n \mid + \beta_{n-1}(l)\sigma_-\sigma_+ \mid n\rangle\langle n \mid \\ & \gamma_n(l)\sigma_+ \mid n\rangle\langle n+1 \mid + \delta_n(l)\sigma_- \mid n+1\rangle\langle n \mid \big], \end{aligned} \tag{5.43}$$

where $\mid n\rangle = (n!)^{-1/2}a^{\dagger n} \mid 0\rangle$ is the n-photon state (for the details of this unitary operator, see Chapter 4). The unitarity of this operator leads to the conditions: $|\alpha_n|^2 + |\gamma_n|^2 = 1$, $|\alpha_n| = |\beta_n|$, $|\gamma_n| = |\delta_n|$, and $\alpha_n\delta_n^* + \beta_n^*\gamma_n = 0$. From the polar forms, $\alpha_n = |\alpha_n|\exp(i\theta_{\alpha_n})$ and so on, follows the condition on the phases: $\theta_{\alpha_n} + \theta_{\beta_n} - \theta_{\gamma_n} - \theta_{\delta_n} = (2m+1)\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$. Here, we choose $\mid e\rangle \mid n\rangle$ as the initial state. $|\alpha_n|$, θ_{α_n}, θ_{β_n}, and θ_{γ_n} form as a set of independent coordinate variables. The metric is

$$ds^2 = \frac{d|\alpha_n|^2}{1 - |\alpha_n|^2} + |\alpha_n|^2(1 - |\alpha_n|^2)(d\theta_{\alpha_n} - d\theta_{\gamma_n})^2, \tag{5.44}$$

which does not seem to be the one of a familiar space. The θ_{β_n}-dependence disappears due to the above choice of the initial state. The transformed Hamiltonian is

$$\begin{aligned} H(l) = \sum_{n=0}^{\infty} \big[& A_n(l)\sigma_+\sigma_- \mid n\rangle\langle n \mid + B_{n-1}(l)\sigma_-\sigma_+ \mid n\rangle\langle n \mid \\ & C_n(l)\sigma_+ \mid n\rangle\langle n+1 \mid + C_n^*(l)\sigma_- \mid n+1\rangle\langle n \mid \big], \end{aligned} \tag{5.45}$$

and so

$$H_{\text{o-d}}(l) = \sum_{n=0}^{\infty} \big[C_n(l)\sigma_+ \mid n\rangle\langle n+1 \mid + C_n^*(l)\sigma_- \mid n+1\rangle\langle n \mid \big], \tag{5.46}$$

where the coefficients, $A_n(l)$'s and $B_{n-1}(l)$'s are real, and $C_n(l)$'s turn out to be also real (as can be seen from the flow equations). They are expressed in terms of the physical coefficients contained in the original Hamiltonian as well as the coefficients appearing in $U(l)$ (for details, see Chapter 4). Let us consider

Wegner's choice:

$$\eta^W(l) = \sum_{n=0}^{\infty}[A_n(l) - B_n(l)]C_n(l)(\sigma_+ \mid n\rangle\langle n+1 \mid -\sigma_- \mid n+1\rangle\langle n \mid). \quad (5.47)$$

Then, we find that Eq. (5.17) vanishes. The flow equations give rise to

$$\frac{d\theta_{\alpha_n}(l)}{dl} = 0, \quad (5.48)$$

$$\frac{d\theta_{\gamma_n}(l)}{dl} = 0, \quad (5.49)$$

which explicitly make the variations of the arc-length functional with respect to $|\alpha_n|$, θ_{α_n}, and θ_{γ_n} all vanish. In addition, comparing $\eta(l) = [H_{\rm d}(l), H_{\text{o-d}}(l)]$ with $\eta^W(l)$ above, we see that Case-B in the preceding section is realized. Thus, again Wegner's flow is geodesic.

Bibliography

[1] F. Wegner, Ann. Phys. (Leipzig) **3**, 77 (1994).

[2] F.J. Wegner, Phys. Rep. **348**, 77 (2001).

[3] F. Wegner, J. Phys. A: Math. Gen. **39**, 8221 (2006).

[4] S.K. Kehrein and A. Mielke, Ann. Phys. (NY) **252**,1 (1996).

[5] A.B. Bylev and H.J. Pirner, Phys. Lett. B **428**, 329 (1998).

[6] C. Raas, U. Low, G.S. Uhrig, and R.W. Kuhne, Phys. Rev. B **65**, 144438 (2002).

[7] V. Hankevych and F. Wegner, Eur. Phys. J. B **31**, 333 (2003).

[8] T. Stauber, Phys. Rev. B **67**, 205107 (2003).

[9] T. Domanski and J. Ranninger, Phys. Rev. B **70**, 184513 (2004).

[10] S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004).

[11] S. Dusuel, J. Vidal, J.M. Arias, J. Dukelsky,and J.E. Garcia-Ramos, Phys. Rev. C **72**, 064332 (2005).

[12] S.D. Glazek, Nucl. Phys. B (Proc. Suppl.) **161**, 59 (2006).

[13] G. Aprea, C. Di Castro, M. Grilli,and J. Lorenzana, Nucl. Phys. B **744**, 277 (2006).

[14] Y. Itto and S. Abe, Physica A **387**, 1063 (2008).

[15] Y. Itto and S. Abe, When is Wegner's flow geodesic?, e-print quant-ph/0806.4425.

[16] S. Abe, Phys. Rev. A **46**, 1667 (1992); S. Abe, Phys. Rev. A **48**, 4102 (1993).

[17] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol. II, Interscience, New York, 1969.

[18] D.N. Page, Phys. Rev. A 36 (1987) 3479.

[19] A.K. Pati, J. Phys. A: Math. Gen. **25**, L1001 (1992).

[20] E.T. Jaynes, F.W. Cummings, Proc. IEEE 51 (1963) 89.

Conclusion

We have reviewed the fundamental property of the Jaynes-Cummings model and the basics of the method of flow equations. Then, we have discussed the flow equation approach as a novel technique of quantum-state engineering for establishing stationary photon-atom entanglement. To illustrate the approach, we have employed the Jaynes-Cummings model as a prototype example and performed its complete flow equation analysis. We have also calculated the positive operator-valued measures for the photons and atom, evaluated the degree of stationarity, and analyzed flow of the entanglement entropy. In addition, we have presented a procedure for implementing the unitary operation associated with the flow equation.

Next, we have developed a new theory for the method of flow equations, which generalizes Wegner's method itself. Through this development, we have found a condition, under which the corresponding flow of a quantum state becomes geodesic in a quantum evolution submanifold, independently of specific

initial conditions at $l = 0$. We have illustrated this by employing the generalized harmonic oscillator, a spin in a constant external magnetic field, and the Jaynes-Cummings model.

The present result implies that the method of flow equations is not just a mathematical tool for diagonalizing a Hamiltonian but provides the optimal strategy in quantum-state engineering for realizing a stationary state from a given initial state in each of a wide class of systems. In fact, a formal solution, $U(l) = P_l \exp \int_0^l dl' \eta^W(l')$ with P_l being Dyson's "l-ordering" symbol, can be divided into the product of many unitary operators, each of which defines a small translation along the geodesic flow and may represent each quantum operation performed experimentally. In the present study, we have discussed the metric structure and associated geodesic nature of the flow of a state in a quantum evolution submanifold. Other basic quantities such as curvature are not explicitly treated here. However, curvature becomes relevant, for example, when the deviation of two geodesic curves with different initial conditions at $l = 0$ is considered. This kind of considerations may cast further light on geometry of the method of flow equations.

Furthermore, we have noted that there exist invariant quantities with respect to l-evolution in the class of exactly solvable systems. This may imply that the method of flow equations admits the underlying continuous symmetry. This fact leads us to examine Noether's theorem in the context of the flow equations. This issue is, however, nontrivial, and needs further elaborations.

Finally, we have studied the invariant in the time-dependent harmonic oscillator based on Noether's theorem. The topic discussed here nicely interconnects dynamics of the time-dependent harmonic oscillator, the concept of invariant, Noether's theorem and symmetry.

In the present dissertation, we have shown how the method of flow equations offers a useful approach for quantum-state engineering. We believe that this method may be able to play a particularly key role in quantum-information technology.

Appendix A

Invariant of the time-dependent harmonic oscillator derived from Noether's theorem

In the flow equation analyses of several physical examples, which are the generalized harmonic oscillator, a spin in a constant external magnetic field, and the Jaynes-Cummings model, we became aware of the fact that there seem to exist invariant quantities in the system of flow equations. This may imply that the method admits the underlying continuous symmetry. Then, this implication leads us to examine Noether's theorem [1-3], which establishes a profound relation between a conservation law and a symmetry contained in a system, in the context of

the flow equations. This issue is, however, nontrivial in connection with the flow equations, and needs further elaborations.

Here, we consider the exact invariant of the time-dependent harmonic oscillator given by Lewis [4,5], who has derived it by making use of the complicated transformations of the variables for the action integral [6]

$$I = \frac{1}{2\pi} \oint p dx \tag{A.1}$$

concerning a periodic orbit in the phase space. Then, Lutzky [7] has shown in a transparent manner how to derive Lewis'invariant based on Noether's theorem. Existence of an invariant is of basic importance in dynamics, since it plays a role of an integral of motion and puts a constraint, for example, on the initial condition.

In this appendix, we present a plain review [8] of the work in [7].

A.1 Exact invariant of the time-dependent harmonic oscillator

The Lagrangian of the time-dependent harmonic oscillator (with the unit mass) reads

$$L(t) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x^2, \tag{A.2}$$

where the over-dot stands for the differentiation with respect to time t, and $\omega(t)$ is an arbitrary function of t. The Hamiltonian associated with the above Lagrangian

is

$$H(t) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)x^2, \tag{A.3}$$

where $p \equiv \partial L/\partial \dot{x}$ stands for the canonical momentum conjugate to the canonical coordinate x. The time evolution of a quantity $A = A(p, x, t)$ is given by [6]

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{H(t), A\}_0, \tag{A.4}$$

where the subscript "0" indicates that the Poisson bracket is defined at "equal time". Explicitly, the equal-time Poisson bracket for physical quantities, $A = A(p, x, t)$ and $B = B(p, x, t)$, is defined by

$$\{A, B\}_0 = \frac{\partial A(p,x,t)}{\partial p}\frac{\partial B(p,x,t)}{\partial x} - \frac{\partial A(p,x,t)}{\partial x}\frac{\partial B(p,x,t)}{\partial p}. \tag{A.5}$$

The equation of motion,

$$\ddot{x} + \omega^2(t)x = 0, \tag{A.6}$$

can be obtained from the Euler-Lagrange equation, $d(\partial L/\partial \dot{x})dt - \partial L/\partial x = 0$, with (A.2) or the canonical equations, $\dot{x} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial x$, with (A.3).

In Refs. [4, 5], Lewis has employed complicated transformations of the variables in Eq. (A.1) and obtained the following exact expression for the action integral as an invariant:

$$I(t) = \frac{1}{2}\left[(\rho p - \dot{\rho}x)^2 + \frac{x^2}{\rho^2}\right], \tag{A.7}$$

where $\rho = \rho(t)$ is an auxiliary variable satisfying

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}. \tag{A.8}$$

Using Eqs. (A.6) and (A.8), we find that $I(t)$ satisfies

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} + \{H(t), I(t)\}_0 = 0, \tag{A.9}$$

which implies in view of Eq. (A.4) that $I(t)$ is in fact an invariant.

To see the physical meaning of the invariant, it may be helpful to consider the situation when $\omega(t)$ slowly changes in time. In such an adiabatic case, the time derivative in Eq. (A.8) is negligible in the lowest-order approximation, yielding $\rho^{(0)}(t) = \omega^{-1/2}(t)$. The higher-order effects can systematically be evaluated by performing the expansion around $\rho^{(0)}(t)$. For example, one can determine the first-order correction $\rho^{(1)}(t)$ by substituting $\rho^{(0)} + \rho^{(1)}$ into Eq. (A.8) and solving the linearized equation for $\rho^{(1)}$. The result is

$$\begin{aligned} \rho(t) &= \rho^{(0)}(t) + \rho^{(1)}(t) + \cdots \\ &= \omega^{-1/2}(t) + \frac{1}{8}\omega^{-7/2}(t)\ddot{\omega}(t) - \frac{3}{16}\omega^{-9/2}(t)\dot{\omega}^2(t) + \cdots . \end{aligned} \tag{A.10}$$

In the adiabatic limit, $\rho(t) = \omega^{-1/2}(t)$, the invariant in Eq. (A.7) becomes

$$I(t) = \frac{H(t)}{\omega(t)}, \tag{A.11}$$

showing that it is the action variable (of the action-angle variables [6]).

A.2 Noether's theorem

As mentioned earlier, Lewis has derived in [4] the exact invariant (A.7) by making use of the complicated transformations of the variables for the quantity in Eq.

(A.1). A more transparent approach has been elaborated by Lutzky [7], who has derived the invariant by making use of Noether's theorem. We devote this section to reviewing this approach in detail.

Let us consider a generic action functional

$$S = \int dt L(x, \dot{x}, t), \tag{A.12}$$

where the Lagrangian has explicit time dependence. Define the following generator of transformation:

$$G = \xi(x,t)\frac{\partial}{\partial t} + \zeta(x,t)\frac{\partial}{\partial x}, \tag{A.13}$$

where $\xi(x,t)$ and $\zeta(x,t)$ are some differentiable functions to be determined later. The associated changes of the relevant variables are given as follows:

$$\delta_G t = \xi, \tag{A.14}$$

$$\delta_G(dt) \equiv d(\delta_G t) = \dot{\xi} dt, \tag{A.15}$$

$$\delta_G x = \zeta, \tag{A.16}$$

$$\delta_G \dot{x} = \left(\delta_G \frac{d}{dt}\right) x + \frac{d}{dt}(\delta_G x) = -\dot{\xi}\dot{x} + \dot{\zeta}, \tag{A.17}$$

where $\dot{\xi} = \partial\xi/\partial t + \dot{x}\partial\xi/\partial x$ and so on, that is, the over-dot denotes the total derivative with respect to time. In Eq. (A.17), $\delta_G(d/dt) = -\dot{\xi}d/dt$ is used, which results from a simple relation: $\delta_G(dt/dt) = 0$. Therefore, the change of the action

in Eq. (A.12) is calculated to be

$$\begin{aligned}\delta_G S &= \int \delta_G(dt) L + \int dt \delta_G L \\ &= \int dt \left[\dot{\xi} L + \zeta \frac{\partial L}{\partial x} + (\dot{\zeta} - \dot{\xi}\dot{x}) \frac{\partial L}{\partial \dot{x}} + \xi \frac{\partial L}{\partial t} \right].\end{aligned} \tag{A.18}$$

If the integrand has the form of the total time derivative of some function, say $f(x,t)$,

$$\dot{\xi} L + \eta \frac{\partial L}{\partial x} + (\dot{\zeta} - \dot{\xi}\dot{x}) \frac{\partial L}{\partial \dot{x}} + \xi \frac{\partial L}{\partial t} = \dot{f}, \tag{A.19}$$

then the action is said to be invariant under the transformation generated by G.

Applying the above discussion to the Lagrangian in Eq. (A.2), we immediately obtain

$$\begin{aligned}&-\omega\dot{\omega}\xi x^2 - \omega^2 \eta x - \frac{1}{2}\omega^2 \frac{\partial \xi}{\partial t} x^2 - \frac{\partial f}{\partial t} \\ &+ \left(\frac{\partial \zeta}{\partial t} - \frac{1}{2}\omega^2 \frac{\partial \xi}{\partial x} x^2 - \frac{\partial f}{\partial x} \right) \dot{x} + \left(\frac{\partial \eta}{\partial x} - \frac{1}{2}\frac{\partial \xi}{\partial t} \right) \dot{x}^2 - \frac{1}{2}\frac{\partial \xi}{\partial x} \dot{x}^3 = 0.\end{aligned} \tag{A.20}$$

This identical relation with respect to $\dot{x}$ (the equation of motion for x is not used yet at this stage) yields the following set of equations:

$$\omega\dot{\omega}\xi x^2 + \omega^2 \zeta x + \frac{1}{2}\omega^2 \frac{\partial \xi}{\partial t} x^2 + \frac{\partial f}{\partial t} = 0, \tag{A.21}$$

$$\frac{\partial \zeta}{\partial t} - \frac{1}{2}\omega^2 \frac{\partial \xi}{\partial x} x^2 - \frac{\partial f}{\partial x} = 0, \tag{A.22}$$

$$\frac{\partial \zeta}{\partial x} - \frac{1}{2}\frac{\partial \xi}{\partial t} = 0, \tag{A.23}$$

$$\frac{1}{2}\frac{\partial \xi}{\partial x} = 0. \tag{A.24}$$

From Eq. (A.24), we see that $\xi = \xi(t)$, i.e., ξ is independent of x. So, Eq. (A.23) is integrated as

$$\zeta(x,t) = \frac{1}{2}\dot{\xi}(t)x + \Psi(t), \tag{A.25}$$

where $\Psi(t)$ is an arbitrary differentiable function of time. This allows us to rewrite Eq. (A.22) in the form

$$\frac{\partial f}{\partial x} = \frac{\partial \zeta}{\partial t} = \frac{1}{2}\ddot{\xi}(t)x + \dot{\Psi}(t), \tag{A.26}$$

which gives rise to

$$f(x,t) = \frac{1}{4}\ddot{\xi}(t)x^2 + \dot{\Psi}(t)x + c(t), \tag{A.27}$$

where $c(t)$ is an arbitrary function of time. Accordingly, Eq. (A.21) becomes

$$\left(\omega\dot{\omega}\xi + \omega^2\dot{\xi} + \frac{1}{4}\dddot{\xi}\right)x^2 + (\ddot{\Psi} + \omega^2\Psi)x + \dot{c} = 0. \tag{A.28}$$

This identical relation for x leads to

$$\dddot{\xi} + 4\omega^2\dot{\xi} + 4\omega\dot{\omega}\xi = 0, \tag{A.29}$$

$$\ddot{\Psi} + \omega^2\Psi, \tag{A.30}$$

$$\dot{c} = 0. \tag{A.31}$$

A simple choice

$$\Psi = 0, \quad c = 0, \tag{A.32}$$

makes Eqs. (A.25) and (A.27) be

$$\zeta(x,t) = \frac{1}{2}\dot{\xi}(t)x \tag{A.33}$$

and

$$f(x,t) = \frac{1}{4}\ddot{\xi}(t)x^2, \tag{A.34}$$

respectively. On the other hand, Eq. (A.29) has the first integral

$$\xi\ddot{\xi} - \frac{1}{2}\dot{\xi}^2 + 2\xi^2\omega^2 = c_1, \tag{A.35}$$

where c_1 is a constant, whose value can be controlled by rescaling ξ. Setting $c_1 = 2$ and

$$\xi = \rho^2(t), \tag{A.36}$$

we obtain the auxiliary Eq. (A.8).

Now, we are at the stage of deriving Noether's invariant. Recall that $\delta_G S = \int dt \dot{f}$. From this and Eq. (A.18), we have

$$\begin{aligned} 0 &= \int dt\dot{f} - \int dt \left\{ \dot{\xi}L + \zeta\frac{\partial L}{\partial x} + (\dot{\zeta} - \dot{\xi}\dot{x})\frac{\partial L}{\partial \dot{x}} + \xi\frac{\partial L}{\partial t} \right\} \\ &= \int dt \left\{ (\xi\dot{x} - \zeta)\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] + \frac{d}{dt}\left[(\xi\dot{x} - \zeta)\frac{\partial L}{\partial \dot{x}} - \xi L + f\right]\right\}, \\ &= \int dt \frac{d}{dt}\left[(\xi\dot{x} - \zeta)\frac{\partial L}{\partial \dot{x}} - \xi L + f\right], \end{aligned} \tag{A.37}$$

where *the equation of motion (i.e., the Euler-Lagrange equation) has been used.* Therefore, the invariant is found to be given by

$$I(t) = (\xi\dot{x} - \zeta)\frac{\partial L}{\partial \dot{x}} - \xi L + f. \tag{A.38}$$

Finally, substituting Eqs. (A.2), (A.33), (A.34), and (A.36) into Eq. (A.38) and using Eq. (A.8), we finally obtain the exact invariant in Eq. (A.7) in terms of the canonical variables.

Bibliography

[1] E. Noether, Nachr. v. d. Ges. d. Wiss. zu Gottingen 235 (1918).

[2] R.M. Marinho Jr., Eur. J. Phys. **28**, 37 (2007).

[3] S. Weinberg, The Quantum Theory of Fields Volume I (Cambridge: Cambridge University Press) 1995.

[4] H.R. Lewis Jr., Phys. Rev. Lett. **18**, 510 (1967).

[5] H.R. Lewis Jr., 1968 J. Math. Phys. **9**, 1976 (1968).

[6] L.D. Landau and E.M. Lifshitz, Mechanics 3rd edition (Oxford: Pergamon) 1976.

[7] M. Lutzky, Phys. Lett. A **68**, 3 (1978).

[8] S. Abe, Y. Itto, and M. Matsunaga, Eur. J. Phys, **30**, 1337 (2009).